General Quadratics, Cubics, and Quartics

By Dave Slomer

It’s hard to not know just about all there is to know about the characteristics of the graph of the general quadratic (2^{nd} degree polynomial) function. It’s a parabola that is either always concave up or always concave down. We can easily find its *zeroes* via the quadratic formula. And, as you probably learned in algebra, there is a formula for the *vertex*. If we define **y2(x)=a*x^2+b*x+c**, find the derivative and set it to zero and solve, possibly all in one step with our TI-89 in hand, we find that the vertex occurs at the point where $x = -b/(2a)$. (See figure 0.)

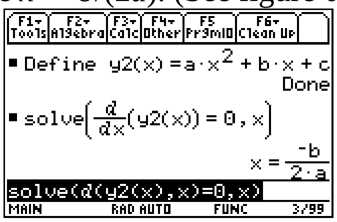


Fig. 0

But what do we know about cubic (3^{rd} degree) and quartic (4^{th} degree) functions?

Exercise 0:

- In figure 1 you see a cubic function that has three *x*-intercepts. How many critical points are there? How many inflection points?
- In figure 2 you see a cubic function that has 1 *x*-intercept. How many critical points are there? How many inflection points?
- In figure 3 you see a cubic function that has 1 *x*-intercept. How many critical points are there? How many inflection points are there?

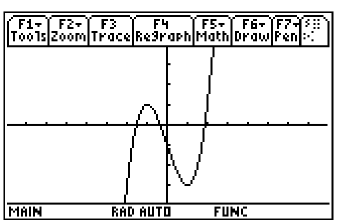


Fig. 1

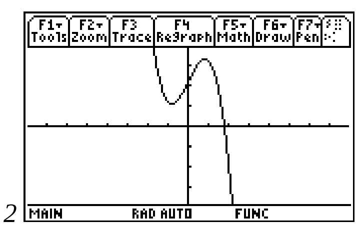


Fig. 2

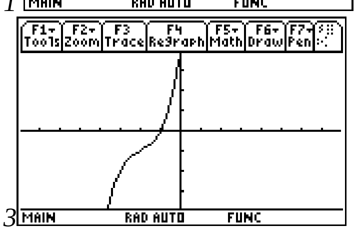


Fig. 3

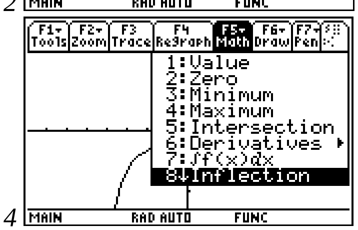


Fig. 4

You undoubtedly noticed that each cubic in figures 1 through 3 appears to have a single inflection point.

Define y3(x)=a*x^3+b*x^2+c*x+d, the general cubic function, whose coefficients are **a**, **b**, **c**, and **d**. Give the command **DelVar a,b,c,d** before proceeding.

Exercise 1: Prove that all cubic functions have exactly one inflection point. To do so, get a formula for its *x*-coordinate in terms of the coefficients of **y3**. [You will want to give the command ***d*(y3(x),x,2)**, which computes the **2**nd derivative of **y3**.] Put your formula to use to find the inflection point of $y1(x) = 4x^3 - 6x^2 - 5x - 3$. Then use your TI-89's **Inflection** point feature (see figure 4) to confirm that your formula worked.

You might call the local maximum and minimum of the cubics in figures 1 and 2 "vertices" because of their similarity in appearance to the vertices of parabolas, but they are not generally called that. Nevertheless, just as we found a formula for the vertex of a parabola earlier, you can find a formula for these *local extrema* (for a cubic that has them), while at the same time establishing the conditions under which a cubic will have them.

Exercise 2: Prove that, in general, if a cubic function has a local maximum, then it has a local minimum, and vice versa. [It can't have one without the other.] To do so, get a formula (in terms of the coefficients) for the critical points. Under what conditions (in terms of the coefficients) will there be both a local maximum and minimum? [To begin to answer all of these questions, you will want to give the command ***d*(y3(x),x)**, which computes the derivative of **y3**.] Use your formula to find the extrema for $y1 = 4x^3 - 6x^2 - 5x - 3$. Then use your TI-89's **Minimum** and **Maximum** features (see figure 4, menu options 3 and 4) to confirm that your formula worked**.**

Do you see the rather striking similarity between the quadratic formula and the formula you just derived for finding the extrema of a cubic? Math just keeps getting better, doesn't it!

Exercise 3: Find the *x*-coordinate of the midpoint of the line segment connecting the local maximum and local minimum for any cubic that has them. [You will have gotten results in Exercise 2 that say **x=… or x=…**. If you move the cursor up so that result is highlighted and press $\boxed{\text{ENTER}}$, you can use the cursor to insert parentheses and change the "**or**" to a "**+**", like so: **(x=…) + (x=…)**. After getting that result, you can then divide by 2.]

Compare your result of Exercise with the result of Exercise 1. Is this surprising? Find the midpoint of the segment connecting the max and min points of the cubic $y1 = 4x^3 - 6x^2 - 5x - 3$ of Exercise 2. Did you get the inflection point in Exercise 1? Trace to this midpoint for that function. Does it look right?

Exercise 4: A common misconception is that if a function's derivative is zero, then it has a local maximum or a minimum there. Produce a specific cubic function (choose specific values for the coefficients) for which the derivative is zero somewhere but that point is neither a maximum nor a minimum. Keep it simple. [You might want to use Exercises 2 and 1 (in that order?) to help.] What kind of point is the critical point? Graph. Does it look right?

Exercise 5: Use the results of Exercise 2 to determine, in general, the conditions under which **y3'(x)=0** yet **y3** has neither a maximum nor a minimum there. Give an example of such a cubic, specifying nonzero values for all of the coefficients. What <u>is</u> going on at such a point on such a cubic? Graph. Does it look right?

Exercise 6: If **y3'(z)=0** and **y3"(z)=0**, then what is going on at the point **(z,y3(z))**? Graph one such cubic.

Now **Define y4(x)=a*x^4+b*x^3+c*x^2+d*x+e**, the general *quartic* function, whose coefficients are **a**, **b**, **c**, **d**, and **e**. Give the command **DelVar a,b,c,d,e** before proceeding.

Exercise 7:

- In figure 5 is a quartic function which has 4 *distinct* *x*-intercepts. How many critical points are there? How many inflection points?
- In figure 6 is a quartic function which has 2 distinct *x*-intercepts. How many critical points are there? How many inflection points?
- In figure 7 is a "complete graph" (all important characteristics are showing) of a quartic function with no *x*-intercepts. How many critical points are there? How many inflection points?

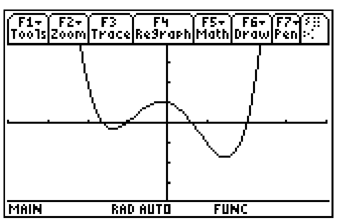
Fig. 5


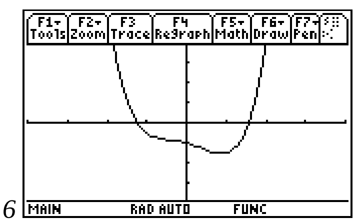
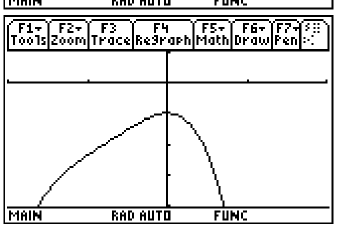
Fig. 6


Fig. 7


Clearly, quartics are more complex than cubics. The number of critical points can range from 1 to 3, while the number of inflection points can range from 0 to 2. And both can be a little hard to see (if they even exist), as figures 6 and 7 show. Unfortunately, because of the difficulty in analyzing cubic equations in general, there isn't much to say about the critical points of a quartic, in general.

Exercise 8: In terms of its coefficients, under what conditions will **y4** have:

- zero inflection points
- one inflection point

- two inflection points

You will want to find the second derivative of **y4** and see where it equals 0. This will give you a formula for determining the *x*-coordinates of any quartic that has inflection points.

Exercise 9: While the formula you found in Exercise 8 is unwieldy at best, you can observe that there will be:

- no inflection point only if the *radicand* is negative,
- one (repeated) inflection point only if the *radicand* is zero, and
- two distinct inflection points only if the *radicand* is positive

Show that, if the radicand is zero, there can be no inflection point; hence the only possibilities (for quartics) are either 0 or 2 inflection points. [Hint: The second derivative is quadratic. Sometimes quadratics don't change sign. When?]

Exercise 10: Use the results of Exercises 8 and 9 to produce equations and graphs of quartics with 0 and 2 inflection points. Use the formula that you found and used to determine the number of inflection points in Exercise 8. Be sure to observe the necessary change in concavity for each of your functions.

Exercise 11: While it is not possible to find a general formula for the critical points, play with the coefficients of **y4** until you find graphs whose characteristics match those in figures 5 through 7. Then, if there are any, find the extrema and inflection points. Be sure to use your formula from Exercise 9 to find the inflection points. By all means use your '89 to confirm your answers.

Exercise 12: If you have access to a powerful Computer Algebra System (such as Derive™), find a general solution to the general cubic equation, **y3=0** (with **y3** as defined earlier). [It's a big mess, all right, but are you surprised that there <u>is</u> a formula? It is called ***Cardan's Formula***.] If your CAS can solve the general cubic equation, it can find a formula for the critical points of the general quartic. Do you think this formula is useful and/or usable?

Calculus Generic Scope and Sequence Topics: Derivatives
NCTM Standards: Number and operations, Algebra, Geometry, Measurement, Problem solving, Connections, Communication, Representation