NUMB3RS Activity: Markov Chain Links
Episode: Harvest

Topic: Markov Chains **Grade Level:** 9 - 12
Objective: Learn what a Markov chain is, how it is used, and how to use matrices to simplify the computations.
Time: 20 - 30 minutes

Introduction

In "Harvest," Charlie is trying to identify possible destinations and unrecorded routes of an ambulance driver suspected of transporting stolen organs. Charlie tells Colby that they can use a Hidden Markov Model to help narrow down these possibilities.

This activity will show students what a Markov chain is and how to use a Markov chain to calculate and interpret probabilities. Andrei Markov (1856 – 1922) found a graphic way to display the probability of events that affect future probabilities. This information can then be organized in a matrix, which allows for easier computations.

This activity will present Markov chains using the classic example of predicting weather.

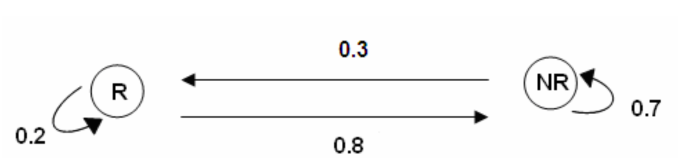
Discuss with Students

We will use weather to illustrate some components of a Markov chain, and will show its representation and perform calculations using a matrix.

Understand Matrix Representation

Suppose that today's weather in a certain town is recorded as rain (R) or no rain (NR). The probability that it will rain tomorrow or not depends on today's weather. Given today's weather, tomorrow's weather can be recorded with a certain probability. The probability of tomorrow's weather, given today's weather, is shown in the diagram and matrix below.

DIAGRAM



MATRIX

Tomorrow

$$\text{Today}\quad \begin{matrix} & \text{R} & \text{NR} \\ \text{R} & 0.2 & 0.8 \\ \text{NR} & 0.3 & 0.7 \end{matrix}$$

i.e. $\begin{bmatrix} 0.2 & 0.8 \\ 0.3 & 0.7 \end{bmatrix}$

DIAGRAM

The diagram on the left shows the possibilities for today and tomorrow's weather. To understand it, start at R, which can be interpreted as rain today. By following the arrow pointing to the right, the probability it will not rain (NR) tomorrow is 0.8. The probability that it will rain tomorrow is 0.2, as shown with the arrow looping back to R.

MATRIX

Start on the far left which represents today. If today was a day with rain (R), then the probability that it will rain tomorrow is 0.2. Another way to look at this is that (row 1, column 1) means (rain today, rain tomorrow), for which the probability is 0.2. Similarly, (row 1, column 2) means (rain today, no rain tomorrow), for which the probability is 0.8.

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Work with students to understand all the possibilities described in both the matrix and the diagram.

Matrix calculations

Suppose today is Monday, and it rains today. What if we want to find the probability of no rain on Wednesday?

Look at the 4 possible weather combinations for the three days, given that it rains on Monday. The weather combinations are ordered as Monday, Tuesday, Wednesday.

R, R, R
R, R, NR
R, NR, R
R, NR, NR

Only two of the combinations have rain on Monday and no rain on Wednesday. Because each day's weather is assumed to be an independent event, use multiplication to find the probability of each combination. Remember, it rained Monday, so the probability for rain Monday is 1.

P(R, R, NR) = 1 × 0.2 × 0.8 = 0.16
P(R, NR, NR) = 1 × 0.8 × 0.7 = 0.56

Then, because rain and no rain are mutually exclusive events (i.e., it cannot "rain and not rain" at the same time), add the probabilities:

P(R, R, NR or R, NR, NR) =
0.16 + 0.56 = 0.72

Notice that this is the same as the sum of the products of the corresponding elements in the first row and second column of the matrix.

$$\begin{bmatrix} \mathbf{0.2} & \mathbf{0.8} \\ 0.3 & 0.7 \end{bmatrix} \times \begin{bmatrix} 0.2 & \mathbf{0.8} \\ 0.3 & \mathbf{0.7} \end{bmatrix} = \begin{bmatrix} 0.28 & \mathbf{0.72} \\ 0.27 & 0.73 \end{bmatrix}$$

So, if today is Monday and it rained, the probability that it will not rain on Wednesday is 0.72. Work with students to understand what the rest of the probabilities in the matrix represents.

Although the matrix multiplication in this activity will be done using a calculator, it is important to know why this method of finding the probability works, so as to understand the connection between the Markov graph and the matrix.

Student Page Answers: 1. *0.38* **2.** $\begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.3 \\ 0.2 & 0.5 & 0.3 \end{bmatrix}$ **3.** $\begin{bmatrix} 0.5 & 0.32 & 0.18 \\ 0.42 & 0.36 & 0.22 \\ 0.38 & 0.36 & 0.26 \end{bmatrix}$ **4.** *the probability of a certain kind of weather on Wednesday, given the weather on Monday; for example, the entry in row 2, column 3 means that the probability that it is cloudy on Monday and rainy on Wednesday is 0.22.* **7.** *0.224* **8.** *the probabilities in each column become closer to each other – the matrix stabilizes* **9.** *Answers may vary. In the long run, for any given day, the probability of sunny weather is about 0.45, the probability of cloudy weather is about 0.34, and the probability of rainy weather is about 0.21.*

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Name: ______________________________ Date: ______________

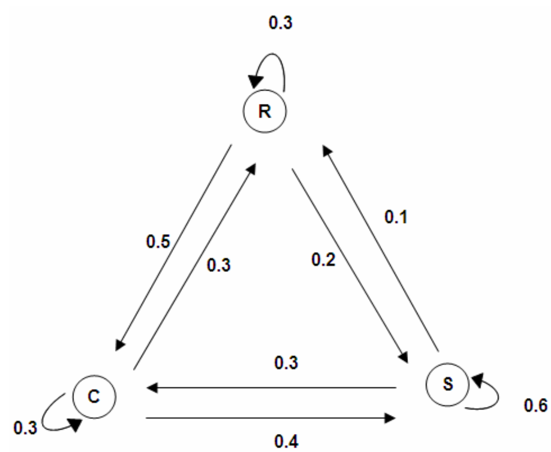
NUMB3RS Activity: Markov Chain Links

In "Harvest," Charlie tells Colby that they can use a Hidden Markov Model to narrow the possibilities of likely destinations and unrecorded routes of the ambulance driver who is suspected of transporting stolen organs. Andrei Markov was a mathematician who developed a method for predicting the probability of certain states in the future from what we know of the present, commonly called a Markov Chain. This activity uses a Markov Chain to predict future weather conditions based on a set of probabilities that are known.

For example, suppose weather consisted of only sunny (S), cloudy (C), and rainy (R) days. Suppose that today is Monday. The graph below shows the probability of tomorrow's weather (Tuesday's weather) if we know today's weather.

If today is rainy (R), the arrows show that the probability of a cloudy day (C) tomorrow is 0.5, the probability of a sunny day (S) tomorrow is 0.2, and the probability of a rainy day (R) tomorrow is 0.3. The graph shows *all* of the possible weather conditions from today to tomorrow.

Notice that the sum of the probabilities of all of the possibilities for each transition state is always equal to 1. For example, the sum of the probabilities for the transitions for R is 0.5 + 0.2 + 0.3 = 1.



What if it is sunny on Monday and we want to know the probability of rain on Wednesday (the day after Tuesday)? Look at the possible weather combinations for the three days given that it was sunny on Monday. The weather combinations are ordered as Monday, Tuesday, Wednesday.

S, S, S	S, C, C
S, S, R	S, R, S
S, S, C	**S, R, R**
S, C, S	S, R, C
S, C, R	

Because we know Monday was sunny and we want to see rain on Wednesday, there are *three* sequences of weather conditions to examine.

Monday was sunny so the probability for sunny weather on Monday is 1, the probability of Tuesday being sunny is 0.6, and the probability of Wednesday being rainy is 0.1. So, $P(S, S, R) = 1 \times 0.6 \times 0.1 = 0.06$. Similarly, $P(S, C, R) = 1 \times 0.3 \times 0.3 = 0.09$ and $P(S, R, R) = 1 \times 0.1 \times 0.3 = 0.03$. So, the probability that the weather for these three days is one of these outcomes is 0.06 + 0.09 + 0.03 = 0.18.

1. What is the probability that if it is raining on Monday, it will be sunny on Wednesday? __________

2. While the graph is helpful for visualizing the problem, the actual computations can be made easier by using a matrix to organize the data. Let the rows represent today's weather, and let the columns tomorrow's weather. Use the graph from this activity to complete matrix *A*.

$$A = \text{Today}\ \begin{matrix} \\ S \\ C \\ R \end{matrix} \overset{\text{Tomorrow}}{\begin{matrix} \quad S \quad\ C \quad\ R \\ \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ \_\_\_ & \_\_\_ & 0.3 \\ \_\_\_ & \_\_\_ & 0.3 \end{bmatrix} \end{matrix}}$$

On the first page of the activity, the probability of sunny weather on Monday and rainy weather on Wednesday by computing *P*(S, S, R) + *P*(S, C, R) + *P*(S, R, R). This probability can also be found using matrix multiplication. The probability of sunny weather on Monday and rainy weather on Wednesday can also be found by finding the product of row 1 and column 3 of matrix *A*.

3. Finding the products of the rows and columns of a matrix is the same as squaring the matrix. Complete A^2 below. (Hint: 0.6 × 0.1 + 0.3 × 0.3 + 0.1 × 0.3 = 0.18)

$$\underset{A}{\begin{bmatrix} \mathbf{0.6} & \mathbf{0.3} & \mathbf{0.1} \\ \_\_\_ & \_\_\_ & 0.3 \\ \_\_\_ & \_\_\_ & 0.3 \end{bmatrix}} \times \underset{A}{\begin{bmatrix} 0.6 & 0.3 & \mathbf{0.1} \\ \_\_\_ & \_\_\_ & \mathbf{0.3} \\ \_\_\_ & \_\_\_ & \mathbf{0.3} \end{bmatrix}} = \underset{A^2}{\begin{bmatrix} \_\_\_ & \_\_\_ & \mathbf{0.18} \\ \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \end{bmatrix}}$$

4. What do the entries of A^2 represent? ______________________________________

5. What is the probability that if it is cloudy on Monday, it will also be cloudy on Wednesday? ___________

6. What do you think the entries of A^3 would represent? __________________________

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7. Use a calculator to compute A^3. What is the probability that if it is raining on Monday, it will also rain on Thursday?

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8. Find higher powers of matrix *A*; for example, find A^{10}. What do the entries of this matrix represent? What do you notice?

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9. What does your answer to question #8 suggest about long term weather patterns according to this model?

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The goal of this activity is to give your students a short and simple snapshot into a very extensive mathematical topic. TI and NCTM encourage you and your students to learn more about this topic using the extensions provided below and through your own independent research.

Extensions

Introduction

In "Harvest," Charlie uses a Hidden Markov Model. This type of model can be used when one does not observe the actual state, but only an outcome that is connected to the state by a certain probability distribution (e.g., one does not observe if it is raining or not, but does know the sales of umbrellas; or one does not know the pitch that is being thrown, but does know the percentages of home runs, strikeouts, etc.) For more about Hidden Markov Models, visit **http://en.wikipedia.org/wiki/Hidden_Markov_model**.

For the Student

Charlie is a baseball fan. Suppose a pitcher has three pitches – fastball (FB), curveball (CB), and change-up (CU). Charlie has observed the following probabilities for the next pitch, where the row is the first pitch and the column is the next pitch.

$$\begin{array}{c} \\ FB \\ CB \\ CU \end{array} \begin{array}{c} \begin{array}{ccc} FB & CB & CU \end{array} \\ \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.5 & 0.2 & 0.3 \\ 0.7 & 0.2 & 0.1 \end{bmatrix} \end{array}$$

If he observes the first pitch was a fastball, what is the probability that the third pitch is a curve?

Additional Resources

Here are a couple of rather complex applications of Markov chains to baseball.

- Project Scoresheet includes large sets of data from major league baseball and discusses questions such as whether a team benefits from the "sacrifice bunt." **http://www.pankin.com/markov/intro.htm**
- This Web site gives an intuitive approach to Markov chains using baseball situations. **http://ite.pubs.informs.org/Vol5No1/Sokol/index.php**

Markov chains can also be applied to music. This link is a master's thesis on applying Markov chains to improvisation in jazz. The thesis itself is very long, but just reading the first page opens up a whole world of math applications in an area where you may not expect to find them.
http://scholar.lib.vt.edu/theses/available/etd-61098-131249/unrestricted/dmfetd.pdf

Related Topic

Markov Chains are used in many different fields. In linguistics, one can use the probability of one letter being followed by another (using a 26×26 matrix). These probabilities vary by language, and have applications to code breaking. For example in the English language, the probability that *q* would be followed by *u* is greater than the probability that *t* would be followed by *h*. However, both of those probabilities would be much greater than the probability that *b* would be followed by *j*.

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