

Logistic Growth

by Dave Slomer

Many populations, such as that of our world, grow exponentially. Food is generally plentiful; disease is generally not a threat. But in many populations, where space or habitat is limited, population growth is limited, possibly because of insufficient food or spread of disease aggravated by the crowded conditions. Populations in this kind of environment often show what is known as *logistic growth*.

We are going to look at laboratory data and explore rates of change even though we have no equation that defines the data. We will be numerically approximating the derivative. Depending upon the size the denominator of the difference quotient, we may get a very good or a not-so-good approximation of instantaneous growth rates, but we will have an average rate, from which we may make intelligent observations and from them make the best possible decisions.

Before starting, turn off or clear all [Y=] functions and get into **Function Graph** [MODE].

Table 1 represents the number of bacteria in a culture over a 20-hour period. Load it into two **lists** on your TI. [In this activity, the lists will be called **ti** and **pop**.]

- The **ti** list can be created by typing **seq(x,x,1,20,1)→ti** on the home screen.
- You can type **{210, 232, 260, 293, 333, 382, 438, 500, 562, 618, 667, 707, 740, 768, 790, 809, 825, 839, 850, 860}→pop** [or use the **Data/Matrix Editor** ([APPS] [6])].
- Next, **scatterplot** the data. To do so, type **NewPlot 1,1,ti,pop**.
- Finally, on the [GRAPH] screen, **Zoom Data** ([F2] [9]).

The screens will look like those in figures 1a and 1b.

Table 1:

ti	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
pop	210	232	260	293	333	382	438	500	562	618	667	707	740	768	790	809	825	839	850	860

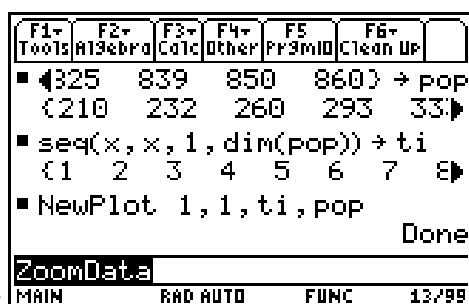


Fig. 1a

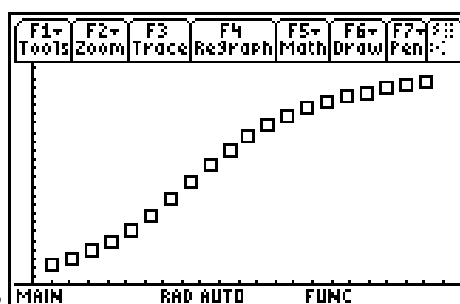


Fig. 1b

Exercise 1: On the [GRAPH] screen, **Trace** ([F3]) the scatterplot to get a “graphic feel” for the data. Notice that the growth rate seems to be somewhat exponential for awhile then begins to start slowing down. Trace to the point where it appears that the population growth rate first begins to slow down. What are the coordinates of that point?

Just where the population growth first begins to slow down is a tough call.

Exercise 2: For what values of **ti** does the graph of population vs. time seem to be concave up? Concave down? Do you see clearly where the inflection point is? Where do you think it is?

Now you know why it's not clear just where the population growth begins to stop increasing exponentially. As with all differentiable curves, the curve is almost linear near the inflection point. If we had an equation for the population vs. time curve, we could find the second derivative and know where the inflection point was, but we have no equation—yet.

Even if it was clear, it is always advisable to get a “second opinion” or to take a look at a problem from a different perspective.

So, plot *instantaneous rate of population growth* vs. time. This rate is impossible to compute exactly without an equation for the function, but there is an easy way to get rate data for approximate rate of growth.

- On the home screen, type **seq(x,x,1,19,1)→ti2** [ENTER].
- Noting that you can find the character Δ by pressing [2nd][+][1][5], type **Δ List(pop)/ Δ List(ti)→popgro** [ENTER].
- Then type **NewPlot 2,1,ti2,popgro** [ENTER] to prepare a new scatterplot.

The screen should look like the one in figure 2a.

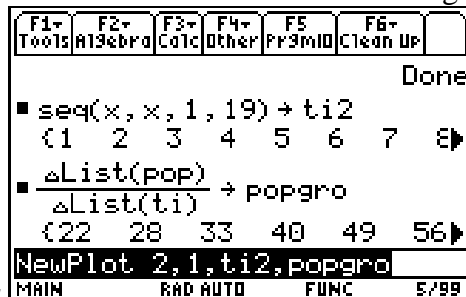


Fig. 2a

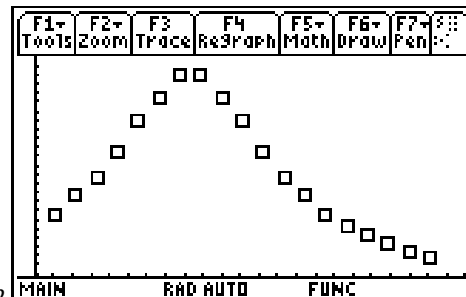


Fig. 2b

You'll have to think about what's on the screen for a bit. First, observe that **Δ List(pop)/ Δ List(ti)** returned 22, 28, 33, 40, 49, 56... and then review the population data (Table 1). Do you see that each number returned is the change in population per hour? So a list of values of the form Δ pop/ Δ ti has been stored into **popgro**, which will be plotted against **ti2**, as the **NewPlot** command requests. [Note: you can't plot **popgro** vs. **ti** because **ti** has one more list element in it than **popgro**.] You might also want to fill in the 3rd row of Table 1 at this time, but it's not necessary.

On the [Y=] screen, cursor up (\uparrow) to stat **Plot 1** and turn it off ([F4]). Then **ZoomData** [press [F2][9]] and study the screen (fig. 2b).

Exercise 3: Trace the new scatterplot and answer the following questions about the information contained therein:

- What are the units for this new scatterplot?
- Approximately where is the absolute maximum point on this graph?
- How does the maximum relate to the graph of **pop** vs. **ti** (fig.
- For which times is the population increasing? decreasing?
- For which times is the population's rate of growth increasing? decreasing?
- At what time was the population's rate of growth fastest? slowest?
- At what time did the population's rate of growth begin to slow?
- Where do you now think the inflection point is on the **pop** vs. **ti** graph?

The numeric and graphic has served us well, but on to the symbolic. There is a way to come up with an equation that might closely fit the data. The concept is known as *regression*.

Exercise 4: Do a *logistic regression* statistical calculation. On the home screen, type **Logistic ti,pop** [ENTER]. Once the calculation is over, the screen will look like the one in figure 3a. If you then type **showstat**, you will see the results of the regression (fig. 3b).

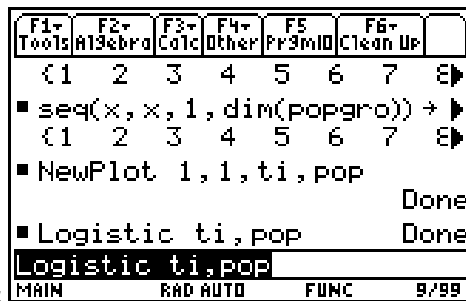


Fig. 3a

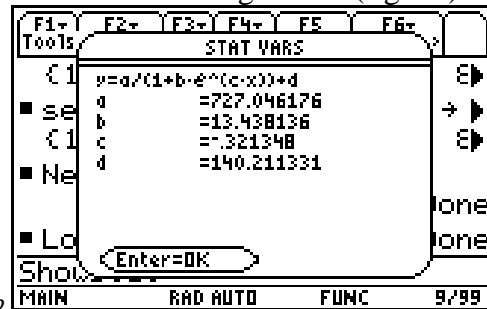


Fig. 3b

The screen is telling you that the equation $y = \frac{727}{1 + 13.4 \cdot e^{-0.321x}} + 140$ is the best logistic

model fit for the data. If you type **regeq(x)** [ENTER], you will see an algebraically equivalent version of the right-hand side of this equation on the screen (fig. 2c). [This screen was displayed with [MODE] **Display Digits Float 5** so the entire right-hand side could be viewed.]

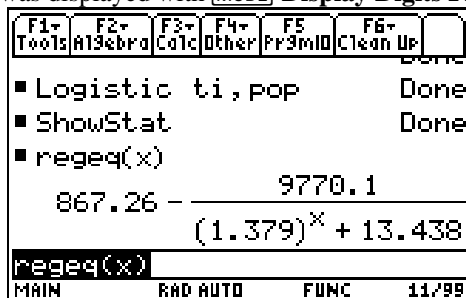


Fig. 3c

Note that **regeq(x)** looks nothing like the information contained in the **showstat** screen, but they are algebraically equivalent. [If you have some free time, try to convert one into the other. With your '89, it might not be so bad. Think logs.] So, take your pick about which version of the regression model to use for **y1** and type it in [feel free to round off to 5 significant digits]. The

horizontal asymptote is easier to “see” in the version on the home screen in figure 3c. [You can also cursor up and “grab” the **ans** for the **regeq** command and store it to **y1**.]

- What is the horizontal asymptote for the logistic regression model?
- What does the horizontal asymptote mean in terms of the growing bacteria?
- Does the model look like a perfect fit? Is it? Why?

The regression model can be used to *interpolate* (predict populations within the given time interval) and possibly to *extrapolate* (predict outside the given time interval).

- Use the regression model to estimate the population (by interpolating) at time 7.5 hours and (by extrapolating) at 0, 40, 400, and 4000 hours. [You’ll have to do the last 3 on the home screen unless you change the window. To do so, type **y1(40)**, for example.] Do all the figures seem reasonable? Do you think there is any reason to question any of them?

Exercise 5: Now that we have a good-fitting model in **y1**, find the inflection point on the regression function’s graph and see if it’s where you expected it to be (in your answers to Exercises 1 through 4) You have three choices:

1. Find where **y1'** has a maximum (analytically, numerically, or graphically).
 2. Find where **y1''** has a zero (analytically, numerically, or graphically).
 3. On the graph screen, use **[F5] (Math) [8] (Inflection)**, which finds inflection points interactively. Like some other graph screen mathematical processes, it needs you to “surround” the likely inflection point. Be sure you’re to the left of it when it asks for **Lower bound** and be sure you’re to the right for **Upper bound**.
- The inflection point for the model is at approximately (_____, _____).

Exercise 7: Store the derivative of **regeq(x)** into **y2**. [You could just store $d(y1(x),x)$ to **y1**.] Turn on Plot 2 and turn off Plot 1 and y1. Then **Zoom Data**. How well does **y2** fit the data? Where is the maximum point on **y2**? Does this affect your previous conclusions about the inflection point? How/Why?

What are your impressions of this mathematical process? Write a few paragraphs about the whole thing and about what you learned during this activity.

Calculus Generic Scope and Sequence Topics: Applications of Derivatives

NCTM Standards: Number and operations, Algebra, Geometry, Measurement, Problem solving, Connections, Communication, Representation