

What's Half of a Half of a Half?

Overview – Activity ID: 8934

Students will use a physical model to determine what happens when they repeatedly halve a piece of paper, and then they reassemble the pieces into a whole. They then use an algebraic model to analyze the same situation, which leads to an introductory discussion of limits.

Math Concepts

- fractions
- decimals
- limits
- rounding

Materials

- TI-34 MultiView™
- pencil
- paper
- scissors

Activity

Have each student take a piece of paper and cut it in half, first folding it to ensure the resulting pieces are equal in size. The student should set aside one of the halves and fold the remaining piece in half and cut it. It is important that students cut in the same direction each time, for example, vertically rather than horizontally. Have students repeat the process until they can no longer cut the remaining strip in half.

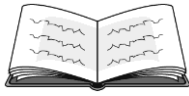
Let's first try to label the pieces of paper according to what fractions of the whole piece they represent. The largest piece is $\frac{1}{2}$, the next largest piece is $\frac{1}{4}$, etc.

Have the students attempt to label the rest of the pieces. They may work together, alone, or with their calculators, but have them attempt the fractions without teacher assistance.

How did you label the remaining strips of paper? How did you know what fraction of the whole piece of paper each strip represented? Were you able to label all the strips?

Discuss the concept that each strip of paper is $\frac{1}{2}$ the size of the previous strip. In terms of mathematical operations, this is the same as multiplying by $\frac{1}{2}$ each time. While the idea makes sense, likely some of the students struggled with fractions much smaller than $\frac{1}{4}$. Show the students how to use their calculators to halve the fractions.

We can easily use the TI-34 MultiView to calculate these fractions. The original piece of paper is a whole piece, so we can use the number 1 to represent its size. When we made the initial cut, we saw that the remaining piece was $\frac{1}{2}$ of the original size, so now we multiply 1 by $\frac{1}{2}$. For the next cut, we again multiply by $\frac{1}{2}$. But we made that cut in a piece only half as big as the original. So the resulting piece would be $\frac{1}{2}$ times $\frac{1}{2}$, or $\frac{1}{4}$ as big as the original. Since we're repeatedly multiplying by $\frac{1}{2}$, let's use the Constant feature of the calculator to make the process simpler.



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Have the students complete the first two columns of the table below, using the Constant feature of the TI-34 MultiView. (Note: The table for the students is provided at the end of the activity, for classroom projection and use. There, it is blank except for the first few entries. Here, the center column has been completed. The whole table is complete on a later page, for teacher reference.)

The first column represents the cut number. The first cut would be 1, the second cut would be 2, etc.

Cut	Fraction of the initial paper	Sum of the cut pieces
1	$\frac{1}{2} \cdot 1 = \frac{1}{2}$	
2	$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$	
3	$\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$	
4	$\frac{1}{2} \cdot \frac{1}{8} = \frac{1}{16}$	
5	$\frac{1}{2} \cdot \frac{1}{16} = \frac{1}{32}$	
6	$\frac{1}{2} \cdot \frac{1}{32} = \frac{1}{64}$	
7	$\frac{1}{2} \cdot \frac{1}{64} = \frac{1}{128}$	

What pattern(s) can you identify?

Identifiable patterns include the fraction getting smaller as the piece of paper gets smaller, each fraction being the previous fraction multiplied by $\frac{1}{2}$, and the denominators getting larger by a factor of 2 each time.

Next, have the students take their cut pieces of paper and reassemble them, starting with the largest strip and progressively adding the next largest strip. Discuss each step of the process.

Let's return to the table we started earlier and complete the third column as we reassemble the strips of paper.

Follow these steps:

1. Press **[2nd]** **[set op1]** to access the Constant feature.
2. Clear anything currently entered in the screen.
3. Now press **[x]** **[1]** **[2]** **[enter]**. The screen should show this:

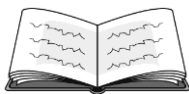
DEG
OP1 = $\times \frac{1}{2}$

4. Press **[2nd]** **[quit]** to return to the home screen.
5. Press 1 **[op1]**. The screen should show this:

DEG \leftrightarrow
1 $\times \frac{1}{2}$ n=1 $\frac{1}{2}$

6. Press **[op1]** again to repeat the process. The screen should show this:

DEG \leftrightarrow
1 $\times \frac{1}{2}$ n=2 $\frac{1}{4}$
 $\frac{1}{2} \times \frac{1}{2}$



What's Half of a Half of a Half?

Now that we have the strips labeled, let's reassemble them and see what fraction of the initial piece of paper we have reassembled at each step. Remember, we started with a whole piece of paper, and we called that 1 initially, which represents one whole. With this in mind, what do you think the sum of all the strips will be?

Expect the students to say "one" or "one piece of paper," since that's what you just reminded them.

Walk through each addition as a class.

After one cut, we were left with $\frac{1}{2}$ a sheet of paper. After two cuts, we had $\frac{1}{4}$ a sheet of paper, so the sum of those two pieces would be $\frac{1}{2}$ plus $\frac{1}{4}$, totaling $\frac{3}{4}$. The next piece added would be $\frac{1}{8}$ the initial sheet of paper, so we add that to the previous answer of $\frac{3}{4}$.

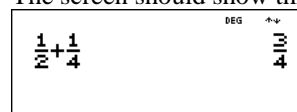
Show the students how to use their calculators to more easily see the pattern.

Cut	Fraction of the initial paper	Sum of the cut pieces
1	$\frac{1}{2} \cdot 1 = \frac{1}{2}$	$\frac{1}{2}$
2	$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$	$\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$
3	$\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$
4	$\frac{1}{2} \cdot \frac{1}{8} = \frac{1}{16}$	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$
5	$\frac{1}{2} \cdot \frac{1}{16} = \frac{1}{32}$	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{31}{32}$
6	$\frac{1}{2} \cdot \frac{1}{32} = \frac{1}{64}$	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{64} = \frac{63}{64}$
7	$\frac{1}{2} \cdot \frac{1}{64} = \frac{1}{128}$	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{128} = \frac{127}{128}$

Follow these steps:

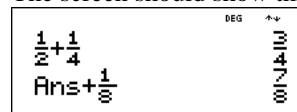
1. Press 1 $\frac{1}{2}$ 2 $\frac{1}{4}$ + 1 $\frac{1}{4}$ 4 $\frac{1}{8}$ **enter**.

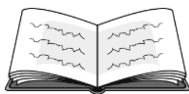
2. The screen should show this:



3. Press + 1 $\frac{1}{4}$ 8 $\frac{1}{8}$ **enter** to copy and paste the previous answer and continue the pattern of adding.

4. The screen should show this:





What's Half of a Half of a Half?

Continue using the Previous Answer feature of your calculator to copy and paste the previous answer, then add the fractions that represent the increasingly smaller strips of paper. As you add more of the fractions together, what pattern do you see?

Identifiable patterns include the following: the numerator is always one less than the denominator, every denominator is double the previous denominator, both the numerator and denominator get increasingly larger, and soon both parts of the fraction are really large.

As we add more fractions, what number does the sum seem to be approaching?

If the students have difficulty realizing the answer is 1, remind them of the physical model they used. As they reassemble the strips of paper, they get closer and closer to having the whole piece of paper (or 1) together.

What will the next fraction in the pattern be? Will we ever actually get to the number 1?

Of course, it is mathematically impossible to reach the number 1, since the fraction will continue in the same pattern (numerator being 1 less than the denominator), but the students should realize that they are getting closer and closer to 1 and that they will never see a sum greater than 1, as this would equate to having more than one sheet of paper.

What's Half of a Half of a Half?

Name _____

Date _____



Directions: Use a paper model similar to the one you used in class and your TI-34 MultiView™ to complete the tables and answer the questions. The first few entries have been done for you.

1. Find the pattern if a piece of paper is repeatedly cut into thirds.

Cut	Fraction of the initial paper	Sum of the cut pieces
1	$\frac{1}{3} \cdot 1 = \frac{1}{3}$	$\frac{1}{3}$
2	$\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$	$\frac{1}{3} + \frac{1}{9} = \frac{4}{9}$
3	$\frac{1}{3} \cdot \frac{1}{9} =$	$\frac{1}{3} + \frac{1}{9} + =$
4		
5		
6		

2. As more fractions are added, what number does the sum of the cut pieces seem to be approaching? How do you know?
3. Assume a piece of paper was cut into thirds, then one of those pieces was set aside. Then one of the remaining thirds was again cut into thirds, continuing until the piece of paper was too small to be cut. If those strips were reassembled, what fraction of the initial piece of paper could be recreated? Explain your answer.


What's Half of a Half of a Half?

Name _____

Date _____



4. Find the pattern if a piece of paper is repeatedly cut into fourths. NOTE: This problem requires decimal approximation because the calculator can display unit fractions only up to $\frac{1}{1,024}$. Use

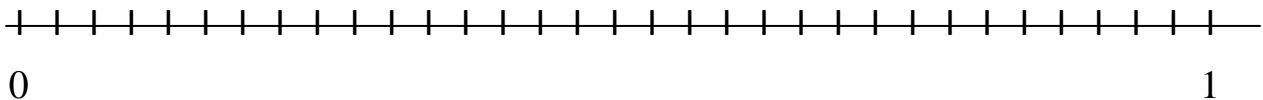
the  key to toggle between fractions and their decimal equivalents on the calculator when possible. You will be able to find the sum as a fraction *and* a decimal in cuts 1 through 4. For cuts 5 through 7, you should write the addends as fractions, and then compute only the decimal approximation.

Cut	Fraction of the initial paper	Sum of the cut pieces	Decimal approximation
1	$\frac{1}{4} \cdot 1 = \frac{1}{4}$	$\frac{1}{4}$	0.25
2	$\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$	$\frac{1}{4} + \frac{1}{16} = \frac{5}{16}$	0.3125
3	$\frac{1}{4} \cdot \frac{1}{16} =$	$\frac{1}{4} + \frac{1}{16} + =$	
4			
5			
6			
7			

5. As more fractions are added, what number does the sum of the cut pieces seem to be approaching? How do you know?

6. Place the following fractions at the appropriate place on the number line below:

$$\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{7}{8}, \frac{1}{16}, \frac{15}{16}, \frac{1}{32}, \frac{31}{32}$$





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ANSWER KEY

1. Find the pattern if a piece of paper is repeatedly cut into thirds.

Cut	Fraction of the initial paper	Sum of the cut pieces
1	$\frac{1}{3} \cdot 1 = \frac{1}{3}$	$\frac{1}{3}$
2	$\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$	$\frac{1}{3} + \frac{1}{9} = \frac{4}{9}$
3	$\frac{1}{3} \cdot \frac{1}{9} = \frac{1}{27}$	$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} = \frac{13}{27}$
4	$\frac{1}{3} \cdot \frac{1}{27} = \frac{1}{81}$	$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} = \frac{40}{81}$
5	$\frac{1}{3} \cdot \frac{1}{81} = \frac{1}{243}$	$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} = \frac{121}{243}$
6	$\frac{1}{3} \cdot \frac{1}{243} = \frac{1}{729}$	$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \frac{1}{729} = \frac{364}{729}$

2. As more fractions are added, what number does the sum of the cut pieces seem to be approaching? How do you know? $\frac{1}{2}$
3. Assume a piece of paper was cut into thirds, then one of those pieces was set aside. Then one of the remaining thirds was again cut into thirds, continuing until the piece of paper was too small to be cut. If those strips were reassembled, what fraction of the initial piece of paper could be recreated? Explain your answer.

One-half of the initial piece of paper could be recreated from the strips. The math in the table above shows that each sum is getting closer to $\frac{1}{2}$. Additionally, by creating a model and actually cutting strips, one can see that because each new strip is only $\frac{1}{3}$ of the previous strip, the amount of the total piece of paper recreated each time a strip is added changes by a very small amount after the first three strips.



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4. Find the pattern if a piece of paper is repeatedly cut into fourths. Note: This problem requires decimal approximation because the calculator can display unit fractions only up to $\frac{1}{1,024}$. Use the $\leftarrow \rightarrow$ key to toggle between fractions and their decimal equivalents on the calculator when possible. You will be able to find the sum as a fraction *and* a decimal in cuts 1 through 4. For cuts 5 through 7, you should write the addends as fractions, and then compute only the decimal approximation.

Cut	Fraction of the initial paper	Sum of the cut pieces	Decimal approximation
1	$\frac{1}{4} \cdot 1 = \frac{1}{4}$	$\frac{1}{4}$	0.25
2	$\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$	$\frac{1}{4} + \frac{1}{16} = \frac{5}{16}$	0.3125
3	$\frac{1}{4} \cdot \frac{1}{16} = \frac{1}{64}$	$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} = \frac{21}{64}$	0.328125
4	$\frac{1}{4} \cdot \frac{1}{64} = \frac{1}{256}$	$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} = \frac{85}{256}$	0.3320313
5	$\frac{1}{4} \cdot \frac{1}{256} = \frac{1}{1,024}$	$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \frac{1}{1,024} = \frac{341}{1,024}$	0.333007813
6	$\frac{1}{4} \cdot \frac{1}{1,024} = \frac{1}{4,096}$	$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \frac{1}{1,024} + \frac{1}{4,096} = \frac{1,365}{4,096}$	0.333251953
7	$\frac{1}{4} \cdot \frac{1}{4,096} = \frac{1}{16,384}$	$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \frac{1}{1,024} + \frac{1}{4,096} + \frac{1}{16,384} = \frac{5,461}{16,384}$	0.333312988

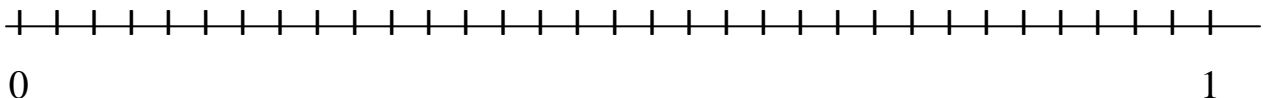
5. As more fractions are added, what number does the sum of the cut pieces seem to be approaching? How do you know?

$$\frac{1}{3}$$

6. Place the following fractions at the appropriate place on the number line below:

$$\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{7}{8}, \frac{1}{16}, \frac{15}{16}, \frac{1}{32}, \frac{31}{32}$$

$$\frac{1}{32}, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}$$



What's Half of a Half of a Half?

Transparency

Cut	Fraction of the initial paper	Sum of the cut pieces
1	$\frac{1}{2} \cdot 1 = \frac{1}{2}$	$\frac{1}{2}$
2	$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$	$\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$
3	$\frac{1}{2} \cdot \frac{1}{4} =$	$\frac{1}{2} + \frac{1}{4} + =$
4		
5		
6		
7		