***NUMB3RS* Activity: Slippery Paths**
Episode: "Toxin"

Topic: Steiner Paths and Trees **Grade Level**: 8 - 12

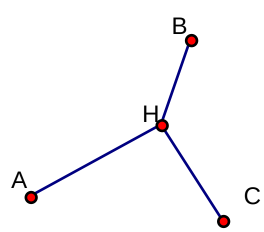
Objective: Minimizing Distances

Time: 20-30 minutes

Introduction

The works of Jakob Steiner (1796–1863), a Swiss mathematician, and Vincenzo Viviani (1622–1703), an Italian mathematician, come into play in this episode. Charlie looks for paths that a poisoner might take in the San Gabriel Mountains. He knows that there are three places where the criminal has been several times. So, he looks for a single site that is the shortest combined distance to the three places. He expects a hideout located at that site or somewhere very near it.

The simplest version of the problem is in the first exercise in which there is an equilateral triangle *ABC* formed using the three locations, *A, B,* and *C,* the criminal visited. The job is to find the single point *H* within the triangle such that *AH* + *BH* + *CH* is as small as possible. In mathematical language, Charlie tries to find the Steiner point of the triangle such that the sum of the lengths of the segments from *H* to *A*, *B*, and *C* is minimal. The segments *AH*, *BH*, and *CH* form a **Steiner tree** in which the sum of the branch lengths is minimal.



In the activity, students will start with triangle *ABC*, place a point *H* in the interior of the triangle, and look for patterns when a Steiner tree is found. Once found, point *H* is a **Steiner point**.

Discuss with Students

In this activity, you will help Charlie find the shortest combined distance to three places from a single point. For this activity, you will need to review: how to find missing sides and angles of a 30-60-90 right triangle; the formula for finding area of a triangle; the Pythagorean Theorem; and similar triangles.

Johnny W. Lott, The University of Montana, Missoula, MT

Student page answers: ***1. a.*** *Answers will vary.* ***b.*** $\sqrt{3}$ ***c.*** *120°* ***d.*** *For part b. Hint: Use a 30-60-90 triangle**.** For part c. Hint: Consider congruent triangles with H being a vertex of all three.* ***e.*** $s\sqrt{3}$
2. a. *Answers will vary.* ***b.*** *The sum is the length of the altitudes.* ***c.*** *Point H could be anywhere in the interior of the triangle; the sum of the lengths is still the length of an altitude. Consider the area of* $\triangle ABC$ = *area of* $\triangle BCH$ + *area of* $\triangle ABH$ + *area of* $\triangle ACH$.
If D, E, and F are the feet of the perpendiculars, then:
$\frac{1}{2}(BC)(AG) = \frac{1}{2}(BC)(HE) + \frac{1}{2}(AC)(HF) + \frac{1}{2}(AB)(HD)$. *Because all of sides of the triangle are equal lengths, we can substitute BC for AC and AB. Then simplifying, we find that AG = HE + HF + HD, where AG is an altitude of the triangle.* ***3.*** *Yes, a Steiner point could be found with more locations.* ***4.*** *There may be many different points that work on a grid but the distances should all be the same.*

Johnny W. Lott, The University of Montana, Missoula, MT

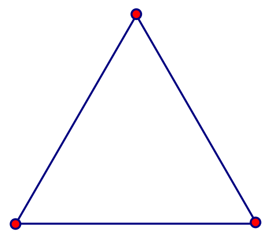
Name: ______________________________ Date: ______________

NUMB3RS Activity: Slippery Paths

Charlie Eppes has found three locations, *A*, *B*, and *C* in the San Gabriel Mountains that a poisoner frequents. Charlie expects that his hideout *H* might be the shortest distance from these three locations. This point is called a Steiner point. Your job is to locate *H*.

1. Suppose that points *A*, *B*, and *C* form an equilateral triangle as shown.

a. Use a ruler and protractor, or create the equilateral triangle with sides of length 1 on a geometry utility on your computer to try and find point *H*.



b. Find the sum of the distances: *HA* + *HB* + *HC.*

c. What are the angle measures for angles *AHB, BHC,* and *CHA?*

d. Prove your answers in parts b and c.

e. Suppose the triangle has sides of length *s.* What is the sum of the distances then?

2. Suppose that in Question 1, the sides of the equilateral triangle represented roads. The poisoner wants a location that is the minimum distance from each of three roads. Remember that the shortest distance from a point to a line is the perpendicular distance.

a. Find point *H* in this case.

b. Once you have found point *H*, measure an altitude of the triangle, *AG*. Also measure *HE, HF,* and *HG* where *E, F,* and *G* are the feet of the perpendiculars from *H* to the respective sides. Compare *AG* and *HE* + *HF* + *HG*. What do you find?

c. Does it matter where *H* is located inside the triangle? Explain. (Hint: Use the areas of triangles to help.)

3. Could you find a Steiner point using more than three locations?

4. If the problem is placed on a set of square city blocks where one can only move horizontally or vertically, then there may be more than one path that is of minimal length. Use a checkerboard to examine a situation where each site the criminal visits is at a vertex of a square on the board.

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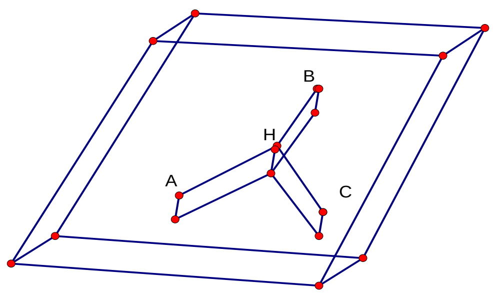
The goal of this activity is to give your students a short and simple snapshot into a very extensive math topic. TI and NCTM encourage you and your students to learn more about this topic using the extensions provided below and through your own independent research.

Extension 1

Charlie relates the solution to this problem as one that has been solved with soap bubbles. When two plates (connected with pegs *A*, *B*, and *C*) are dipped into soapy water and then removed, the soap bubbles formed between the plates move to reach an equilibrium achieved when the distance between the pegs *A*, *B*, and *C* and the intersection of the soap bubbles is at a minimum. When this happens, a new point, *H*, is created by the bubbles, giving the shortest path of travel from *H* to the three pegs. An illustration with three parallelograms, having *AH, BH,* and *CH* respectively as one side, representing the soap bubbles is shown below.



Use the website below to create the soap bubble experiment to find the shortest distance from a point to three other points.
http://www.shout.net/~mathman/html/bubbles.html

Extension 2

Suppose you wanted to locate an airline hub so that the hub was the shortest distance from three cities. Which of the student activity problems might help you find the "best" location for the hub?

Read *The Man Who Loved Only Numbers: The Story of Paul Erdös and the Search for Mathematical Truth* (pp. 164-165) by Paul Hoffman. Discuss the problem of Delta Air Lines being charged by AT&T for a 2,000-mile set of telephone lines connecting three locations and challenging the charge by finding a place (a Steiner point) from which the lines could be run at a distance of about 1,640 miles, and a savings of about 13.4%.

Other Resources

Boys, C. V. *Soap Bubbles: Their colors and Forces which Mold Them*. New York: Dover Publications, Inc., 1959.

Hoffman, Paul. *The Man Who Loved Only Numbers: The Story of Paul Erdös and the Search for Mathematical Truth.* New York: Hyperion, 1998.

Peterson, Ivars. *Soap Films and Grid Walks*, Mathematical Association of America online journal, 1996. **http://www.maa.org/mathland/mathland_4_8.html**

Isenberg, Cyril. *The Science of Soap Films and Soap Bubbles*. New York: Dover Publications, 1992.

Johnson, Jerry. "Minimum Distance," In *Teaching with Student Math Notes, Vol. 2* pp. 51-56. (Edited by Evan M. Maletsky). Reston, VA: National Council of Teachers of Mathematics, 1993.

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