<u>Systems Approach to Recursion</u>

Systems of equations may also be applied to finding the <u>*closed recursive (or explicit) form*</u> of a function. The recursive requires that we know the previous value in a sequence to find the subsequent. For instance: Consider the sequence 4, 7, 10, 13, 16, … We can see immediately that the values are changing by a positive 3 at each step (that is – we add three to the present value to arrive at the next (4+3=7, 10+3=13, and so on).

In a linear sequence such as this one, we can see that the difference of one value to the next is constant in the very first iteration of our search:

```
4       7       10      13      16
 \     / \     /  \    /  \    /
    3       3       3       3
```

Where each new value is a consistent distance from the previous, so we would say:

$a_n = a_{n-1} + d$

a_n = Value of nth term
a_{n-1} = Value of the immediately previous term
d = The common difference between successive terms

And we have a well-known formula that we teach to pre-algebra students to arrive at a function we can use to predict the anyth (nth) term:

$a_n = a_1 + (n – 1) d$

a_n = Value of nth term
a_1 = Value of the very first term
n = Number in sequence of the value sought
d = The common difference between successive terms

There is, however no such formula to use to describe the explicit form of the function when it is not linear – but there is a process we can follow and this involves the use of the calculator's matrix solving capabilities.

Consider the following: -1, 5, 15, 29, 47, 69, … Let's first line up the values

```
–1       5       15       29       47       69
  \     / \     /  \     /  \     /  \     /
     6       10       14       18       22
      \     /  \     /  \     /  \     /
         4        4        4        4
```

Nope – It's not a common difference yet – Not linear

There it is! – In the second line (iteration) so it must be Quadratic!

Now, remember that the Standard Form of the Quadratic is $Ax^2 + Bx + C = y$
We'll use this to get the coefficients for a system of equations.

We know the y-values (-1, 5, 15, etc), and we know the placement in the sequence of each of the answers (1, 2, 3, etc), so we can replace x with 1 (and then 2, then 3, etc) to get the correct number of variables in a system of equations.

How many equations do we need? One for each variable, of course.

That gives us:

$$a(1)^2 + b(1) + c = -1$$
$$a(2)^2 + b(2) + c = 5$$
$$a(3)^2 + b(3) + c = 15$$

Which becomes

$$a + b + c = -1$$
$$4a + 2b + c = 5$$
$$9a + 3b + c = 15$$

Processing the matrix on the NSpire family of handhelds is much more direct. Bring up the calculation area (Home button, then #1) and:

1. Type rref and open the parentheses (note that the parentheses is open and that the closing symbol is shadowed – make sure what you type next is inside the symbols)
2. Open the Template menu and select the n by n matrix, which looks like a 3 by 3 - Tell it you want a 3 row by 4 column matrix, $\begin{bmatrix} [\ ] & [\ ] & [\ ] \\ [\ ] & [\ ] & [\ ] \\ [\ ] & [\ ] & [\ ] \end{bmatrix}$
3. Fill in the matrix (carefully) and press [ENTER]

In this example, the solution appears as:

$$\begin{matrix} [1 & 0 & 0 & 2] \\ [0 & 1 & 0 & 0] \\ [0 & 0 & 1 & -3] \end{matrix}$$

- but how do we read this?

It means that 1A, no Bs, and no Cs = 2, and that 1 B, no As, and no Cs = 0 and that 1C, no As and no Bs = -3.

In our function, that's 2x2+0x-3 = y , or more properly 2x2-3 =y.making the solution to our function $2x^2 + 0x - 3 = y$ or more properly $2x^2 - 3 = y$.

Now try this:

Can you determine the explicit form of the equation used to generate this sequence?

2, 5, 10, 17, 26, …

1. Use the difference list function to subtract each term from the next one in the sequence.

ΔList({2,5,10,17,26}) ▸ {3,5,7,9}

Is the difference constant? If it is we have a linear function, if not, we'll need to do the same thing for the new set of numbers. Get ΔList from the List Operations submenu in the Statistics menu. Use next page.

What does the system of equations look like?

$A(\ )^2 + B(\ )^1 + C(\ )^0 =$

$A(\ )^2 + B(\ )^1 + C(\ )^0 =$

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So the matrix looks like . . .

$$\begin{bmatrix} (\ ) & (\ ) & (\ ) & (\ ) \\ (\ ) & (\ ) & (\ ) & (\ ) \\ (\ ) & (\ ) & (\ ) & (\ ) \end{bmatrix}$$ Copy matrix on next page

And the solution comes from performing the reduced row echelon form

$$rref\left(\begin{bmatrix} (\ ) & (\ ) & (\ ) & (\ ) \\ (\ ) & (\ ) & (\ ) & (\ ) \\ (\ ) & (\ ) & (\ ) & (\ ) \end{bmatrix}\right)$$

What does the quadratic look like? ____

$y = x^2 + 1$

Note to Instructor:

Linear value in ID matrix 1.3E-13 which is rounding issue with TI – Means Zero

Now try this one:

10, 26, 58, 112, 194,…

```
10        26        58        112       194
 \        / \       / \       / \       /
     16        32        54        82
      \        / \       / \       /
          16        22        28
           \        / \       /
               6         6
```

$$ax^3 + bx^2 + cx + d = 10$$
$$ax^3 + bx^2 + cx + d = 26$$
$$ax^3 + bx^2 + cx + d = 58$$
$$ax^3 + bx^2 + cx + d = 112$$

$$(1)^3 a + (1)^2 b + (1)c + d = 10$$
$$(2)^3 a + (2)^2 b + (2)c + d = 26$$
$$(3)^3 a + (3)^2 b + (3)c + d = 58$$
$$(4)^3 a + (4)^2 b + (4)c + d = 112$$

$$a + b + c + d = 10$$
$$8a + 4b + 2c + d = 26$$
$$27a + 9b + 3c + d = 58$$
$$64a + 16b + 4c + d = 112$$

$$rref\left(\begin{bmatrix} 1 & 1 & 1 & 1 & 10 \\ 8 & 4 & 2 & 1 & 26 \\ 27 & 9 & 3 & 1 & 58 \\ 64 & 16 & 4 & 1 & 112 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Curve: $y = x^3 + 2x^2 + 3x + 4$

And finally this one:

1. In a store display, grapefruit are stacked 4 levels high in the shape of a pyramid with a square base. What expression can be used to determine how many grapefruit can be stacked in a pyramid *n* layers high?

```
1         5         14        30        55
\        / \        / \       / \        /
     4         9         16        25
     \        / \        / \       /
          5          7         9
          \         / \       /
               2          2
```

$$a+b+c+d=1$$
$$8a+4b+2c+d=5$$
$$27a+9b+3c+d=14$$

$$\frac{1}{3}x^3+\frac{1}{2}x^2+\frac{1}{6}x+0=y$$
$$\left(\frac{2}{2}\right)\frac{1}{3}x^3+\left(\frac{3}{3}\right)\frac{1}{2}x^2+\frac{1}{6}x=y$$
$$\frac{2x^3+3x^2+x}{6}=y$$
$$\frac{x(x+1)(2x+1)}{6}=y$$