Constructing an Ellipse

ID: 9980

Time required
60 minutes

Activity Overview

This activity introduces ellipses from a geometric perspective. Two different methods for constructing an ellipse are presented and explored. Students discover that the sum of the distances from a point on an ellipse to its foci is always constant. This fact is then used as the basis for an algebraic derivation of the general equation for an ellipse centered at the origin.

Topic: Analytic Geometry — Conics & Polar Coordinates

- *Derive the equation (in rectangular form) of an ellipse as the locus of a point that moves so that its total distance from two fixed points (–f, 0) and (f, 0) is a constant.*
- *Write the equation of an ellipse with center at (0, 0) given its vertices and co-vertices and graph it.*

Teacher Preparation and Notes

- *This activity is appropriate for an Algebra 2 or Precalculus classroom.*
- *Students should have experience using the distance formula and solving radical equations.*
- *This activity is intended to be* ***teacher-led*** *with students in* ***small groups****.*
- *Notes for using the TI-Nspire™ Navigator™ System are included throughout the activity. The use of the Navigator System is not necessary for completion of this activity.*
- ***To download the student TI-Nspire document (.tns file) and student worksheet, go to education.ti.com/exchange and enter "9980" in the keyword search box.***

Associated Materials

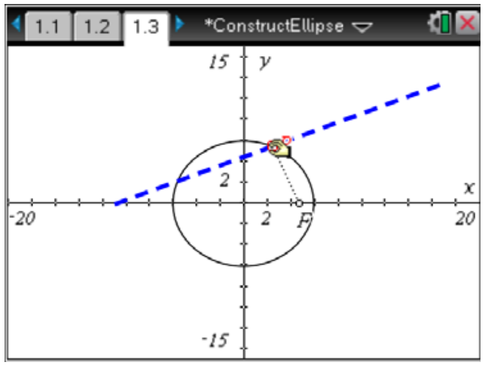
- *ConstructEllipse_Student.doc*
- *ConstructEllipse.tns*

Suggested Related Activities

To download any activity listed, go to education.ti.com/exchange and enter the number in the keyword search box.

- *Properties of an Ellipse (TI-Nspire technology) — 8430*
- *Orbit Of Jupiter (TI-Nspire technology) — 10035*
- *Ellipse: Envelope of Lines (Cabri Jr.) (TI-84 Plus family) — 7291*
- *Ellipse: Locus of Points (Cabri Jr.) (TI-84 Plus family) — 7290*
- *NUMB3RS – Season 2 – "Harvest" – Waxing Elliptical (TI-84 Plus family) — 6522*

Problem 1 – Envelope construction

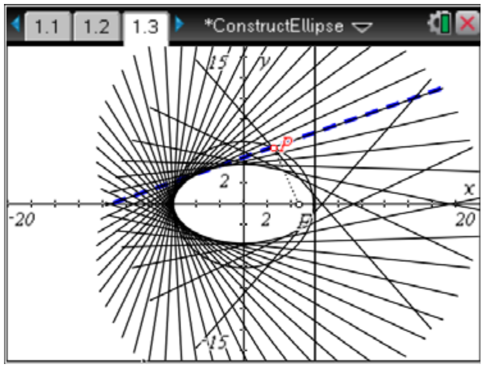


Students should begin by examining the circle on page 1.3. They will see that point *F* lies on a diameter of the circle, segment *FP* connects *F* to a point *P* on the circle, and a perpendicular line to segment *FP* through point *P* is drawn.

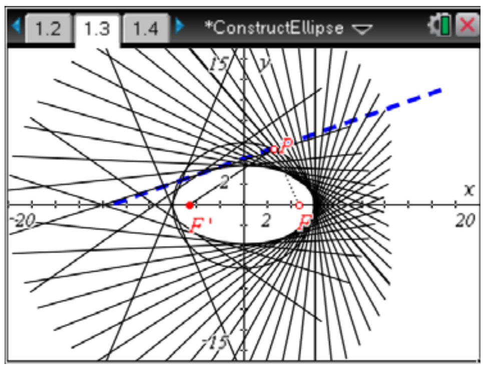
Students are directed to drag point *P* around the circle and then display the locus (**MENU > Construction > Locus**) of the perpendicular line as *P* travels along the circle, which generates the shape of an ellipse.

TI-Nspire Navigator Opportunity: *Live Presenter*

See Note 1 at the end of this lesson.



This method for constructing an ellipse is called the "envelope method." Explain to students that the diameter of the circle is equal to the width of the ellipse along its *major* axis (the longer of its two axes). The point *F* is a special type of fixed point that can be used to generate the ellipse. Tell students that ellipses have two such fixed points, called *foci* (singular: *focus*).



Next, students will use the diagram on page 1.3 to explore the other focus. They will need to first hide the locus of lines. Then students should reflect *F* over the *y*-axis using the **Reflection** tool from the Transformation menu. Label the image point *F'*.

After constructing a segment from *F'* to *P* and a line perpendicular to segment *F'P* through point *P*, students should then drag point *P* and notice that this new line also traces out an ellipse. The locus will confirm this, and students should also notice that both ellipses traced out are identical.

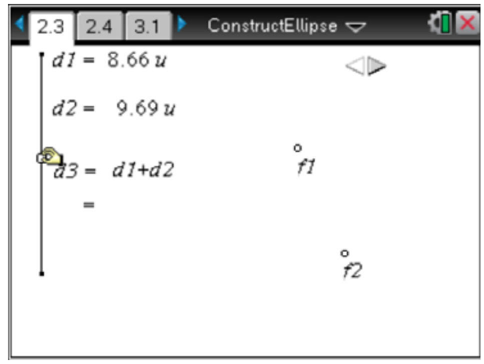
By dragging the focus *F* (which in turn moves *F'*), students should find that the location of the foci affects the shape of the ellipse.

TI-Nspire Navigator Opportunity: *Screen Capture*

See Note 2 at the end of this lesson.

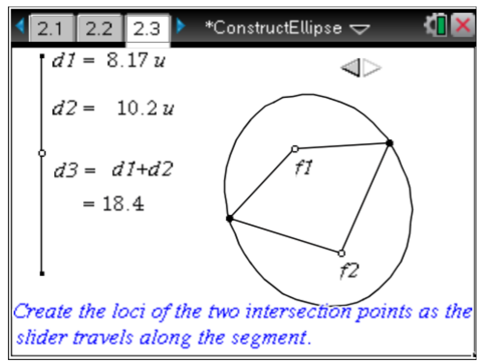
Problem 2 – String and pins construction

An **ellipse** is defined as the set of points in a plane such that the sum of the distances from two fixed points (foci) in that plane is constant. Students will now use this definition to construct an ellipse.



Page 2.3 contains a segment with a slider and two additional points, *f1* and *f2*, which will become the foci of the ellipse. The values of *d1* and *d2*, determined by the slider, will be the distances from *f1* and *f2* (respectively) to the point on the ellipse.

Students should calculate *d3*, which is the sum of *d1* and *d2*. They will find that the value of *d3* remains constant as they move the slider.



Page 2.4 directs students to construct two circles. After selecting the **Compass** tool from the Construction menu, students should click on the measurement for the radius, followed by the point to use as the center of the circle. Next, students should mark the intersections of the circles and then construct four segments (the radii of the circles).

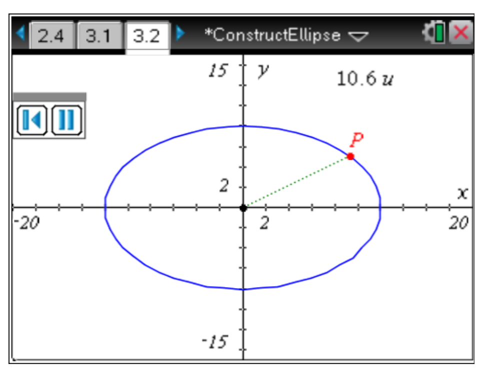
Displaying the loci of the each intersection point as the slider travels along the segment reveals an ellipse.

This construction is called the "string and pins" construction because it is traditionally performed by wrapping a piece of string (represented here by the segment and slider) around two pins driven into a flat surface at the foci.

TI-Nspire Navigator Opportunity: *Screen Capture*

See Note 3 at the end of this lesson.

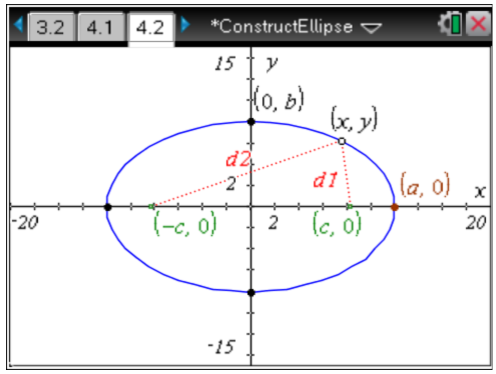
Problem 3 – Semimajor and semiminor axes



Page 3.2 shows an animated point on an ellipse. Students are told that as the point travels around the curve, its distance from the center of the ellipse changes.

They should determine that the semiminor axis (minimum distance) occurs when the point is on the *y*-axis at the co-vertices of the ellipse and the semimajor axis (maximum distance) occurs when the point is on the *x*-axis at the vertices of the ellipse.

Problem 4 – Deriving the equation of an ellipse



Page 4.2 shows an ellipse centered at the origin. The lengths of the segments connecting (*x*, *y*) to the foci are *d1* and *d2*. Since the foci are equidistant from the center of the ellipse, the coordinates of the foci are (*c*, 0) and (–*c*, 0) Let *a* be the distance from the center of the ellipse to the vertex (*a*, 0).

Students will follow the steps on pages 4.3–4.7 to derive a general equation for such an ellipse.

1. First, students drag (*x*, *y*) so that *y* = 0. Since the distance from (–*a*, 0) to (*a*, 0) is equal to the sum of (–*a*, 0) to (–*c*, 0) plus the distance from (–*c*, 0) to (*a*, 0), the equation is 2*a* = *d1* + *d2*.

2. Now students will drag (*x*, *y*) to any point on the ellipse. Using the distance formula, students should write the following for *d1* and *d2*.

$$d1=\sqrt{(x-c)^2+(y-0)^2}=\sqrt{(x-c)^2+y^2}$$

$$d2=\sqrt{(x-(-c))^2+(y-0)^2}=\sqrt{(x+c)^2+y^2}$$

3. Substitute these into the equation 2*a* = *d1* + *d2*.

$$2a=\sqrt{(x-c)^2+y^2}+\sqrt{(x+c)^2+y^2}$$

4. Simplify the radical expression by isolating a radical on one side and then squaring both sides.

$$\sqrt{(x+c)^2+y^2}=2a-\sqrt{(x-c)^2+y^2}$$

$$\left(\sqrt{(x+c)^2+y^2}\right)^2=\left(2a-\sqrt{(x-c)^2+y^2}\right)^2$$

$$(x+c)^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$$

Isolate the remaining radical on the left side and simplify the right side.

$$-4a\sqrt{(x-c)^2+y^2}=-4a^2+(x+c)^2+y^2-(x-c)^2-y^2$$

$$\sqrt{(x-c)^2+y^2}=-\frac{1}{4a}(-4a^2+(x^2+2xc+c^2)-(x^2-2xc+c^2))$$

$$\sqrt{(x-c)^2+y^2}=-\frac{1}{4a}(-4a^2+4xc)$$

$$\sqrt{(x-c)^2+y^2}=a-\frac{xc}{a}$$

Square both sides again.

$$\left(\sqrt{(x-c)^2+y^2}\right)^2=\left(a-\frac{xc}{a}\right)^2$$

$$(x-c)^2+y^2=a^2-2xc+\frac{c^2}{a^2}x^2$$

$$x^2-2xc+c^2+y^2=a^2-2xc+\frac{c^2}{a^2}x^2$$

$$x^2+c^2+y^2=a^2+\frac{c^2}{a^2}x^2$$

5. Factor out *x,* then simplify.

$$x^2-\frac{c^2}{a^2}x^2+y^2=a^2-c^2$$

$$x^2\left(1-\frac{c^2}{a^2}\right)+y^2=a^2-c^2$$

$$x^2\left(\frac{a^2}{a^2}-\frac{c^2}{a^2}\right)+y^2=a^2-c^2$$

$$x^2\left(\frac{a^2-c^2}{a^2}\right)+y^2=a^2-c^2$$

6. Divide both sides by a^2-c^2.

$$x^2\left(\frac{a^2-c^2}{a^2}\right)+y^2=a^2-c^2$$

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1$$

7. Students will now return to the diagram and drag (*x*, *y*) so that *x* = 0. This forms two right triangles with two congruent sides, i.e., two congruent triangles. Since the triangles are congruent, students can conclude that *d1* = *d2*.

8. Recall that 2*a* = *d1* + *d2*. Therefore 2*a* = *d1* + *d1*, and *a* = *d1*.

9. Define *b* as the distance from the center of the ellipse to this point. Students can use the Pythagorean Theorem to write an expression for b^2 in terms of *a* (which equals *d1*) and *c*.

$$b^2=a^2-c^2$$

Note: Students may think $c^2=a^2+b^2$, but in this case the hypotenuse has length *a* and the sides have lengths *b* and *c*, producing the equation $a^2=b^2+c^2$.

10. So we can write the equation in Step 6 as $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. This is the general equation for an ellipse centered at the origin with semimajor axis *a* and semiminor axis *b*.

TI-Nspire Navigator Opportunities

Note 1
Problem 1, *Live Presenter*

Use *Live Presenter* to guide students through pages 1.2 to 1.5. Use the displayed handheld screen to guide the discussion through these pages.

Note 2
Problem 1, *Live Presenter*

Use *Live Presenter* to help students describe the relationship between the foci and the resulting loci. If there is time, ask students what the location is of the foci when the loci results in a:

- circle
- ellipse
- hyperbola

Note 3
Problem 2, *Screen Capture*

Use *Live Presenter* to monitor student progress as they work through the construction on page 2.3, offering help as needed.