Making Piecewise Functions *Continuous* and *Differentiable*

by Dave Slomer

Piecewise-defined functions are applied in areas such as Computer Assisted Drawing (CAD). Many piecewise functions in textbooks are neither *continuous* nor *differentiable*—the graph is likely to be "broken" (as in figure 1a) or "pointy" (as in figure 3a). While this may sometimes be desirable, careful analysis can enable us to slightly modify the function, making it become both continuous <u>and</u> differentiable, if need be. To do so requires <u>precise</u>, analytic definitions, not vague words such as "broken" and "pointy".

f is ***continuous*** at $x = a$ if (and only if) $\lim_{x \to a} f(x) = f(a)$. [This will happen if the left- and right-hand <u>limits</u> at $x = a$ equal each other <u>and</u> equal $f(a)$. (See figures 2 {not continuous} and 3b {continuous}.)]

f is ***differentiable*** at $x = a$ if (and only if) $f'(a)$ <u>exists</u>. [This means that the left- and right-hand <u>derivatives</u> at $x = a$ must exist and equal each other. (See figures 5c {not differentiable} and 6e {differentiable}.) Their common value will be $f'(a)$.]

A very important theorem about the relationship between continuity and differentiability says, **"If *f* is differentiable at *x* = *a*, then it is continuous at *x* = *a*."** [This can be shortened to "**Differentiability implies continuity**" or reworded as "Differentiability is a sufficient condition for continuity" or "Continuity is a necessary condition for differentiability." An important consequence of it [called its *contrapositive*] is, "If *f* is <u>not</u> continuous at $x = a$, then *f* is <u>not</u> differentiable at $x = a$."] Note that the converse, "If *f* is continuous at $x = a$, then *f* is differentiable at $x = a$" is <u>**not**</u> necessarily true, as Example 1 will now convince you.

Example 1: As figure 1a shows, the piecewise function $y_1(x) = \begin{cases} x+1, \text{if } x < 2 \\ \sin x, \text{if } x \ge 2 \end{cases}$ is not continuous at $x = 2$. Making this function continuous is not so hard. Making it differentiable isn't much harder. We'll do both, one at a time, continuity first.

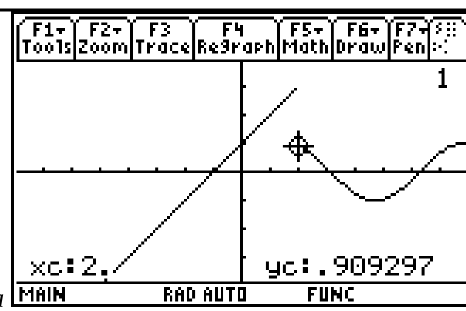


Fig. 1a

To enlist your TI-89's help, use the **when** command, whose basic syntax is **when(*condition, function defintion for that condition, function defintion elsewhere*)**. For the piecewise function in this example, give the command shown in figure 1b.

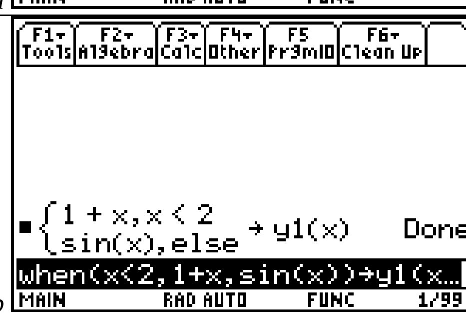


Fig. 1b

*(Note that <u>you</u> do not type the word **else** in the **when** command that defines **y1(x).** The '89 displays **else** rather than displaying **x≥2** [or making <u>you</u> type it], which would be redundant since the first line already says **x<2**. A better word might have been **otherwise** or **elsewhere**, but space is limited.)*

In figure 2, we see the analytic reason for the discontinuity: the limit as x approaches 2 does not exist, because the left- and right-hand limits are not equal, equaling 3 and sin(2), respectively. The definition of continuity, then, proves that **y1** is not continuous at $x = 2$.

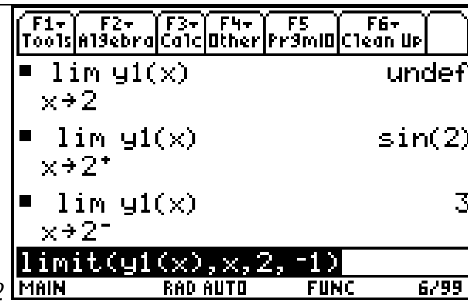


Fig. 2

To make **y1** continuous at $x = 2$, we could raise the sine graph or lower the line. If we decide to raise the sine, it would need to be raised by the exact amount that the left and right-hand limits differ by, 3-sin(2). To do this, we modify the definition of **y1** via a vertical shift, like so: $y_1(x) = \begin{cases} x+1, & \text{if } x < 2 \\ \sin x + (3 - \sin(2)), & \text{otherwise} \end{cases}$. This modified function's graph is shown in figure 3a, while confirmation that the limit exists is shown in figure 3b.

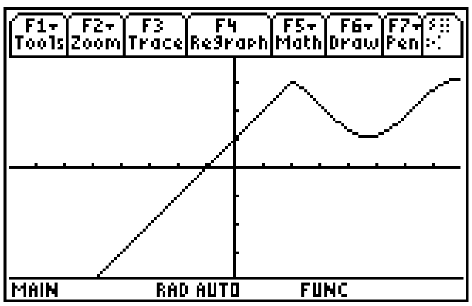


Fig. 3a

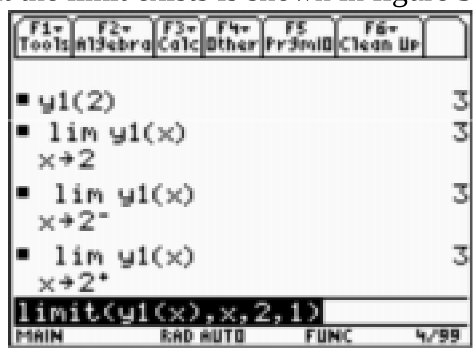


Fig. 3b

[It is not necessary to look at left- and right-hand limits if "the" limit exists, though it might be reassuring!]

Exercise 1: By applying a vertical shift to one piece or the other, make the function $y_1(x) = \begin{cases} 2x - 3 & \text{if } x > 0 \\ \cos(x) & \text{otherwise} \end{cases}$ become continuous at $x = 0$. Supply written analytic (including symbolic) evidence that you succeeded. Don't worry about making it differentiable. Is it differentiable at $x = 0$? Why?

Example 2: We suspect that the modified function of Example 1 is not differentiable since it does not look smooth at the point (2,3) in figure 3a. And, it is not differentiable at $x = 2$, as figure 4 shows. No shift is going to fix the lack of differentiability, since the problem is one of slope—the left-hand derivative at $x = 2$ is the slope of the line, 2, while the right-hand derivative is different, and equals cos(2). (Why?)

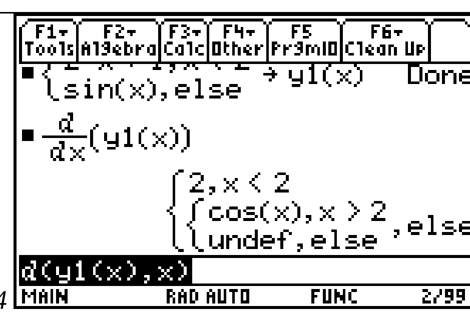


Fig. 4

If the first line in figure 5a, the line that is tangent to the sine piece of the function at $x = 2$ is made to be the left piece of the function. Since (2,sin(2)) is a point on the curve [Why? See figure 1b] and since the right-hand derivative is cos(2) [Why? See figure 4], we make $\boxed{y_1(x) = \begin{cases} \sin(2) + \cos(2)(x-2) \text{ if } x < 2 \\ \sin x \text{ if } x \geq 2 \end{cases}}$.

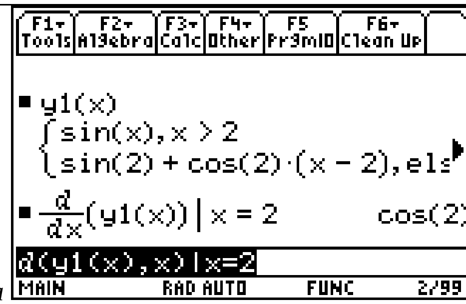


Fig. 5a

Figure 5a shows that **y1'(2)** now exists, equaling **cos(2)**. All <u>looks</u> well, too, as shown in figure 5b. How do you know that **y1** is also continuous at **x = 2**?

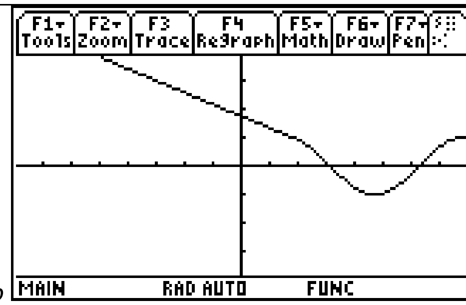


Fig. 5b

Exercise 2: Choose *m* and *b* to make the function $y_1(x) = \begin{Bmatrix} mx + b, \text{ if } x > 0 \\ \sin(x), \text{ otherwise} \end{Bmatrix}$ differentiable at $x = 0$. Let Example 1 guide you. Supply written analytic (including symbolic) evidence that you succeeded. How would you convince someone that the function is continuous at $x = 0$?

Exercise 3: Choose *m* and *b* to make the function $y_1(x) = \begin{Bmatrix} mx + b, \text{ if } x > 2 \\ \sin(x), \text{ otherwise} \end{Bmatrix}$ differentiable at $x = 2$. Supply written analytic (including symbolic) evidence that you succeeded. Why do you not have to use limits to prove that the function is continuous at $x = 2$? (Review the definitions and theorem at the beginning of this activity.)

Example 2: But what if neither piece was linear? What if the left piece were a sine and the right a parabola? In particular, what if $y_1(x) = \begin{Bmatrix} x^2, & \text{if } x > 0 \\ \sin(x) & \text{otherwise} \end{Bmatrix}$? What might we do if we wanted to make y_1 differentiable at $x = 0$?

Define **y1** as in figure 5a, observe that the derivative does not exist at 0, and then graph it (see figure 5b).

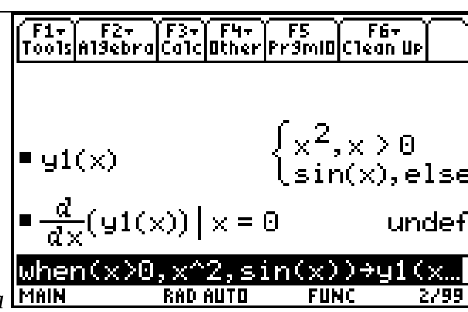


Fig. 5a

In figure 5b, **y1** may <u>look</u> differentiable (smooth), but figure 5a shows that it's not smooth enough.

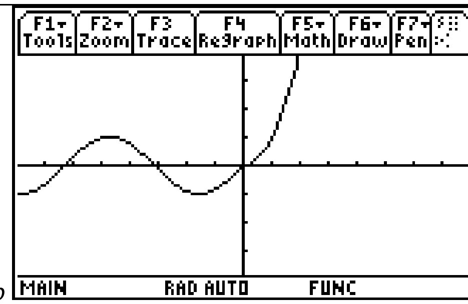


Fig. 5b

Figure 5c tells more precisely why the derivative doesn't exist at $x = 0$: the left-hand derivative is 1 while the right-hand derivative is 0.

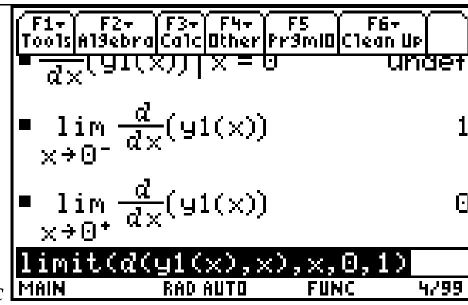


Fig. 5c

In figure 5d, zooming in just a little (with the **Axes OFF**) makes the function look most definitely un-smooth at $x = 0$. [It does not <u>matter</u> what it <u>looks</u> like—figures 5a and 5c say it all—but a look at a graph <u>in the right window</u> can be reassuring—a "second opinion" if you will].

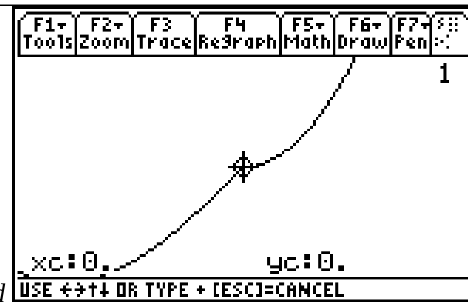


Fig. 5d

So, what do we do in order to let the left piece be a sine, the right a parabola, and make the piecewise function differentiable at $x = 0$? As with continuity, we modify one piece—we arbitrarily choose the sine. (**<u>Before proceeding</u>**, press [2nd][F6] to **Clear variables a-z)**

It's not clear whether moving <u>and</u> distorting the sine horizontally <u>and</u> vertically will be necessary, but, just in case, in figure 6a, we define the left half of **y1** to be the general sine function, leaving the parabola alone. (Surely it would not be necessary to modify <u>both</u>.)

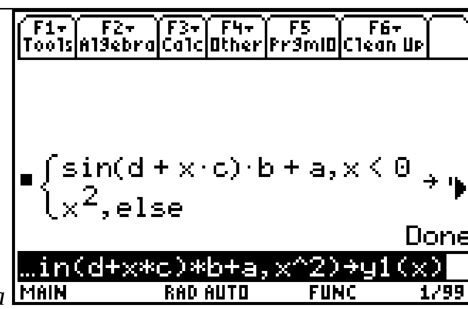


Fig. 6a

In figure 6b, we compute the "continuity" limits. We see that, to have a chance at continuity, **a+b sin(d)** must equal **0** (Why?). (And why "have a chance"?)

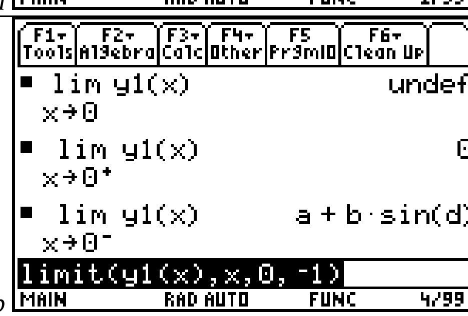
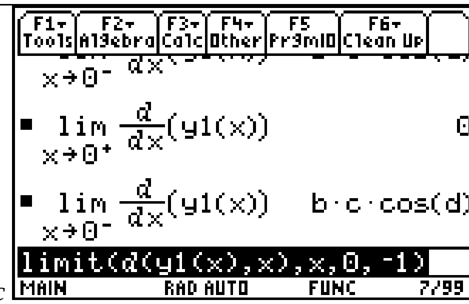


Fig. 6b

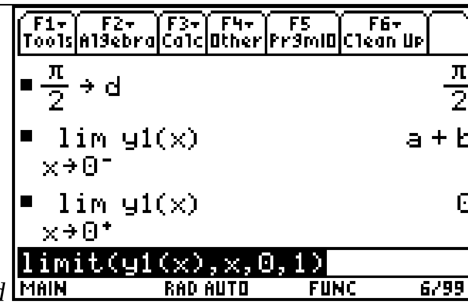
In figure 6c, we compute the left- and right-hand <u>derivatives</u> and see that, for differentiability, **b c cos(d)** must be **0**. (Why?) Since **b** and **c** may <u>not</u> be 0 (Why?), **cos(d)** must be zero. This happens, among other places, at **d=π/2**. (Why there? Name another place or two.)

Fig. 6c



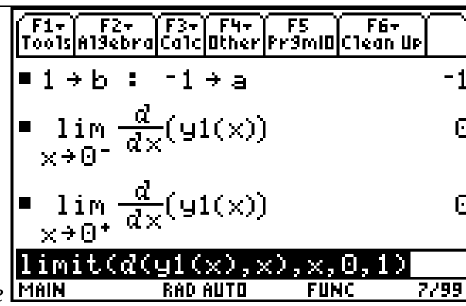
In figure 6d, we store **π/2** into **d** and revisit the "continuity" limits of figure 6b. We see that **a+b** must equal **0** to get continuity (Why? And where did **sin(d)** go?). This does <u>not</u> tell us what either should be, but it <u>does</u> tell us that <u>any</u> nonzero value of **b** <u>will</u> make **y1** continuous at $x = 0$ <u>**if**</u> **a = -b**. (Why?)

Fig. 6d



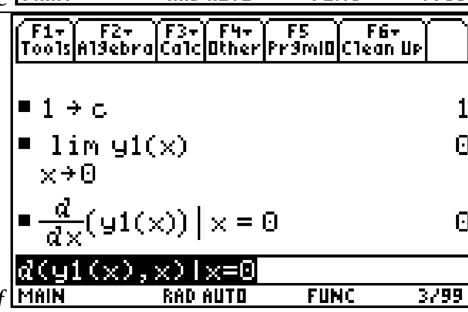
In figure 6e, we decide, arbitrarily, to store 1 into **b** and, hence, -1 into **a**, and revisit the "differentiability" limits of figure 6c. What a nice surprise! No matter what **c** is, we'll have differentiability! (Why?) [You might want to try a different value of **b** and see if the same thing happens. Must it? Why?]

Fig. 6e



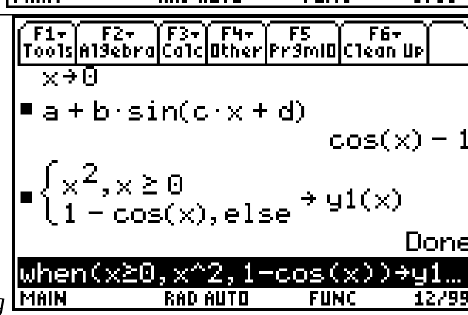
In figure 6f, we arbitrarily make **c = 1** and note that the function is now both differentiable and continuous. (Why is it not necessary to compute the "continuity" limit? Why do we not need to find the left- and right-hand derivatives?)

Fig. 6f



In figure 6g, a look at a simplified version of the sine piece shows something interesting: Where did the sine go? We store this simplified version into **y1** [not that we must] and look at the graph in figure 6h.

Fig. 6g



In figure 6h, a graph of this new function does seem to show smoothness at (0,0). But no matter what it <u>looks</u> like, it is the derivative in figure 6f that makes it differentiable (and therefore continuous—Why?).

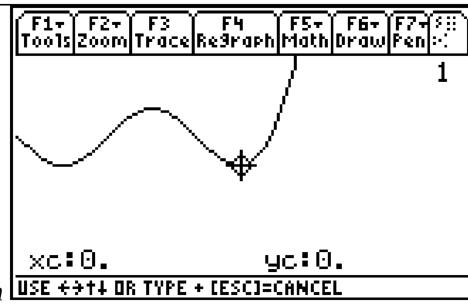


Fig. 6h

Exercise 3: Is the function $y_1(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$ differentiable at $x = 0$? Explain, using a graph of its derivative as part of your argument. Is y_1 discontinuous at any x? Is its derivative? Why?

Exercise 4: The function $y_1(x) = \begin{cases} \dfrac{x-2}{x^2-4}, & \text{if } x \neq \pm 2 \\ 0, & \text{otherwise} \end{cases}$ is neither continuous nor differentiable at $x = \pm 2$. Why? Try to make it continuous and differentiable for all x by leaving the top line of the definition intact while changing the “otherwise” part. Did you succeed with either attempt? If you can’t make it differentiable for all x, can you add a third line to at least make it continuous for all x? Explain.

Press [2nd][F6] to **Clear variables a-z**.

Exercise 5: Find values for b and d so that $y_1(x) = \begin{cases} \sin(x), & \text{if } x < 0 \\ b\ln(x+d) & \text{otherwise} \end{cases}$ will be differentiable at $x = 0$. Why do you suppose the vertical shift (a) was omitted instead of the horizontal? Do you think it matters whether the horizontal distortion (c) is omitted in favor of the vertical (b) or whether c is included while b is omitted? Explore.

Press [2nd][F6] to **Clear variables a-z**.

Exercise 6: Piece together an exponential function and a square root function at $x = 0$ so that the resulting function will be <u>defined</u> and differentiable (and therefore continuous) for <u>**all**</u> x.

Calculus Generic Scope and Sequence Topics: Limits and Continuity, Derivatives
NCTM Standards: Number and operations, Algebra, Geometry, Measurement, Problem solving, Connections, Communication, Representation