

Ages 17-19 – Exploring functions with CAS

a) Consider the functions $c_n(x) = \cos(n \cdot \arccos x)$ with $n \in \mathbb{N}_0$:

F1	F2	F3	F4	F5	F6
←	Algebra	Calc	Other	PrgmIO	Clean Up
■	cos(cos ⁴ (x))				x
■	tExpand(cos(2·cos ⁴ (x)))				2·x ² - 1
■	tExpand(cos(3·cos ⁴ (x)))				4·x ³ - 3·x
■	tExpand(cos(4·cos ⁴ (x)))				8·x ⁴ - 8·x ² + 1
■	tExpand(cos(5·cos ⁴ (x)))				16·x ⁵ - 20·x ³ + 5·x
tExpand(cos(5cos⁴(x)))					
MAIN	RAD AUTO	PAR	5/30		

F1	F2	F3	F4	F5	F6
←	Algebra	Calc	Other	PrgmIO	Clean Up
					16·x ⁵ - 20·x ³ + 5·x
■	tExpand(cos(6·cos ⁴ (x)))				32·x ⁶ - 48·x ⁴ + 18·x ² - 1
■	tExpand(cos(7·cos ⁴ (x)))				64·x ⁷ - 112·x ⁵ + 56·x ³ - 7·x
■	tExpand(cos(8·cos ⁴ (x)))				128·x ⁸ - 256·x ⁶ + 160·x ⁴ - 32·x ² + 1
tExpand(cos(8cos⁴(x)))					
MAIN	RAD AUTO	PAR	8/30		

Conjecture

$c_n(x)$ is a polynomial function of degree n . The function is even if n is even and odd if n is odd.

b) Consider the functions $s_n(x) = \sin(n \cdot \arcsin x)$ with $n \in \mathbb{N}_0$:

F1	F2	F3	F4	F5	F6
←	Algebra	Calc	Other	PrgmIO	Clean Up
■	sin(sin ⁴ (x))				x
■	tExpand(sin(2·sin ⁴ (x)))				2·x·√(1-x ²)
■	tExpand(sin(3·sin ⁴ (x)))				3·x - 4·x ³
■	tExpand(sin(4·sin ⁴ (x)))				4·x·√(1-x ²) - 8·x ³ ·√(1-x ²)
tExpand(sin(4sin⁴(x)))					
MAIN	RAD AUTO	PAR	4/30		

F1	F2	F3	F4	F5	F6
←	Algebra	Calc	Other	PrgmIO	Clean Up
					4·x·√(1-x ²) - 8·x ³ ·√(1-x ²)
■	tExpand(sin(5·sin ⁴ (x)))				16·x ⁵ - 20·x ³ + 5·x
■	tExpand(sin(6·sin ⁴ (x)))				32·x ⁵ ·√(1-x ²) - 32·x ³ ·√(1-x ²) + 6·x·√(1-x ²)
■	tExpand(sin(7·sin ⁴ (x)))				-64·x ⁷ + 112·x ⁵ - 56·x ³ + 7·x
tExpand(sin(7sin⁴(x)))					
MAIN	RAD AUTO	PAR	7/30		

Conjecture

$s_n(x)$ is a polynomial function of degree n if n is odd, $s_n(x) = c_n(x)$ for $n = 1, 5, 9, \dots$ and

$s_n(x) = -c_n(x)$ for $n = 3, 7, 11, \dots$

$s_n(x)$ is $\sqrt{1-x^2}$ times a polynomial of degree $n-1$ if n is even.

The function $s_n(x)$ is odd for all $n \in \mathbb{N}_0$.

c) Consider the functions $t_n(x) = \tan(n \cdot \arctan x)$ with $n \in \mathbb{N}_0$:

F1	F2	F3	F4	F5	F6
←	Algebra	Calc	Other	PrgmIO	Clean Up
■	tan(tan ⁴ (x))				x
■	tExpand(tan(2·tan ⁴ (x)))				$\frac{-2 \cdot x}{x^2 - 1}$
■	tExpand(tan(3·tan ⁴ (x)))				$\frac{x^3 - 3 \cdot x}{3 \cdot x^2 - 1}$
tExpand(tan(3tan⁴(x)))					
MAIN	RAD AUTO	PAR	3/30		

F1	F2	F3	F4	F5	F6
←	Algebra	Calc	Other	PrgmIO	Clean Up
					$\frac{4 \cdot x - 4 \cdot x^3}{x^4 - 6 \cdot x^2 + 1}$
■	tExpand(tan(4·tan ⁴ (x)))				$\frac{x^5 - 10 \cdot x^3 + 5 \cdot x}{5 \cdot x^4 - 10 \cdot x^2 + 1}$
■	tExpand(tan(5·tan ⁴ (x)))				
tExpand(tan(5tan⁴(x)))					
MAIN	RAD AUTO	PAR	2/30		

Conjecture

$t_n(x)$ is an odd rational function, with an odd numerator and even denominator.

If n is even, the denominator has degree n and the numerator has degree $n-1$. If n is odd, the numerator has degree n and the denominator has degree $n-1$.

Proof of the conjectures

(i) Are the functions even or odd?

- $c_n(-x) = \cos(n \cdot \arccos(-x)) = \cos(n \cdot (\pi - \arccos x)) = \cos(n \cdot \pi - n \cdot \arccos x)$

If n is even, $c_n(-x) = \cos(-n \cdot \arccos x) = \cos(n \cdot \arccos x) = c_n(x)$.

If n is odd, $c_n(-x) = \cos(\pi - n \cdot \arccos x) = -\cos(n \cdot \arccos x) = -c_n(x)$.

Conclusion: $c_n(x)$ is even if n is even and odd if n is odd.

- $s_n(-x) = \sin(n \cdot \arcsin(-x)) = \sin(-n \cdot \arcsin x) = -s_n(x)$

Conclusion: the functions $s_n(x)$ and $t_n(x)$ (analogous) are odd.

(ii) $s_n(x) = c_n(x)$ for $n = 1, 5, 9, \dots$ and $s_n(x) = -c_n(x)$ for $n = 3, 7, 11, \dots$

$$s_n(x) = \sin(n \cdot \arcsin x) = \sin\left(n \cdot \left(\frac{\pi}{2} - \arccos x\right)\right) = \sin\left(n \cdot \frac{\pi}{2} - n \cdot \arccos(x)\right)$$

If $n = 1 + 4k$, $s_n(x) = \sin\left(\frac{\pi}{2} + k \cdot 2\pi - n \cdot \arccos(x)\right) = \sin\left(\frac{\pi}{2} - n \cdot \arccos(x)\right) = c_n(x)$

If $n = 3 + 4k$, $s_n(x) = \sin\left(\frac{3\pi}{2} + k \cdot 2\pi - n \cdot \arccos(x)\right) = \sin\left(\frac{3\pi}{2} - n \cdot \arccos(x)\right)$
 $= -\sin\left(\frac{\pi}{2} - n \cdot \arccos(x)\right) = -c_n(x)$

(iii) $c_n(x)$ ($n \in \mathbb{N}_0$) and $s_n(x)$ (n odd) are polynomials of degree n .

De Moivre's theorem states that for $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$:

$$(\cos \alpha + i \cdot \sin \alpha)^n = \cos(n\alpha) + i \cdot \sin(n\alpha).$$

Expanding the left side and equating the real and imaginary parts provides expressions for $\cos(n\alpha)$ and $\sin(n\alpha)$. Then substitute $\alpha = \arccos x$ (or $\alpha = \arcsin x$).

Example:

Substitution of $\alpha = \arcsin x$ in $\sin(5\alpha)$

yields the polynomial

$$s_5(x) = 16x^5 - 20x^3 + 5x$$

The binomial theorem proves the general case:

$$(\cos \alpha + i \cdot \sin \alpha)^n = \sum_{k=0}^n \binom{n}{k} (\cos \alpha)^k \cdot (i \cdot \sin \alpha)^{n-k} = \sum_{k=0}^n \binom{n}{k} (\cos \alpha)^k \cdot (\sin \alpha)^{n-k} \cdot i^{n-k}.$$

Collect the terms with even power $n-k$ of i for the real part $\cos(n \cdot \alpha)$ and substitute

$$\alpha = \arccos x, \text{ then } (\cos \alpha)^k \cdot (\sin \alpha)^{n-k} = x^k \cdot (1-x^2)^{\frac{n-k}{2}}, \text{ a polynomial of degree } n.$$

Collect the terms with odd power $n-k$ of i to find the imaginary part $\sin(n \cdot \alpha)$ and substitute $\alpha = \arcsin x$.

If n is odd, k must be even and $(\cos \alpha)^k \cdot (\sin \alpha)^{n-k} = (1-x^2)^{\frac{k}{2}} \cdot x^{n-k}$ is a polynomial of degree n .

If n is even, k must be odd and $(\cos \alpha)^k \cdot (\sin \alpha)^{n-k} = \sqrt{1-x^2} \cdot (1-x^2)^{\frac{k-1}{2}} \cdot x^{n-k}$.

Therefore $(1-x^2)^{\frac{k-1}{2}} \cdot x^{n-k}$ is a polynomial of degree $n-1$.

(iv) $t_n(x) = \tan(n \cdot \arctan x)$ is a rational function ($n \in \mathbb{N}_0$).

Using the trigonometric formula $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \cdot \tan \beta}$, we find that

$$\tan(n\alpha) = \tan((n-1)\alpha + \alpha) = \frac{\tan((n-1)\alpha) + \tan \alpha}{1 - \tan((n-1)\alpha) \cdot \tan \alpha}.$$

The substitution $\alpha = \arctan(x)$ yields the recursion formula

$$t_n(x) = \frac{t_{n-1}(x) + x}{1 - t_{n-1}(x) \cdot x} \quad (n \geq 2 \text{ and } t_1(x) = x).$$

CAS can be used to produce the sequence $t_1(x), t_2(x), t_3(x), \dots$

$$\frac{x + x}{1 - x \cdot x} = \frac{-2 \cdot x}{x^2 - 1}$$

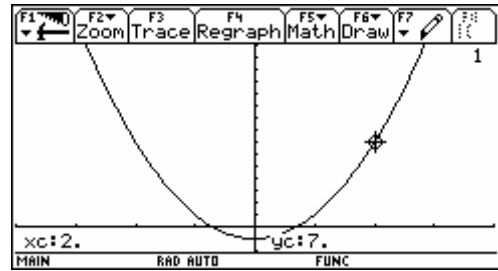
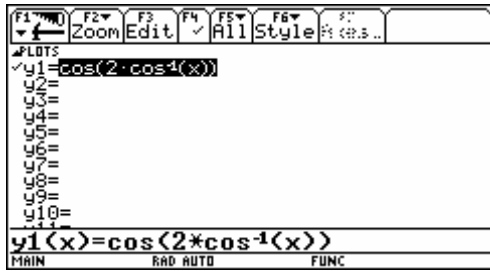
$$\frac{\frac{-2 \cdot x}{x^2 - 1} + x}{1 - \frac{-2 \cdot x}{x^2 - 1} \cdot x} = \frac{x \cdot (x^2 - 3)}{3 \cdot x^2 - 1}$$

$$\frac{1 - \frac{-2 \cdot x}{x^2 - 1} \cdot x}{1 - \frac{x \cdot (x^2 - 3)}{3 \cdot x^2 - 1} \cdot x} = \frac{3 \cdot x^2 - 1}{x^4 - 6 \cdot x^2 + 1}$$

d) Remarks

- (i) Observe how CAS and formal mathematics can cooperate.
- (ii) The real function $c_2(x) = \cos(2 \cdot \arccos x) = 2x^2 - 1$ has domain $[-1, 1]$.

For CAS, the function has domain \mathbb{R} : working with complex functions, the result of $\arccos(2)$ is non-real, but $\cos(2\arccos(2)) = 7$ is real!



(iii) Knowing that $c_n(x)$ ($n \in \mathbb{N}_0$) and $s_n(x)$ (n odd) are polynomials of degree n , it is possible to find these polynomials with their zeros.

To find the zeros of $c_4(x) = \cos(4 \cdot \arccos x)$,

observe that $0 \leq 4 \arccos x \leq 4\pi$ and

$$\text{if } 4 \cdot \arccos x = \frac{\pi}{2} + k \cdot \pi \quad (k = 0, 1, 2, 3)$$

$$\text{or } \arccos x = \frac{\pi}{8} + k \cdot \frac{\pi}{4} \quad (k = 0, 1, 2, 3).$$

The zeros of $c_4(x)$ are $x_k = \cos\left(\frac{\pi}{8} + k \cdot \frac{\pi}{4}\right)$ ($k = 0, 1, 2, 3$), consequently

$$c_4(x) = a(x - x_0)(x - x_1)(x - x_2)(x - x_3).$$

Find a with $c_4(0) = 1$. Result: $c_4(x) = 8x^4 - 8x^2 + 1$.

