Linear Approximation

ID: 8355

Time required
45 minutes

Activity Overview

Students will use the tangent to a function to estimate the value of the function near the point of tangency. They will also observe errors associated with these approximations, decide if these errors represent underestimates or overestimates, and find intervals that guarantee a specified level of precision.

Topic: Application of Derivatives

- *Calculate the equation of the tangent line to a graph at any given point.*
- *Construct a tangent line to the graph of a differentiable function at x = a to approximate its value near x = a.*
- *Construct a tangent line to the graph of a differentiable function at x = 0 to approximate its value near x = 0.*

Teacher Preparation & Notes

- *This investigation offers an opportunity for students to develop an understanding of how a tangent line to a curve can be used to approximate the values of a function near the point of tangency. Linear approximations are often used in scientific applications, including formulas used in physics where sin θ is replaced with its linear approximation, θ.*
- *Students should already know how to find the equation of the tangent to a curve at a specified point. Most likely, they have worked on such problems for the purpose of finding instantaneous rate of change.*
- *In the latter part of this activity, students graphically investigate finding intervals of x that ensure a given level of precision for the linear approximations. As an extension, you may wish to have students investigate how to find these intervals analytically by solving the absolute value inequality $|L(x) - f(x)| < e$, where e is the desired level of accuracy and L(x) is the linear approximation at x = a.*
- *Notes for using the TI-Nspire™ Navigator™ System are included throughout the activity. The use of the Navigator System is not necessary for completion of this activity.*
- ***To download the student TI-Nspire document (.tns file) and student worksheet, go to education.ti.com/exchange and enter "8355" in the keyword search box.***

Associated Materials

- *LinearApprox_Student.doc*
- *LinearApprox.tns*

Suggested Related Activities

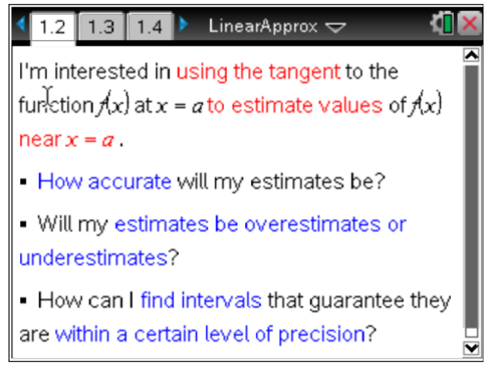
To download any activity listed, go to education.ti.com/exchange and enter the number in the keyword search box.

- *Epislon-Delta Window Challenge (TI-Nspire technology) — 16065*
- *Slopes of Secant Lines (TI-Nspire technology) — 16085*
- *Secant and Tangent Lines (TI-Nspire technology) — 11141*

Introduction

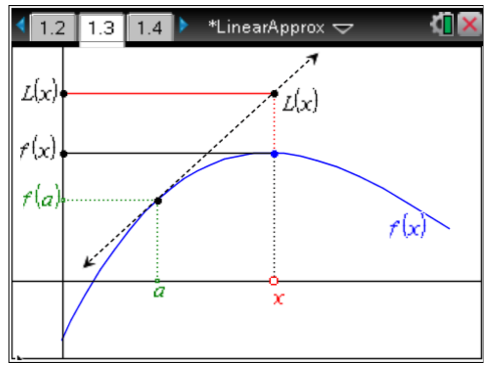
One focus question defines this activity:

How can I use the tangent line to a curve to estimate the values of a function?



Explain to students that the questions posed on page 1.2 will be explored during this activity and are central to the concept of linear approximation.

Use the diagram on page 1.3 to provide a visual reference for the focus question and to allow students to understand the concept behind linear approximation.



Solutions

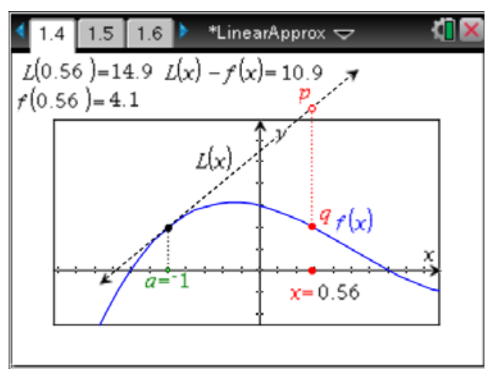
- Labels are shown in the diagram at right.
- $L(x)$ represents the approximation.
- $L(x) - f(x)$ represents the error.
- In this instance, the linear approximation, $L(x)$, is greater than the actual value of the function, $f(x)$, so it is an overestimate.

TI-Nspire Navigator Opportunity: *Live Presenter*

See Note 1 at the end of this lesson.

Investigating linear approximation

Students begin working cooperatively on page 1.4, with the graph of the function **f1**$(x) = x^3 - 3x^2 - 2x + 6$ and its tangent at $a = -1$. The questions posed on the student worksheet lead them through an examination of the important aspects of linear approximation.

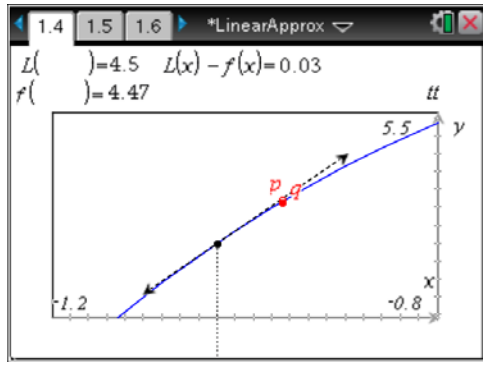


Solutions

- The y-coordinate of point p represents the approximation.
- The difference in the y-coordinates of points p and q, denoted $L(x) - f(x)$, gives the error.
- The y-coordinate of point q gives the true value.
- The tangent line lies above the graph of **f1**(x), so all approximations (within these window settings) are overestimates.
- As point p is dragged closer to the point of tangency, the error decreases.

- The screenshot above shows a point where the linear approximation has an error of less than 0.5. Here, point *p* is within $\left|-0.713-(-1)\right|=0.287$ (horizontal) units from point *a*.
- Dragging point *p* to the left of the point of tangency reveals that the linear approximation continues to be an overestimate.

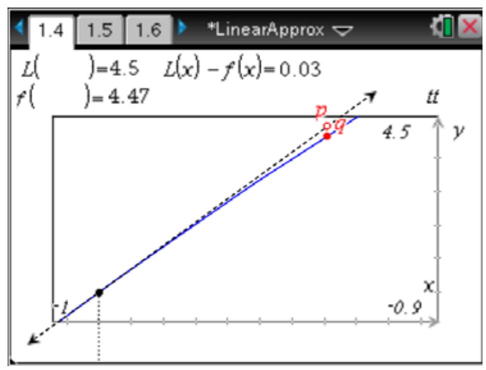
Using the **Zoom – Box** feature, students can better see the relationship between the tangent and the curve as point *p* is dragged closer to the point of tangency.



Solutions

- Near the point of tangency, the linearization and the curve appear to almost coincide.
- Since the tangent and line almost seem to coincide, these linear approximations can be quite accurate near *a*. This observation is often referred to as *local linearization*.

Students should know how to determine the equation of the tangent at *a*, and before proceeding they should check their equation using the **Coordinates and Equations** tool from the Tools menu.

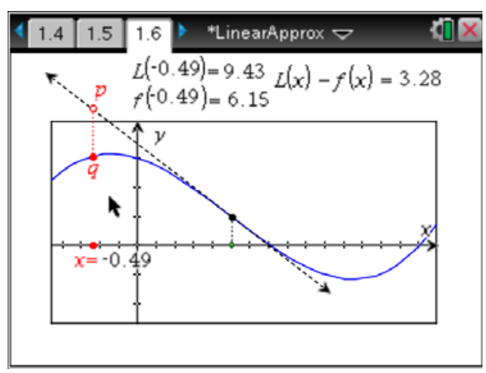


Solutions

- $L(x) = 7x + 11$
- $L(–1.03) = 3.79$; This means that **f1**(–1.03) ≈ 3.79. (See the screenshot at the right.)
- The error associated with the linear approximation is about 0.0054 and that the estimate is still an overestimate.

Underestimates versus overestimates

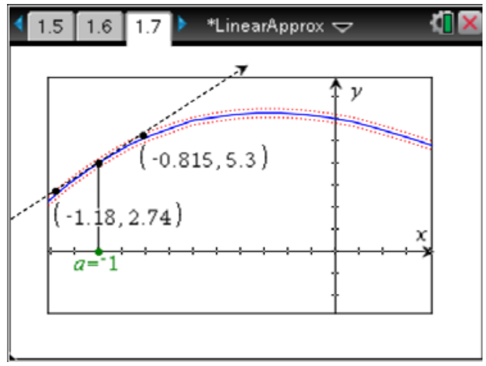
Page 1.6 displays a linear approximation for the same function **f1**, but for a different point of tangency, namely, *a* = 1.



Solutions

- When *p* is to the left of the point of tangency, the approximation is an overestimate.
- When *p* is moved to the *right* of the point of tangency, the error associated with the linear approximation becomes negative, which indicates an *underestimate*.
- The point of tangency in this specific problem is an inflection point of the function, since its concavity switches at *a* = 1.
- In general, linear approximations produce overestimates for values of *x* where the curve is concave down and underestimates when the curve is concave up.

Finding intervals of accuracy

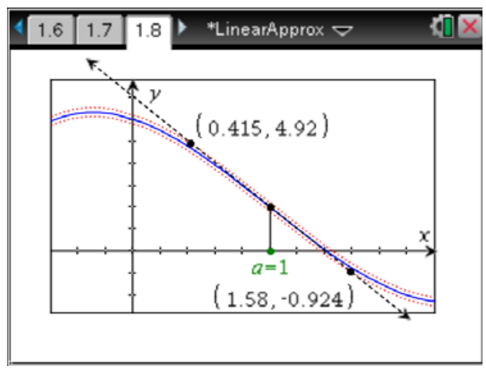


Page 1.7 again displays the graph of f(x), along with the graphs of g(x) = f(x) + 0.2 and h(x) = f(x) – 0.2. The point of tangency is at a = –1. Using this diagram, students can visually explore the x-values for the linear approximation to **f1**(x) with a specified error, in this case, 0.2 units.

Students should note that $L(x)$ only intersects the graph that has been shifted up due to the concavity of the function near a = –1.

Solutions

- The transformed graphs are shifted by the desired amount of accuracy, in this case 0.2 units. The x-coordinates of the intersection points between $L(x)$ and the shifted graph(s) will give the x-interval that ensures the associated errors are within 0.2 units of the actual value of f(x).
- The screenshot above shows that the interval [–1.180, –0.815] guarantees that $L(x)$ gives results within 0.2 units of f(x).



On page 1.8, students consider the same error specifications (within 0.2 units from the true value) for a different point of tangency, namely, a = 1.

Solutions

- Unlike the diagram on page 1.7, here, $L(x)$ intersects both shifted graphs—due to the fact that the concavity of f(x) changes at a = 1. When p is to the left of the point of tangency, the approximation is an overestimate; to the right, it is an underestimate.
- The screen at right shows that the interval [0.415, 1.585] guarantees that $L(x)$ gives results within 0.2 units of f(x).

TI-Nspire Navigator Opportunities

Note 1
Problem 1, *Live Presenter*

It may be helpful to students if *Live Presenter* is used to discuss the interactivity of this page. Try grabbing and dragging point x on the x-axis. Ask students what is happening to $F(x)$ and $L(x)$ as point x is dragged further away/closer to point a. These discussions will help lay the foundation for the activity.

Note 2
Problem 2, *Quick Poll, Live Presenter*

After the students have explored the effects of dragging both sliders, send a *Quick Poll* asking what effect each slider has on the shape and position of the curve. Use *Live Presenter* to clear up any misunderstandings the quick poll shows.

