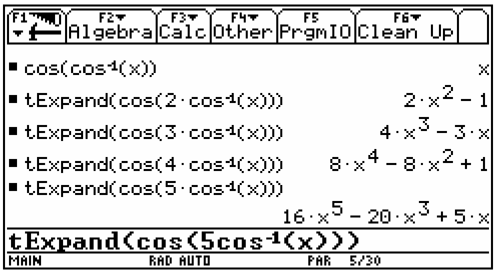
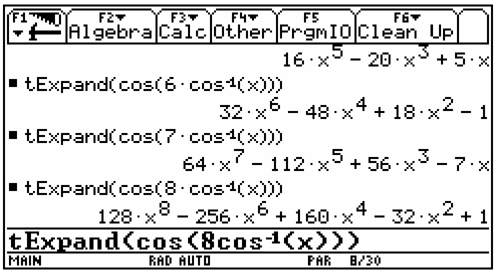
Ages 17-19 – Exploring functions with CAS

a) Consider the functions $c_n(x) = \cos(n \cdot \arccos x)$ with $n \in \mathbb{N}_0$:

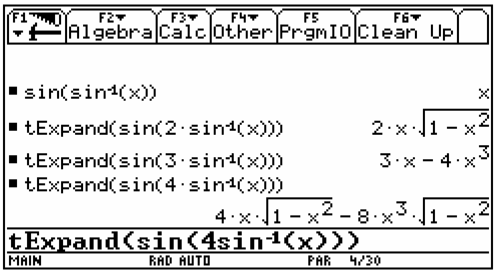


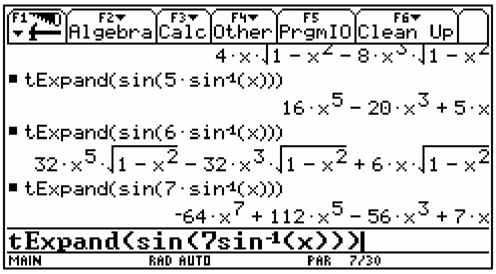


<u>Conjecture</u>

$c_n(x)$ is a polynomial function of degree n . The function is even if n is even and odd if n is odd.

b) Consider the functions $s_n(x) = \sin(n \cdot \arcsin x)$ with $n \in \mathbb{N}_0$:





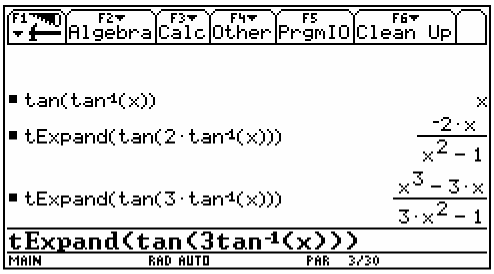
<u>Conjecture</u>

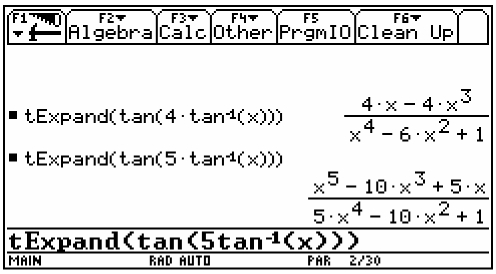
$s_n(x)$ is a polynomial function of degree n if n is odd, $s_n(x) = c_n(x)$ for $n = 1,5,9,\ldots$ and $s_n(x) = -c_n(x)$ for $n = 3,7,11,\ldots$

$s_n(x)$ is $\sqrt{1-x^2}$ times a polynomial of degree $n-1$ if n is even.

The function $s_n(x)$ is odd for all $n \in \mathbb{N}_0$.

c) Consider the functions $t_n(x) = \tan(n \cdot \arctan x)$ with $n \in \mathbb{N}_0$:





Conjecture

$t_n(x)$ is an odd rational function, with an odd numerator and even denominator.

If n is even , the denominator has degree n and the numerator has degree $n-1$. If n is odd, the numerator has degree n and the denominator has degree $n-1$.

Proof of the conjectures

(i) Are the functions even or odd?

- $c_n(-x) = \cos(n \cdot \arccos(-x)) = \cos(n \cdot (\pi - \arccos x)) = \cos(n \cdot \pi - n \cdot \arccos x)$

  If n is even, $c_n(-x) = \cos(-n \cdot \arccos x) = \cos(n \cdot \arccos x) = c_n(x)$.

  If n is odd, $c_n(-x) = \cos(\pi - n \cdot \arccos x) = -\cos(n \cdot \arccos x) = -c_n(x)$.

  Conclusion: $c_n(x)$ is even if n is even and odd if n is odd.

- $s_n(-x) = \sin(n \cdot \arcsin(-x)) = \sin(-n \cdot \arcsin x) = -s_n(x)$

  Conclusion: the functions $s_n(x)$ and $t_n(x)$ (analogous) are odd.

(ii) $s_n(x) = c_n(x)$ for $n = 1,5,9,\ldots$ and $s_n(x) = -c_n(x)$ for $n = 3,7,11,\ldots$

$s_n(x) = \sin(n \cdot \arcsin x) = \sin\left(n \cdot \left(\frac{\pi}{2} - \arccos x\right)\right) = \sin\left(n \cdot \frac{\pi}{2} - n \cdot \arccos(x)\right)$

If $n = 1 + 4k$, $s_n(x) = \sin\left(\frac{\pi}{2} + k \cdot 2\pi - n \cdot \arccos(x)\right) = \sin\left(\frac{\pi}{2} - n \cdot \arccos(x)\right) = c_n(x)$

If $n = 3 + 4k$, $s_n(x) = \sin\left(\frac{3\pi}{2} + k \cdot 2\pi - n \cdot \arccos(x)\right) = \sin\left(\frac{3\pi}{2} - n \cdot \arccos(x)\right)$

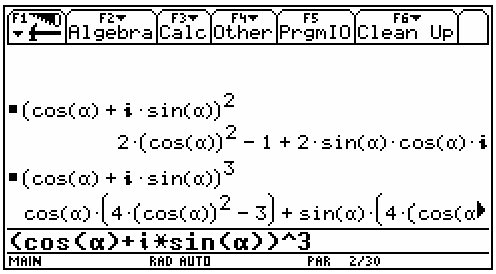
$= -\sin\left(\frac{\pi}{2} - n \cdot \arccos(x)\right) = -c_n(x)$

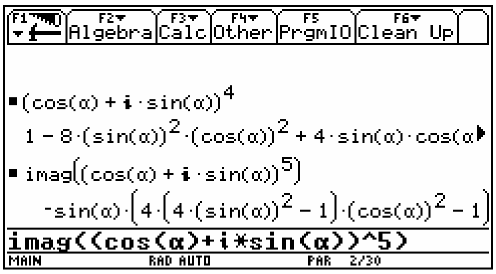
(iii) $c_n(x)$ $(n \in \mathbb{N}_0)$ and $s_n(x)$ (n odd) are polynomials of degree n.

De Moivre's theorem states that for $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$:

$(\cos\alpha + i \cdot \sin\alpha)^n = \cos(n\alpha) + i \cdot \sin(n\alpha)$.

Expanding the left side and equating the real and imaginary parts provides expressions for $\cos(n\alpha)$ and $\sin(n\alpha)$. Then substitute $\alpha = \arccos x$ (or $\alpha = \arcsin x$).

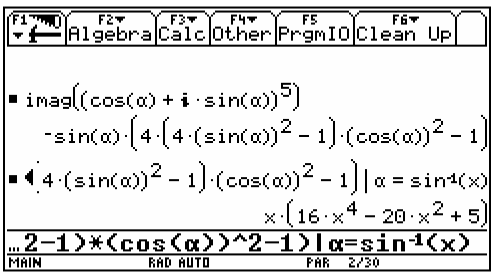




Example:

Substitution of $\alpha = \arcsin x$ in $\sin(5\alpha)$ yields the polynomial

$s_5(x) = 16x^5 - 20x^3 + 5x$



The binomial theorem proves the general case:

$$\left(\cos\alpha + i\cdot\sin\alpha\right)^n = \sum_{k=0}^{n}\binom{n}{k}\left(\cos\alpha\right)^k\cdot\left(i\cdot\sin\alpha\right)^{n-k} = \sum_{k=0}^{n}\binom{n}{k}\left(\cos\alpha\right)^k\cdot\left(\sin\alpha\right)^{n-k}\cdot i^{n-k} \quad .$$

Collect the terms with even power $n-k$ of *i* for the real part $\cos\left(n\cdot\alpha\right)$ and substitute

$\alpha = \arccos x$, then $\left(\cos\alpha\right)^k\cdot\left(\sin\alpha\right)^{n-k} = x^k\cdot\left(1-x^2\right)^{\frac{n-k}{2}}$, a polynomial of degree *n*.

Collect the terms with odd power $n-k$ of *i* to find the imaginary part $\sin\left(n\cdot\alpha\right)$ and substitute $\alpha = \arcsin x$.

If n is odd, *k* must be even and $\left(\cos\alpha\right)^k\cdot\left(\sin\alpha\right)^{n-k} = \left(1-x^2\right)^{\frac{k}{2}}\cdot x^{n-k}$ is a polynomial of degree *n*.

If n is even, *k* must be odd and $\left(\cos\alpha\right)^k\cdot\left(\sin\alpha\right)^{n-k} = \sqrt{1-x^2}\cdot\left(1-x^2\right)^{\frac{k-1}{2}}\cdot x^{n-k}$.

Therefore $\left(1-x^2\right)^{\frac{k-1}{2}}\cdot x^{n-k}$ is a polynomial of degree $n-1$.

(iv) $t_n\left(x\right) = \tan\left(n\cdot\arctan x\right)$ is a rational function $\left(n\in\mathbb{N}_0\right)$.

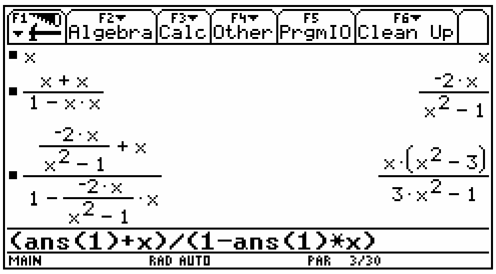
Using the trigonometric formula $\tan\left(\alpha+\beta\right) = \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\cdot\tan\beta}$, we find that

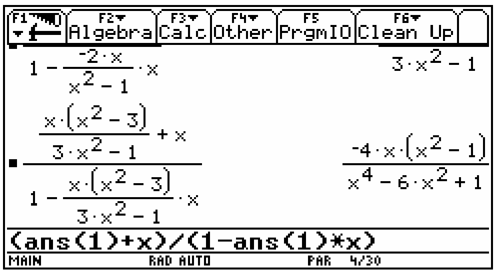
$$\tan\left(n\alpha\right) = \tan\left(\left(n-1\right)\alpha+\alpha\right) = \frac{\tan\left(\left(n-1\right)\alpha\right)+\tan\alpha}{1-\tan\left(\left(n-1\right)\alpha\right)\cdot\tan\alpha} \quad .$$

The substitution $\alpha = \arctan\left(x\right)$ yields the recursion formula

$$t_n\left(x\right) = \frac{t_{n-1}\left(x\right)+x}{1-t_{n-1}\left(x\right)\cdot x} \quad \left(n\geq 2 \text{ and } t_1\left(x\right)=x\right).$$

CAS can be used to produce the sequence $t_1\left(x\right), t_2\left(x\right), t_3\left(x\right), \ldots$



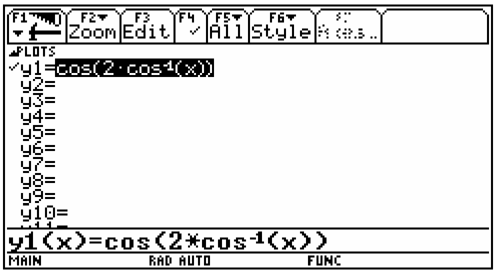


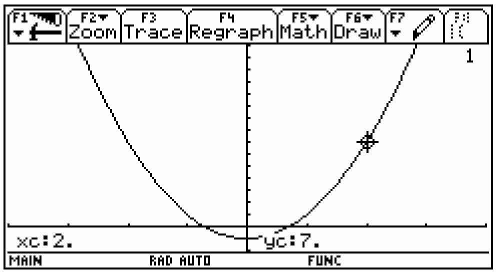
d) <u>Remarks</u>

(i) Observe how CAS and formal mathematics can cooperate.

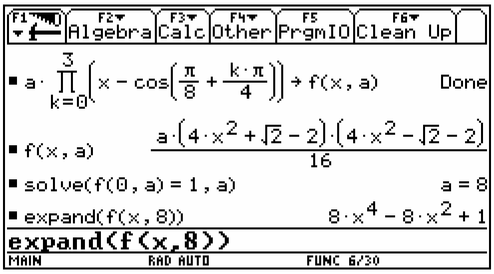
(ii) The real function $c_2\left(x\right) = \cos\left(2\cdot\arccos x\right) = 2x^2-1$ has domain $\left[-1,1\right]$.

For CAS, the function has domain $\mathbb{R}$: working with complex functions, the result of $\arccos(2)$ is non-real, but $\cos\left(2\arccos\left(2\right)\right)=7$ is real!





(iii) Knowing that $c_n\left(x\right)$ $\left(n\in\mathbb{N}_0\right)$ and $s_n\left(x\right)$ (n odd) are polynomials of degree n, it is possible to find these polynomials with their zeros.



To find the zeros of $c_4\left(x\right)=\cos\left(4\cdot\arccos x\right)$,

observe that $0\le 4\arccos x\le 4\pi$ and

if $4\cdot\arccos x=\dfrac{\pi}{2}+k\cdot\pi \quad \left(k=0,1,2,3\right)$

or $\arccos x=\dfrac{\pi}{8}+k\cdot\dfrac{\pi}{4} \quad \left(k=0,1,2,3\right)$.

The zeros of $c_4\left(x\right)$ are $x_k=\cos\left(\dfrac{\pi}{8}+k\cdot\dfrac{\pi}{4}\right)$ $\left(k=0,1,2,3\right)$, consequently

$c_4\left(x\right)=a\left(x-x_0\right)\left(x-x_1\right)\left(x-x_2\right)\left(x-x_3\right)$.

Find a with $c_4\left(0\right)=1$. Result: $c_4\left(x\right)=8x^4-8x^2+1$.