Cyclic Quadrilaterals – ID: 9691

By Judy Hicks

Activity Overview

In this activity, students will explore cyclic quadrilaterals and their properties.

Concepts

- *Circles*
- *Quadrilaterals*
- *Cyclic quadrilaterals*

Teacher Preparation

This activity is designed to be used in a high-school geometry classroom.

- *Students should already be familiar with the parts of a circle (arcs, chords, diameters, etc.) and special quadrilaterals (squares, rectangles, isosceles trapezoids, etc.).*

- *This activity assumes a basic working knowledge of the TI-Nspire device, such as drawing shapes and finding lengths of segments. The student TI-Nspire document contains only a few pages to get students started. Most of the constructions will be performed by the students themselves.*

- *Depending on the level of your students and time restraints, you may wish to only complete Problem 1.*

- *The screenshots on pages 2–7 demonstrate expected student results. Refer to the screenshots on page 8 for a preview of the student TI-.Nspire document.*

- ***To download the student and solution .tns files and student worksheet, go to education.ti.com/exchange and enter "9691" in the quick search box.***

Classroom Management

- *This activity is intended to be mainly **teacher-led**, with breaks for individual student work. Use the following pages to present the material to the class and encourage discussion. Students will follow along using their handhelds.*

- *The student worksheet GeoAct33_CyclicQuads_worksheet_EN helps to guide students through the activity and provides a place to record their answers.*

- *The TI-Nspire solution document GeoAct33_CyclicQuads_Soln_EN.tns shows completed diagrams to illustrate each step of the activity. You may wish to share these diagrams with students upon completion.*

TI-Nspire™ Applications

Graphs & Geometry, Notes

This activity allows students to interactively discover, confirm, and explore the properties of cyclic quadrilaterals. A cyclic quadrilateral is a quadrilateral that has all four of its vertices on a single circle.

Problem 1 – Exploring cyclic quadrilaterals

On page 1.3, students will construct cyclic quadrilateral *ABCD*. They should select **Polygon** from the Shapes menu, and click to place each vertex on the circle. To make sure that the vertices are placed *on* the circle, check that the label while hovering over the circle reads "point on" instead of "point." To complete the polygon, either click a second time on the last vertex, or return to and click on the first vertex.

A vertex may be labeled by typing the label immediately after placing the point, or afterwards by using the **Text** tool from the Actions menu. The shading of the circle or quadrilateral or appearance of the vertex points may be altered using the **Attributes** tool, also from the Actions menu.
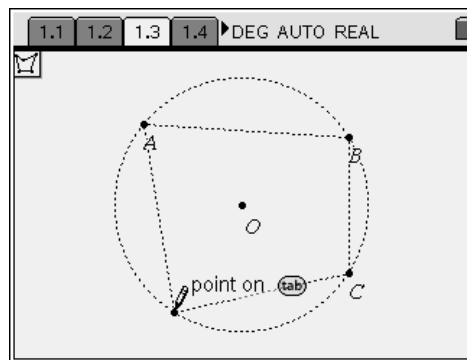
Note: Have students recall the necessary conditions for a figure to be a polygon. One such requirement is that the sides of a polygon *only* intersect at their endpoints. Have students drag a vertex of figure *ABCD* such that two of its sides intersect, as shown to the right. It is <u>*imperative*</u> that students recognize that this figure *ABCD* is no longer a polygon (or quadrilateral) by definition, and therefore dragging vertices in such a manner is a violation of the basic premise of this activity.
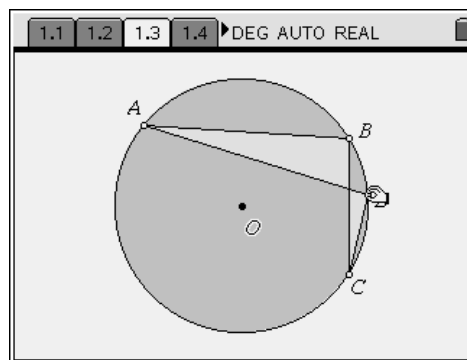
Using the **Angle** tool from the Measurement menu, students should measure each angle of quadrilateral *ABCD*. To measure an angle (such as ∠*DAB*), click on the three points of the angle such that the vertex is the second point chosen (click, in order, points *D*, *A*, and *B*). Then, press ⊙ once more to anchor the measurement to the screen.
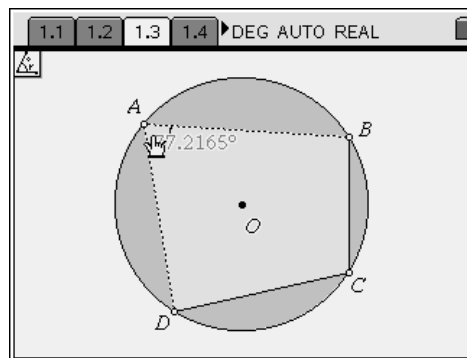
After all four angles have been measured, hover the cursor over each measurement and press ⊖ to decrease the number of displayed decimal places. (Pressing ⊕ will increase the number of displayed decimal places.) One or two decimal places is sufficient for this activity.

Students should now drag vertices (making sure *ABCD* remains a quadrilateral, as discussed earlier) and come to the conclusion that the opposite angles are supplementary. This notion can be strengthened using the **Text** and **Calculate** tools from the Actions menu to display and evaluate **A+C** and **B+D**. To use the **Calculate** tool, click first on the expression to be evaluated and then on the values to use for the variables in the order in which you are prompted.

This dynamic construction may be found on page 1.2 of the TI-Nspire solution document.
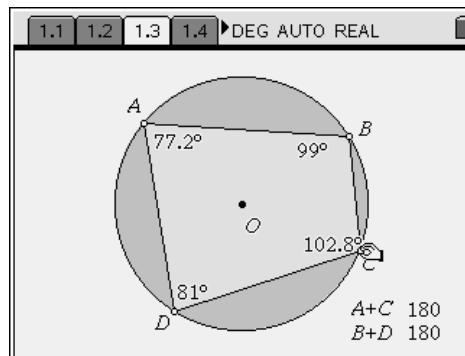
To prove that this is the case, students should be familiar with the Inscribed Angle Theorem, which states that the measure of an angle inscribed in a circle is half the measure of its intercepted arc (i.e., for our diagram above, $m\angle BAD = \frac{1}{2}m\overset{\frown}{BCD}$).

The proof that opposite angles are supplementary follows directly from this theorem and the fact that the measure of a whole circle is 360°.
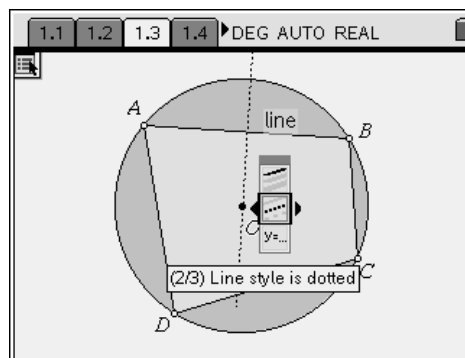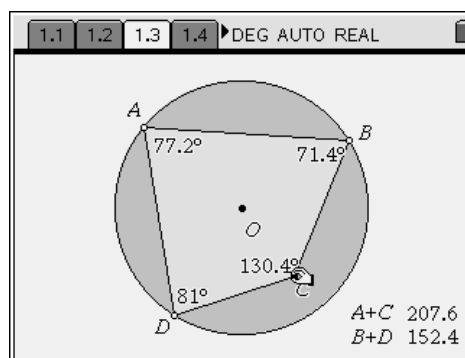
To have students observe that this does not hold true for non-cyclic quadrilaterals, they can use the **Redefine** tool from the Actions menu to redefine one of the vertices *off* of the circle and drag it around. It becomes obvious that the opposite angles are no longer supplementary.

Before continuing, students should return to their original quadrilateral. To do so, they can delete unnecessary measurements, hide the measurements using the **Hide/Show** tool (Actions menu), or undo their work (⌃ctrl + ⓩ).

The next exploration, posed on student page 1.5, asks students to consider the perpendicular bisectors of each of the sides of the quadrilateral. Have them select the **Perpendicular Bisector** tool from the Construction menu and click on a side to display each perpendicular bisector. You might also wish to have them use the **Attributes** tool to change the appearance of the lines.
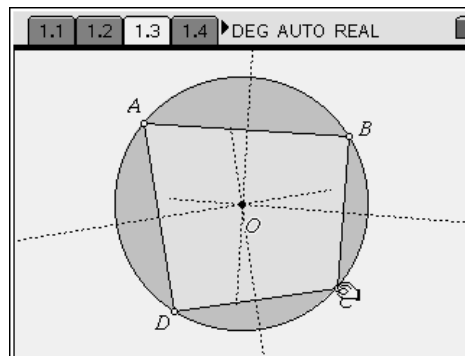
It should be obvious that these lines are concurrent, intersecting at the center of the circle, even as the vertices are dragged around the work area. Students should be able to explain that this behavior occurs as a result of the converse of the Perpendicular Bisector theorem, as well as the uniqueness of a perpendicular bisector. (Consequently, the perpendicular bisector of a chord of a circle always contains the center of the circle.)
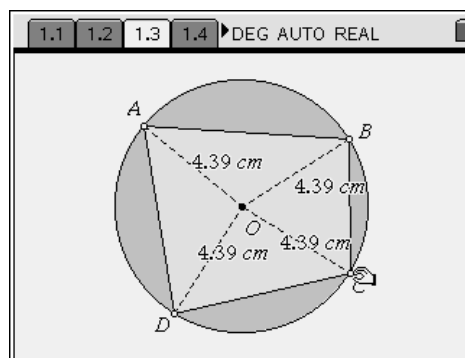
Again, redefining a vertex off of the circle, students will find that the perpendicular bisectors are no longer concurrent.

This dynamic construction may be found on page 1.3 of the TI-Nspire solution document.

The question from student page 1.6 asks students to consider the distance from each vertex of quadrilateral *ABCD* to the center of the circle, *O*. Drawing (**MENU > Points & Lines > Segment**) and measuring (**MENU > Measurement > Length**) the segments, it should come as no surprise that these lengths are equal, since they are each a radius of the circle.

This dynamic construction may be found on page 1.4 of the TI-Nspire solution document.

Student page 1.7 reveals another property for students to explore and explain: When two opposite vertices are collinear with the center of the circle, the angles formed at the other two vertices are right. This follows from the Inscribed Angle Theorem discussed earlier and the measure of a semi-circle being 180°.
(If points *A*, *O*, and *C* are collinear, $\overline{AC}$ is a diameter of circle *O*.)
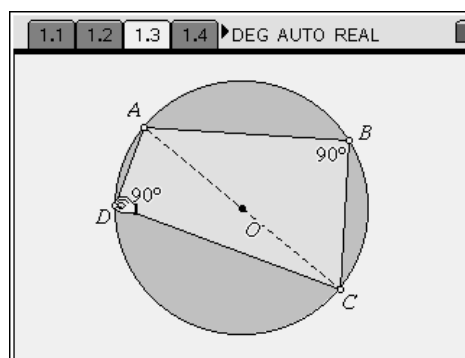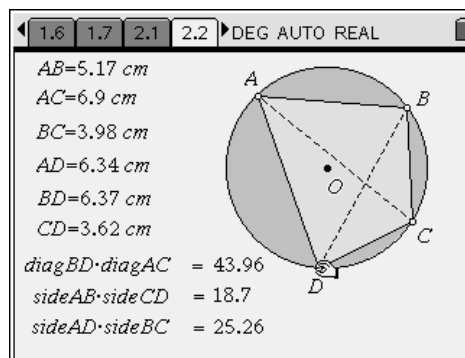

*This dynamic construction may be found on page 1.5 of the TI-Nspire solution document.*

Problem 2 – Ptolemy's theorem

On page 2.2, students find a diagram showing a cyclic quadrilateral *ABCD* and its diagonals. The lengths of the sides and diagonals are displayed, as are certain products of those lengths. Dragging vertices around the circle, students are prompted to discover that $AB \cdot CD + BC \cdot AD = AC \cdot BD$. This is known as Ptolemy's Theorem, which states that, for a quadrilateral inscribed in a circle, the sum of the products of the two pairs of opposite sides is equal to the product of the diagonals.

This dynamic construction may be found on page 2.1 of the TI-Nspire solution document.

Students should be encouraged to redefine vertex *D* off of the circle once again to verify that the relationship expressed in Ptolemy's theorem only holds true for cyclic quadrilaterals. Specifically, the relationship becomes $AB \cdot CD + BC \cdot AD > AC \cdot BD$, and the inequality remains the same regardless of whether point *D* is in inside or outside the circle.
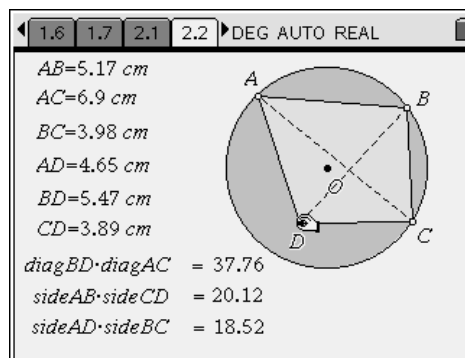
Problem 2 also asks students to consider whether the different types of special quadrilaterals can be cyclic. They can use the diagram on student page 2.2 and take appropriate measurements, or they can create their own constructions.

One conclusion at which students should arrive is that isosceles trapezoids can be cyclic. For two parallel chords of a circle, the chords subtend equal arcs between their endpoints. These equal arcs have congruent chords that form the non-parallel sides of the trapezoid.
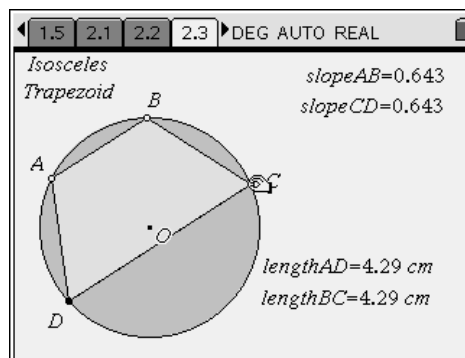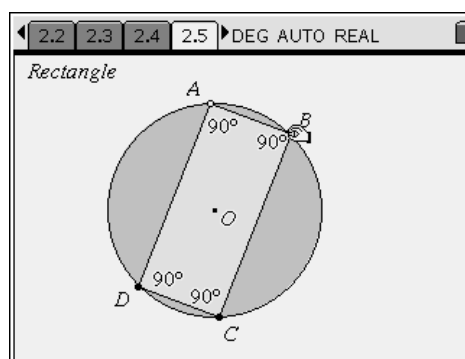
This dynamic construction may be found on page 2.3 of the TI-Nspire solution document.

Convex kites can also be cyclic, the requirement being that opposite angles are supplementary. Since a kite always has one pair of opposite angles that are congruent, one pair of the opposite angles must both be right angles (which are congruent and supplementary). This necessitates that the line of symmetry of the kite lay on the diameter of the circle, and thus, the non-vertex angles each measure 90°.
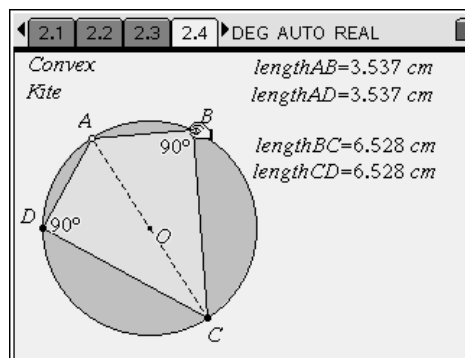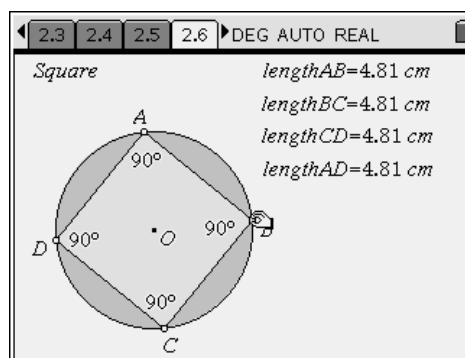
This dynamic construction may be found on page 2.4 of the TI-Nspire solution document.

Next, students should make some conclusion regarding parallelograms. Have students once again recall that it was already shown that for a quadrilateral to be cyclic, its opposite angles must be supplementary. For parallelograms, this only occurs when all angles measure 90°, i.e., the parallelogram is a rectangle. Parallelograms that are *not* rectangles cannot be cyclic.

This dynamic construction may be found on page 2.5 of the TI-Nspire solution document.

Similarly, squares are cyclic, and rhombi that are *not* squares are not cyclic.
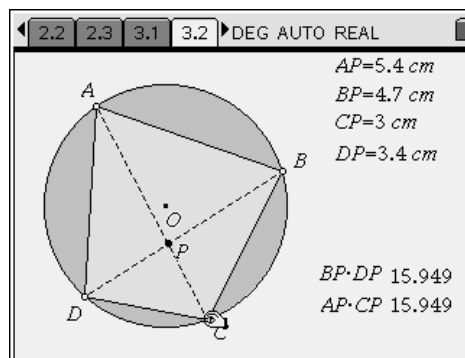
This dynamic construction may be found on page 2.6 of the TI-Nspire solution document.

Problem 3 – Chord-Chord Power theorem

In Problem 3, a cyclic quadrilateral is displayed, along with its diagonals and their point of intersection, *P*. The lengths of the "parts" of the diagonals (from vertex to intersection point) are also displayed. Students should use the **Calculate** tool to evaluate the two expressions near the bottom of the screen, to determine that $AP \cdot CP = BP \cdot DP$.

This property, sometimes referred to as the Chord-Chord Power theorem, is true for any two chords in a circle (including the diagonals of a cyclic quadrilateral) and does not true for quadrilaterals that are *not* cyclic. As before, students should use the **Redefine** tool to free a vertex from the circle to show that the two products are no longer equal.

This dynamic construction may be found on page 3.1 of the TI-Nspire solution document.

Problem 4 – Similar triangles

On page 4.2, students are again shown a cyclic quadrilateral, its diagonals, and their point of intersection of the diagonals, *P*. Direct students' attention to $\triangle ABP$ and $\triangle CDP$.

Before using the **Angle** tool from the Measurement menu to measure any of the angles, students should observe that $\angle ABD \cong \angle ACD$ since both of these inscribed angles intercept the same arc, $\overset{\frown}{AD}$. Similarly, $\angle BAC \cong \angle BDC$, so therefore $\triangle ABP \cong \triangle CDP$.

The same can be said about $\triangle DAP$ and $\triangle BCP$—quite an interesting property!

Of course, students should be encouraged to redefine a vertex off of the circle to find that this is not the case with non-cyclic quadrilaterals.
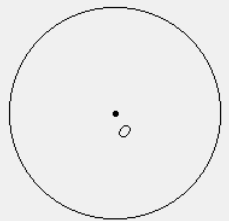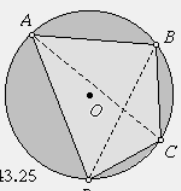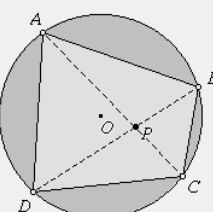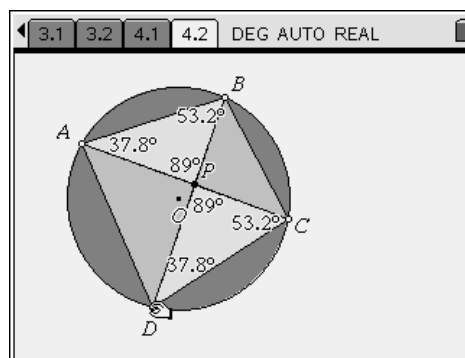

This dynamic construction may be found on page 4.1 of the TI-Nspire solution document.
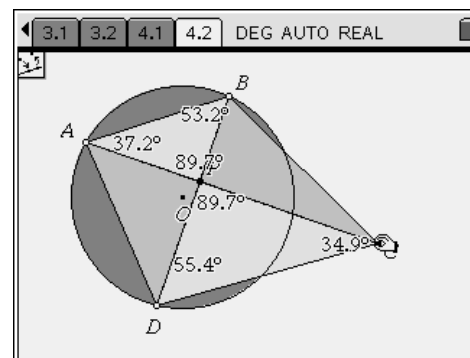
Cyclic Quadrilaterals – ID: 9691

(Student)TI-Nspire File: *GeoAct33_CyclicQuads_EN.tns*

1.1 1.2 1.3 1.4 ▶DEG AUTO REAL

CYCLIC QUADRILATERALS

Geometry

Exploring relationships

1.1 1.2 1.3 1.4 ▶DEG AUTO REAL

On the next page, use the **Polygon** tool from the Shapes menu to construct a quadrilateral *ABCD* that has all four vertices on the given circle. Such a quadrilateral is called a *cyclic quadrilateral*.

Cyclic quadrilaterals have some interesting properties!

1.1 1.2 1.3 1.4 ▶DEG AUTO REAL

1.1 1.2 1.3 1.4 ▶DEG AUTO REAL

Measure the interior angles of the quadrilateral.

▪ What appears to be true about the opposite interior angles of a cyclic quadrilateral? Explain why this occurs.

1.2 1.3 1.4 1.5 ▶DEG AUTO REAL

Next, construct the perpendicular bisectors of each side of the quadrilateral.

▪ What appears to be true about the perpendicular bisectors of the sides of a cyclic quadrilateral? Explain your reasoning.

1.3 1.4 1.5 1.6 ▶DEG AUTO REAL

Now, measure the distance from the center of the circle to each vertex of the quadrilateral.

▪ What conjecture can you make about these distances? Explain why this is the case.

1.4 1.5 1.6 1.7 ▶DEG AUTO REAL

Drag two opposite vertices such that they are collinear with the center of the circle.

▪ What appears to be true about the measures of the angles formed at the other two vertices? Explain why this happens.

1.5 1.6 1.7 2.1 ▶DEG AUTO REAL

On the next page, a cyclic quadrilateral is shown and some measurements and calculations are displayed.

Drag vertices of the quadrilateral and conjecture relationships among these six measurements.

1.6 1.7 2.1 2.2 ▶DEG AUTO REAL

$AB = 5.17\ cm$
$AC = 6.9\ cm$
$BC = 3.98\ cm$
$AD = 6.44\ cm$
$BD = 6.26\ cm$
$CD = 3.41\ cm$
$diag\,BD \cdot diag\,AC = 43.25$
$side\,AB \cdot side\,CD = 17.6$
$side\,AD \cdot side\,BC = 25.64$

1.7 2.1 2.2 2.3 ▶DEG AUTO REAL

▪ What relationship exists between the measurements?

▪ What types of special quadrilaterals can also be cyclic quadrilaterals? Use the diagram on the previous page to help you.

2.1 2.2 2.3 3.1 ▶DEG AUTO REAL

On the next page, a cyclic quadrilateral is shown.

The diagonals are also shown, as are the measurements from the intersection of the diagonals (point *P*) to each vertex.

▪ How are these measurements related?

2.2 2.3 3.1 3.2 ▶DEG AUTO REAL

$AP = 5.5\ cm$
$BP = 3.1\ cm$
$CP = 2.8\ cm$
$DP = 5\ cm$

$BP \cdot DP = ?$
$AP \cdot CP = ?$

2.3 3.1 3.2 4.1 ▶DEG AUTO REAL

On the next page, a cyclic quadrilateral is shown, along with its diagonals and their point of intersection, *P*.

Consider triangles *ABP* and *CDP*. Measure their angles.

▪ What can you say about these two triangles? Explain.

▪ What about triangles *DAP* and *BCP*?

3.1 3.2 4.1 4.2 DEG AUTO REAL