BRING GEOMETRY TO LIFE

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In the following we will use *Geometer's Sketchpad* on a TI-92 Plus to illustrate some interesting geometric properties. The *Sketchpad* program and manual for the 92 Plus can be downloaded for free from the *Texas Instruments* web site (**www.ti.com/calc/flash**).

THE EULER LINE OF A TRIANGLE

THEOREM: In any triangle the centroid, circumcenter, and orthocenter are collinear. The line containing these three points is called the Euler line of the triangle.

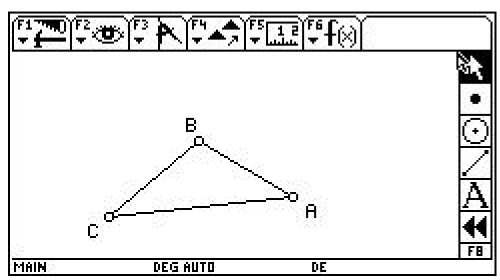
To illustrate this theorem, first start with a new sketch (**F1/9/1/ESC**)*. You may want to turn Sketchpad's automatic labeling feature off (**F1/A/down 5/right 1/1/ENTER/ENTER**) since the labels tend to clutter the screen. In this article, this feature will be turned on since we will need to refer to points by their labels.

Construct a triangle:

a. Use the segment tool (**F8/down 3/left 3/ ENTER**) to construct side $\overline{AB}$. Press **ENTER** to establish point *A*, move the cursor to point *B* and press **ENTER** again.

b. Construct side $\overline{BC}$ by pressing **ENTER** again at point *B*, moving the cursor to *C* and pressing **ENTER** once more.

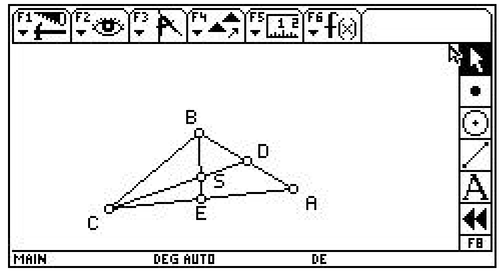
c. Construct side $\overline{CA}$ in the same manner.



* **Note:** As this article progresses, the directions for using *Sketchpad* will become less explicit.

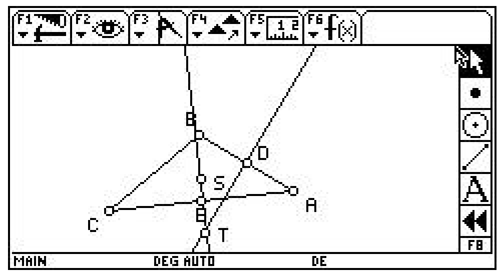
Construct the centriod *S* (intersection of the medians) of the triangle:

a. Find the midpoints of two sides (**ESC**, highlight the side, and press **F3/2**)
b. Construct the medians (**ESC**, highlight a midpoint and the opposite vertex, press **F3/4**)
c. Find the point of intersection of the medians (highlight the medians, press **F3/3**)
d. Hide the medians (**ESC**, highlight medians, press **F2/1**)



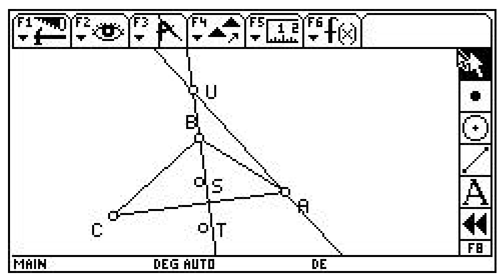
Construct the circumcenter *T* (intersection of perpendicular bisectors) of the triangle:

a. Construct the perpendicular bisectors of $\overline{CA}$ (**ESC**, highlight *E* and $\overline{CA}$, **F3/8**) and $\overline{AB}$
b. Construct the point of intersection of the perpendicular bisectors (**ESC**, highlight bisectors, press **F3/3**)
c. Hide the perpendicular bisectors and the midpoints (**ESC**, highlight bisectors and midpoints *D* and *E*, press **F2/1**)

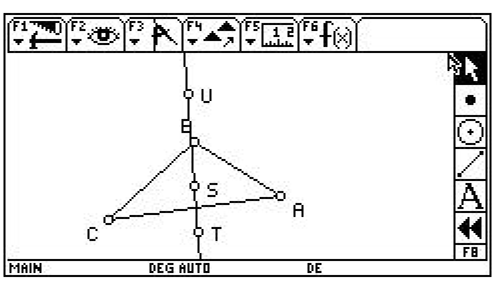


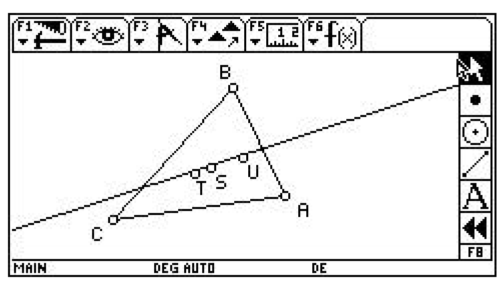
Construct the orthocenter *U* (intersection of the altitudes) of the triangle:

a. Construct the altitudes from *A* (**ESC**, highlight *A* and $\overline{BC}$, press **F3/8**) and *B*
b. Construct the point of intersection of the altitudes (**ESC**, highlight altitudes, press **F3/3**)
   Note: If the point of intersection is off the screen, you can reshape the triangle by dragging its vertices.
c. Hide the altitudes (**ESC**, highlight altitudes, press **F2/1**)

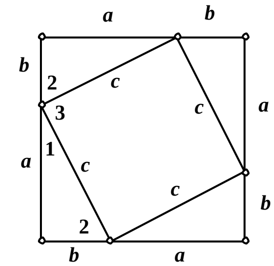


Construct the Euler line (**ESC**, highlight *S*, *T*, and *U*, press **F3/6**). Investigate the placement of the Euler line by dragging the vertices of the triangle (**ESC**, highlight vertex, hold down the **LOCK** key while moving the cursor).



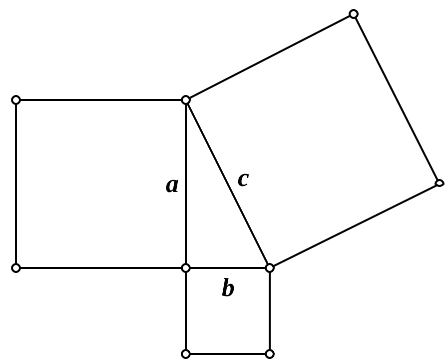


THE PYTHAGOREAN THEOREM



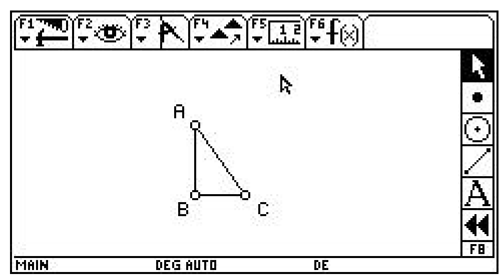
An easy way to algebraically prove the Pythagorean theorem is to start with a large square whose side is the sum of the measures of the legs of the right triangle. Alternate these measures as you go around the perimeter of this square, as pictured at the right. Since the sum of the acute angle of a right triangle is 90° and the sum of the angles forming a straight line is 180°, angle **3** must be a right angle. So the figure in the middle is a smaller square. Thus the area of the large square is equal to 4 times the area of the right triangle plus the area of the smaller square. This gives:

$$(a+b)^2 = 4\left(\tfrac{1}{2}ab\right) + c^2 \;\Rightarrow\; a^2 + 2ab + b^2 = 2ab + c^2 \;\Rightarrow\; a^2 + b^2 = c^2$$



Although this is a nice proof, it does not illustrate the Pythagorean theorem the way the picture at the right does, the picture we are used to seeing. But it is not at all clear from this picture how one can show that the sum of the two smaller squares is equal to the larger square.

Here's where Geometer's Sketchpad helps out. We are going to use sketchpad to actually move the smaller squares into the larger square. To do this, first start with a new sketch (**F1/9/1/ESC**).



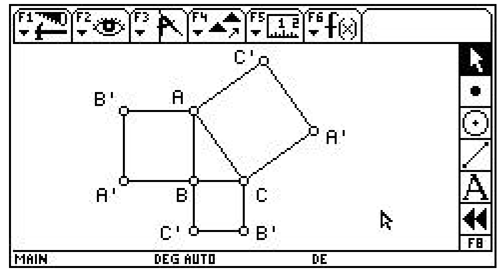
Construct the right triangle:

a. Use the segment tool (**F8**/**down 3/left 3**) to construct leg $\overline{AB}$.

b. To construct leg $\overline{BC}$:
   1. press **ESC** to release the segment tool
   2. press enter to highlight point *B*
   3. press **F3/8** to construct a line perpendicular to $\overline{AB}$ at *B*
   4. press **F3/1** to construct to construct a point *C* on this line
   5. place the cursor on *C*, hold down the **LOCK** key while using the cursor to move *C* to the desired location.
   6. Release the **LOCK** key, press **ESC** to unhighlight *C*, highlight line $\overleftrightarrow{BC}$ and hide it (**F2/1**).
   7. Construct $\overline{BC}$ (highlight *B* and *C* and press **F3/4**).

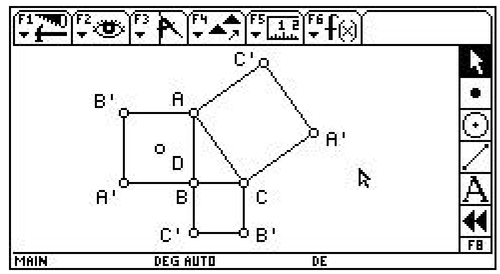
c. Construct the hypotenuse $\overline{AC}$ (**ESC**, highlight *A* and *C*, and press **F3/4**).

Construct the three squares formed by the sides of this triangle:



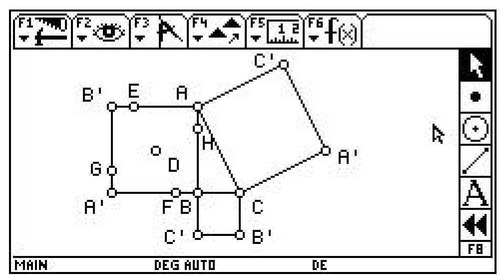
a. Mark point A as the center of rotation (**ESC**, highlight A, **F4/1**)
b. Rotate $\overline{AC}$ 90° about point A (highlight $\overline{AC}$ and C, **F4/8**, enter 90, press **ENTER** twice)
c. Rotate $\overline{AB}$ -90° about A (**ESC**, highlight $\overline{AB}$ and B, **F4/8**, enter -90, press **ENTER** twice)
d. In the same fashion, mark B as the center of rotation and rotate $\overline{AB}$ 90° about B and $\overline{BC}$ -90° about B. Then mark C as the center of rotation and rotate $\overline{CB}$ 90° about B and $\overline{CA}$ -90° about B.
e. Construct the remaining side of each square (**ESC**, highlight the two points, **F3/4**)

Find the center D of the medium sized square:



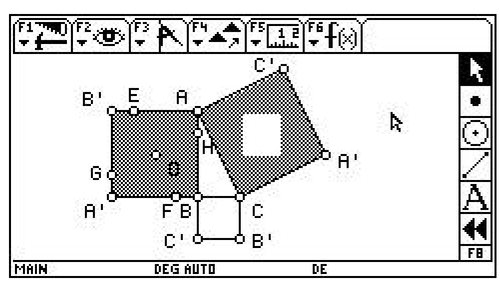
a. Construct segments $\overline{AA'}$ and $\overline{BB'}$ (**ESC**, highlight the endpoints, **F3/4**)
b. Find the point D where the diagonals intersect (highlight the other diagonal, **F3/3**)
c. Hide the diagonals (**ESC**, highlight the diagonals, **F2/1**)

Find where the lines through D, which are parallel to the sides of the large square, intersect the medium sized square:



a. Find the line through D which is parallel to $\overline{CA'}$ (**ESC**, highlight $\overline{CA'}$ and D, **F3/7**)
b. Find the points E and F where this line intersects the sides of the medium sized square (highlight a side of the square, **F3/3**, **ESC**, highlight the parallel line and the other side of the square, **F3/3**)
c. In a similar fashion, find the line through D which is parallel to $\overline{CA}$ and find the points G and H where this line intersects the sides of the medium sized square
d. Hide the two parallel lines (**ESC**, highlight these lines, **F2/1**)

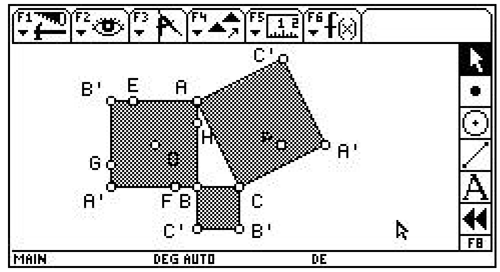
Move the medium sized square into the large square:



a. Set $\overrightarrow{DA'}$ as the vector through which the quadrilateral $GB'ED$ will be translated (**ESC**, highlight D first and then A', **F4/5**)
b. Shade quadrilateral $GB'ED$ and then translate it through $\overrightarrow{DA'}$ (highlight points G, B', E, and D, **F3/C**, **F4/7**, **ESC**)
c. In a similar fashion, translate shaded quadrilaterals $HBFD$, $FA'GD$, and $EDGH$ by vectors $\overrightarrow{DA}$, $\overrightarrow{DC'}$, and $\overrightarrow{DC}$ respectively.

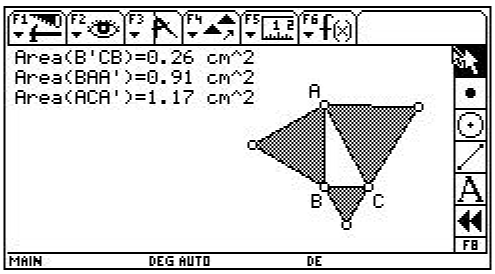
Move the small square into the large square:

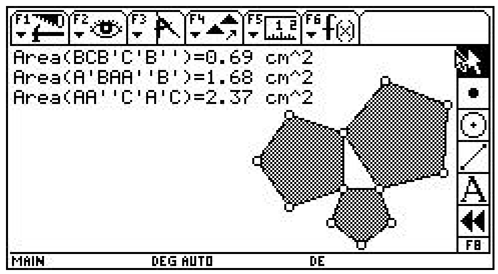
a. Translate point A by vector $\overrightarrow{DC}$ to obtain point P
b. Translate shaded square $BC'B'C$ by vector $\overrightarrow{B'P}$

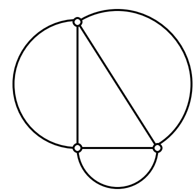


In the above, we saw that when squares are placed on the sides of a right triangle, the sum of the areas of the smaller squares is equal to the area of the larger square. It is interesting to note that this property holds when one places any regular polygon on the sides of the right triangle.

We illustrate this below for equilateral triangles and for regular pentagons. In fact, one can place semicircles on the sides of a right triangle and the sum of the areas of the smaller semicircles will equal the area of the larger semicircle.



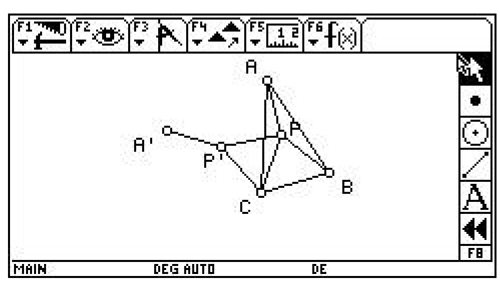




FERMAT'S POINT

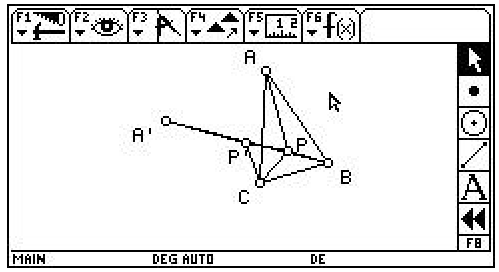
Fermat's point is the point such that the sum of the distances from this point to the vertices of a triangle is minimum. In the following we will show one of many ways of finding this point. First start with a new sketch (**F1/9/1/ESC**).

Construct an arbitrary triangle (**F8**, segment tool).
Construct an arbitrary point P inside the triangle (**F8**, point tool).
Construct the segments joining P to the vertices of the triangle (**F3/4**).
Mark C as the center of rotation (**F4/1**).
Rotate $\overline{CP}$, $\overline{PA}$, and their endpoints 60° about C (**F4/8**).
Construct segment $\overline{PP'}$ (**F3/4**).



Since $\overline{CP}$ was rotated 60° to $\overline{CP'}$, $\triangle CPP'$ is an equilateral triangle. So $m(\overline{PP'}) = m(\overline{CP})$. Thus the length of the broken path $\overline{A'P'} + \overline{P'P} + \overline{PB}$ is the same as the sum of the distances from P to the vertices of the triangle.

Construct segment $\overline{A'B}$ (**F3/4**). Since the shortest distance between two points is a straight line, the path $\overline{A'B}$ is shorter than path $\overline{A'P'} + \overline{P'P} + \overline{PB}$. Drag point P until these two paths are the same. P will then be Fermat's point since $\overline{A'P'} + \overline{P'P} + \overline{PB}$ is a straight line, and thus has minimal length.

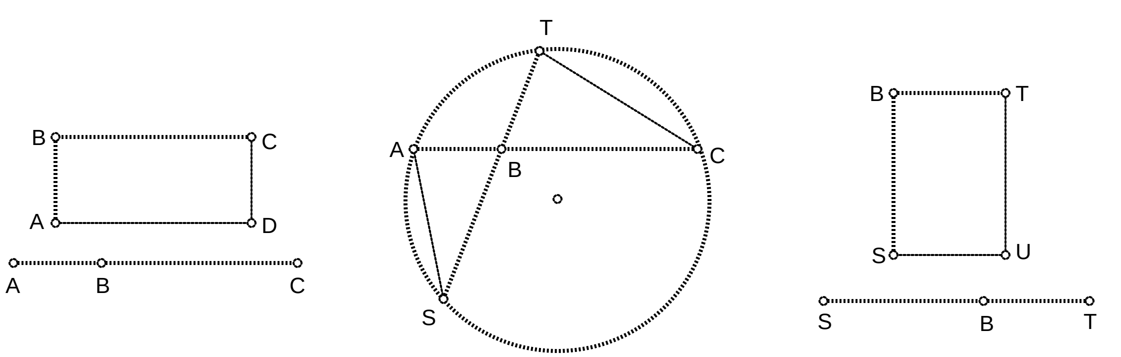


PERIMETER

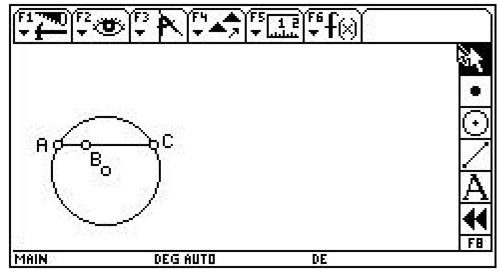
PROBLEM: Find a rectangle of fixed area which has the largest perimeter.

COMMENT: This is a calculus problem which has no simple algebraic or geometric solution. So we will not actually find the desired rectangle. But we will illustrate how to find an infinite number of rectangles having the same area, but different perimeters.

SOLUTION: Let rectangle *ABCD* be any rectangle having the desired fixed area. Then the measure of segment $\overline{AC}$ (i.e., $\overline{AB} + \overline{BC}$) is equal to half of the perimeter of this rectangle. If $\overline{AC}$ (along with point *B*) is the chord of a circle, then any other chord $\overline{ST}$ of the circle which passes through *B* can be used to construct a rectangle *SBTU* having the same fixed area as *ABCD* but a possibly different perimeter. To see that the areas are equal, note that triangles *SAB* and *CTB* are similar since $m\angle SAB = m\angle CTB = \frac{1}{2} m\overset{\frown}{SC}$ and $m\angle ASB = m\angle TCB = \frac{1}{2} m\overset{\frown}{AT}$. So $\frac{m\overline{BC}}{m\overline{SB}} = \frac{m\overline{BT}}{m\overline{AB}}$, and thus $m\overline{AB} \cdot m\overline{BC} = m\overline{SB} \cdot m\overline{BT}$.

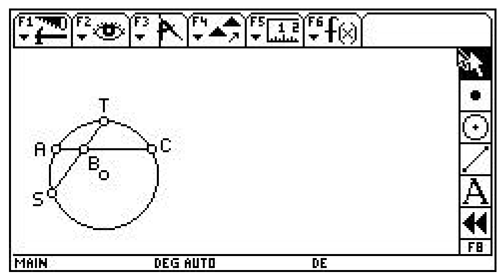


Since a diameter is the longest chord of a circle, we will have found the rectangle of "maximum" perimeter when $\overline{ST}$ is a diameter of the circle. The "maximum" perimeter is then twice the diameter of the circle. By "maximum" perimeter we mean the largest perimeter we can get using the given circle. Clearly a larger circle containing chord $\overline{AC}$ will result in a rectangle of the same fixed area but having larger "maximum" perimeter. To illustrate this:



Construct a circle containing chord $\overline{AC}$ and point B:

a. Construct a circle (**F8**, circle tool).
b. Hide the radius point (**F2/1**). Keep the center.
c. Construct 2 points on the circle (**F3/1**).
d. Move these points to the desired location of $\overline{AC}$ (hold **LOCK**, move cursor).
e. Construct $\overline{AC}$ (**F3/4**).
f. Construct a point B on $\overline{AC}$ (**F3/1**) and move it to the desired location.

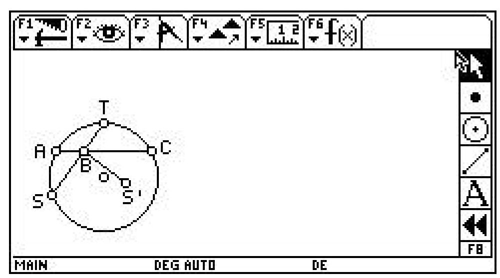


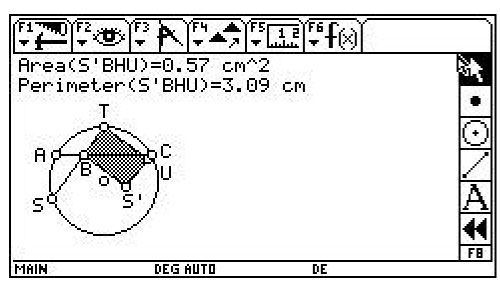
Construct chord $\overline{ST}$:

a. Construct a point on the circle (**F3/1**).
b. Move it to the desired location for T.
c. Construct the line $\overleftrightarrow{BT}$ (**F3/6**).
d. Find the other point S where the line intersects the circle (**F3/3**).
e. Hide the line (**F2/1**).
f. Construct segment $\overline{ST}$ (**F3/4**).

Construct the rectangle $S'BTU$ which will have fixed area:

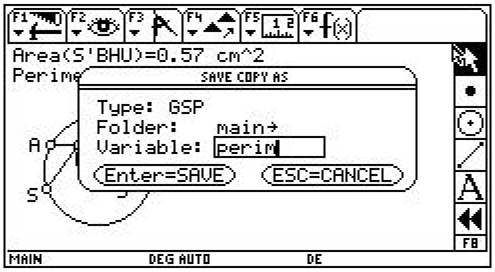
a. Construct a circle centered at B and having radius $\overline{BS}$ (highlight B and then S, (**F3/A**).
b. Construct a line perpendicular to $\overline{ST}$ at B (**F3/8**).
c. Find the points where the line and the new circle intersect (**F3/3**).
d. Hide the line, new circle, and the external point of intersection (**F2/1**).
e. Construct a segment between B and the remaining point S' of intersection (**F3/4**).
f. Mark vector $\overrightarrow{BT}$ (**F4/5**).
g. Translate $\overline{BS'}$ and point S' by $\overrightarrow{BT}$ (**F4/7**).
h. Construct segment $\overline{S'U}$ (**F3/4**).
i. Shade $S'BTU$ (**F3/C**), and compute its area (**F5/6**) and perimeter (**ESC**, rehighlight shaded rectangle, **F5/3**).

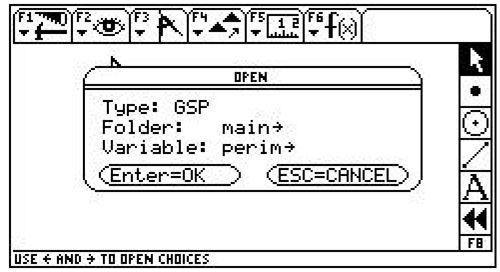




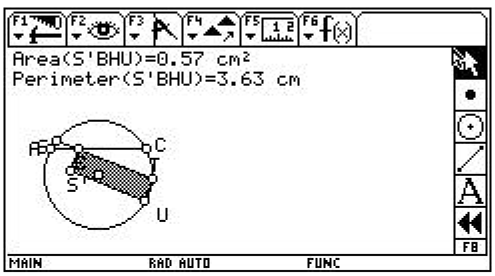
(**Note:** Point H was re-labeled as point T. But sometimes *Sketchpad* reverts back to the original labeling.)

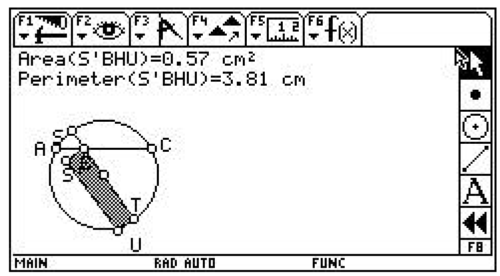
Before proceeding, you may want to save (**F1/9/3**) this construction, and open it (**F1/9/2**) at a later date for a classroom demonstration.





To show how the perimeter changes as the area stays fixed, just drag point *T* around the circle. The perimeter, with respect to the given circle, will be "maximum" when chord $\overline{ST}$ passes through the center of the circle.





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