Number Sums

Student Activity

7 8 9 **10** **11** 12

TI-Nspire Investigation Student 90 min

Introduction

Count the number of cans in the stack shown opposite.

Think about the way you counted the cans. Did you count them from the top down, one at a time? Did you look at the number of cans in each row and add these numbers together? Now imagine a stack 30 rows high; would you count them the same way? Our brains are wired to look for patterns. This investigation uses combinations of numeric, visual and algebraic strategies to explore this and related problems by observing and building patterns.

Numeric – Visual Approach

The can stack can be tackled by working with sums of:

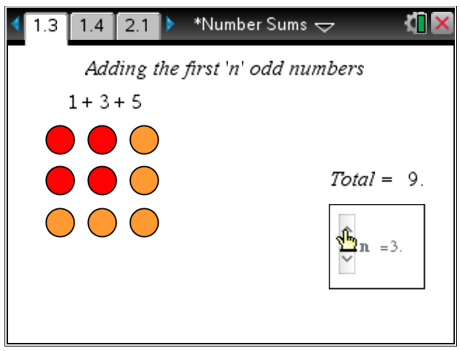
- Odd numbers
- Even numbers
- Whole numbers

Odd Numbers

Open the TI-Nspire document “Number Sums”.

Read the instructions and then navigate to page: 1.3.

Adjust the slider, observe the diagram, expression and total (sum) displayed on the screen.



Question: 1.

Explain how the visual representation relates to the sum of the first n odd numbers.

*Increasing the length of **two** sides of the square plus the **one** corner piece increases the pattern by an odd number each time.*

Question: 2.

Write an equation for the sum ***s*** of the first ***n*** odd numbers. Start your equation “s =”.

Explain how the visual representation helps produce the equation.

[Include your rule on page 1.4 of the TI-nspire document]

The square makes it easier to see the rule as the ‘area’ of the square. $s = n^2$

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Question: 3.

Use your formula to determine and check the sum of the first 20 odd numbers.

Sum of first 20 numbers: $s = n^2$

$$s = 20^2$$

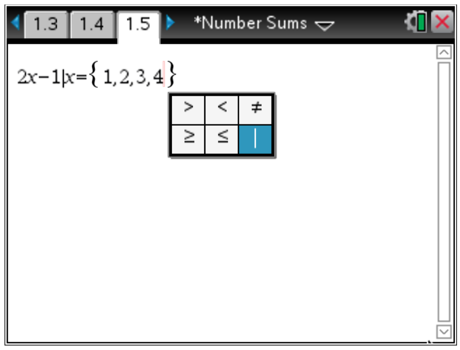
$$s = 400$$

A sequence of numbers can be generated by the rule:

$$2x - 1 \text{ where } x \in \{1,2,3\ldots\}$$

To see how this 'rule' works, enter the rule shown on the calculator page (1.5) and substitute the set of numbers:

{1, 2, 3, 4}



The '|' symbol can be found by pressing [**Ctrl**] + [**=**]

Question: 4.

What numbers are produced for the calculation: $2x - 1 \mid x = \{1,2,3,4\}$?

Show a sample calculation to illustrate and explain why the rule produces these types of numbers.

The expression 2x – 1 produces odd numbers. $2x - 1 = 2(1) - 1 = 1$

$$2x - 1 = 2(2) - 1 = 3$$

$$2x - 1 = 2(3) - 1 = 5$$

The rule first multiplies by 2 producing an even number. When 1 is subtracted an odd number results.

Question: 5.

The word 'sum' is a built in command. Type: **sum(** then copy and paste the previous into the sum command and press [**enter**]. Use this approach to find the sum of the first 10 odd numbers and use it to check your rule from question 2.

$sum(\{1,2,3,4,5,6,7,8,9,10\}) = n^2$

$$100 = 10^2$$

Comment: If the TI-nspire document is collected from students, navigate to page 1.5 to see student calculations.

The use of the 'sum' command is good for small sets of numbers or when the numbers already exist in a list. For larger sets it is more convenient to use a rule to generate the numbers. The Greek letter Σ is used in mathematics to represent the 'sum of'. The calculator has this functionality built in so it can be used to find the 'sum of' a set of numbers generated by a rule:

$$\sum_{start}^{finish} rule$$

The sum template can be found in the template fly-out or by selecting 'sum' from the calculus menu. The rule being used in this example has the variable x. So the sum of the first 20 odd numbers would include:

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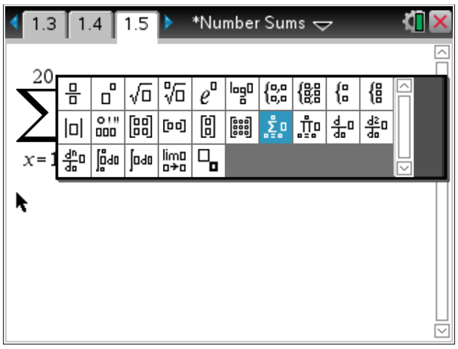
Start: $x = 1$ -- *initial value used by the rule*

Finish: 20 -- *final value used by the rule*

Rule: $2x - 1$ -- *rule used to generate the numbers*



Question: 6.

Use the calculator to determine the answer to: $\sum_{x=1}^{20} 2x - 1$.

How does the answer compare to that obtained in question 3.

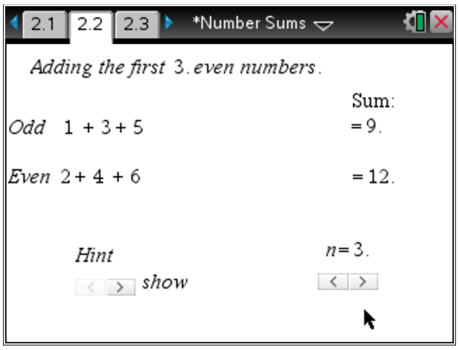
The answer produced is the same for question 3 as both methods are finding the sum of the first 20 odd numbers.

Comment: Page 1.5 of the student's TI-Nspire document will contain evidence of whether or not students used the summation command.

Even Numbers

Navigate to page 2.2, adjust the value of *n* and observe the sequence of even and odd numbers with the corresponding sums.

Use the 'hint' option to reveal more clues for developing a rule for the sum of the first *n* even numbers.



Question: 7.

Describe (in words) the relationship between the sum of the first *n* even numbers and the sum of the first *n* odd numbers.

The each even number is 1 more than the respective odd number. The sum of the first n even numbers is therefore 'n' more than the sum of the first n odd numbers.

Question: 8.

Use your rule for the sum of the first *n* odd numbers to help write a rule for the sum of the first *n* even numbers. [Include your rule on page 2.3 of the TI-nspire document.]

Sum of first n odd numbers: $s = n^2$ *Sum of first n even numbers:* $s = n^2 + n$

Question: 9.

What numbers are produced for the calculation: $2x \mid x = \{1, 2, 3, 4\}$? Show a sample calculation to illustrate. [Use page 2.4 of the TI-nspire document to check your answers.]

Even numbers are produced by the expression 2x. $2x = 2(1) = 2$

$2x = 2(2) = 4$

$2x = 2(3) = 6$

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Question: 10.

Use your calculator to determine $\sum_{x=1}^{20} 2x$ and explain each component of this expression.

$\sum_{x=1}^{20} 2x = 420$

x=1 is the starting value; substitution into the formula produces a 2 for the first value in the sum. Successive terms are added until x=20, which produces a 40 for the last value in the sum.

Question: 11.

Use your formula to calculate the sum of the first 20 even numbers and compare it to: $\sum_{x=1}^{20} 2x$.

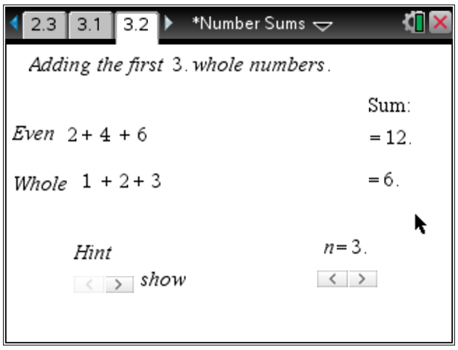
$n^2 + n = \sum_{n=1}^{20} 2n$

$20^2 + 20 = 420$

Whole Numbers

Navigate to page 3.2, adjust the value of *n* and observe the sequence of even and whole numbers with the corresponding sums.

Use the 'hint' option to reveal more clues for developing a rule for the sum of the first *n* whole numbers.



Question: 12.

Describe (in words) the relationship between the sum of the first *n* even numbers and the sum of the first *n* whole numbers.

The sum of the first n whole numbers is 'half' the sum of the first n even numbers.

Question: 13.

Write a rule for the sum of the first *n* whole numbers. [Include your rule on page 3.3]

$$s = \frac{n^2 + n}{2}$$

Question: 14.

Explain how the sum of the first *n* whole numbers relates to the can stacking problem and hence determine the number of cans in a stack 30 rows high.

First row has 1 can, second row has 2 cans, third has 3 etc… and the nth row will have n cans; so a stack of cans n cans high will equal 1 + 2 + 3… n.

$$s = \frac{n^2 + n}{2}$$

$$s = \frac{30^2 + 30}{2}$$

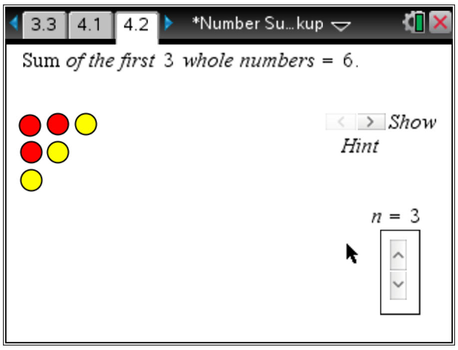
$$s = 465$$

Author: P. Fox

Visual Approach

Looking at things from a different angle or imagining extra pieces can sometimes help to see patterns that you may not have previously noticed.

Navigate to page 4.2, adjust the value of *n* and observe the way the pattern develops.



Use the 'hint' option to reveal items that add to the visualisation of this pattern.

Question: 15.

How does this pattern relate to the original can stacking problem?

The cans are represented by circles and the stack has been rotated by 45^o.

Use the 'Hint' option to reveal additional visual clues for this problem.

Question: 16.

Describe the overall shape produced as *n* is changed. (Hint: on)

Overall shape is a rectangle.

Question: 17.

When *n* = 4, calculate the 'area' of the shape produced. (Hint: on)

The shape is a rectangle with side lengths 4 and 5. Area = 4 x 4 = 20

Question: 18.

Adjust the value of *n* and describe the length and width of the shape produced. (Hint: on)

The length of the square is n and the width is n + 1 .

Question: 19.

Remove the 'hint' and write a rule for the pattern. [Include your rule on page 4.3]

$s=\frac{1}{2}\cdot n\cdot(n+1)$ *...based on the area of a triangle; half the area of the rectangle.*

Numerical Approach

The focus here is on numerical strategies only. Consider calculating the sum of the first 10 whole numbers. This can be done quickly and easily in your head; 1 + 2 + 3....9 + 10, however this technique will be very time consuming when trying to calculate the sum of the first 100 whole numbers.

Consider a more alternative, more efficient strategy which incorporates multiplication via grouping:

(1 + 10) + (2 + 9) + (3 + 8) ....

This strategy is displayed on the page 5.2 of the TI-Nspire document. The numbers are purposefully grouped so that multiplication can be used in conjunction with addition.

Author: P. Fox

Question: 20.

Use the grouping strategy to calculate the sum of the first 10 whole numbers. (Show working)

(1 + 10) + (2 + 9) + (3 + 8) + (4 + 7) + (5 + 6) = 5 x 11 = 55

Question: 21.

Use the grouping strategy to calculate the sum of the first 100 whole numbers. (Show summary)

(1 + 100) + (2 + 99) + (3 + 98) + (4 + 97) … (50 + 51) = 50 x 101 = 5050

Question: 22.

Generalise the grouping strategy to write a rule for the sum of the first *n* whole numbers.

$s = \frac{n}{2} \cdot (n+1)$ *Number of groups is n/2, sum of each group is (n + 1).*

Question: 23.

Compare the formulas created in each key question: 13, 19 and 22.

$$\frac{n^2+n}{2} = \frac{1}{2} \cdot n \cdot (n+1) = \frac{n}{2} \cdot (n+1)$$

Each of these formulas is the same. The formula first uses the sum of the first n even numbers $n^2 + n$ and then halves the result. The second formula works out the area of the rectangle n (n + 1) then halves the result for the area of the triangle. The third formula shows there are n/2 groups of (n + 1).

Author: P. Fox