Dog Pen Problem (Maximum Area of a Rectangle Space)

Activity Overview
In this activity, students explore various approaches to "solving" the problem of maximizing the area of a rectangle space with a fixed perimeter in the context of a farmer's "dog pen".

Concepts
- Fixed Perimeter of a Rectangle
- Maximum Area of a Rectangle
- Quadratic Regression
- Local Maximum

Teacher preparation
This activity allows students at different proficiency levels to explore the problem of maximizing the area of a rectangle with a fixed perimeter. Students should have familiarity finding the perimeter and area of a rectangle.

Classroom management tips
- This activity is designed to be student-centered with the teacher acting as a facilitator while students work cooperatively. The student worksheet is intended to guide students through the main ideas of the activity and provide a place to record their observations and reflections.
- For the most part, students will manipulate pre-made sketches, rather than constructing the diagrams themselves. Therefore, a basic working knowledge of the TI-Nspire CAS handheld is needed.
- You may choose to use Problem 3—Symbolic Proof as an extension activity or the subject for a whole-class discussion.
- The ideas contained in the following pages are intended to provide a framework as to how the activity will progress. Suggestions are also provided to help ensure that the objectives for this activity are met.

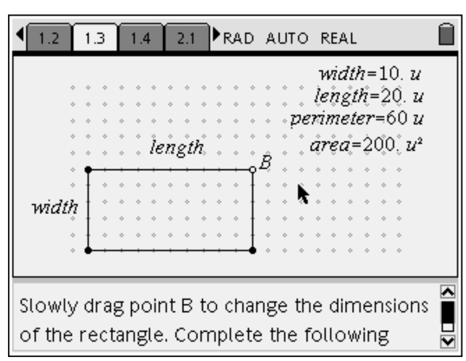
TI-Nspire CAS Applications
Calculator, Graphs & Geometry, Lists & Spreadsheet, Notes

One problem defines this activity: *A farmer wants to make the largest possible rectangular pen for his dogs. He has 60 feet of fencing. What is the largest area the pen can have? What should the length and width of the pen be?*

To launch this activity, present this problem to the class and discuss with the students what they think the dimensions of the pen should be (you may want to give students color tiles, graph paper, and/or geo-boards to accommodate their level of understanding of this problem). Make sure that they understand that the perimeter is the sum of the lengths of the sides of the rectangle and that area is the number of square units needed to cover the surface of the entire rectangular region.

Problem 1—Launch the Problem

Step 1: Remind students to slowly drag point B to change the dimensions of the rectangle. At each new location the students should record the values of width, length, perimeter, and area in the table provide on the Student Handout. They should do this for at least 6 different locations of B. *Note: If students get a "dependant object locked" message as they drag point B the handheld can is telling them that it can not keep up with the speed of their entries. They are probably pressing and holding a direction on the NavPad to make the point B move. To troubleshoot this problem, tell them to not hold down the direction on the NavPad.*

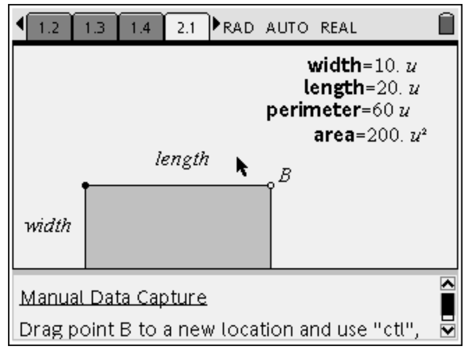


Students are asked to answer the questions on the Student Handout. The teacher should then facilitate a discussion of their findings. This can be done as a large group or small group discussion.

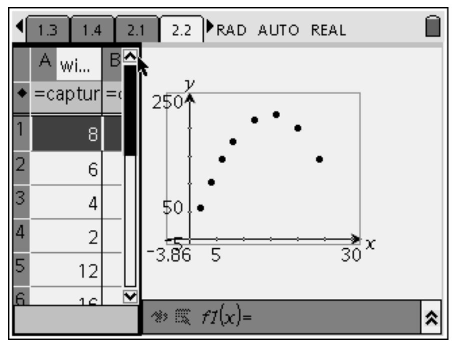
During this discussion, students should notice that the perimeter of the rectangle does not change and that as the width changes the area changes. (Note: some students may notice that the area does not change at a constant rate). Also, students should make conjectures about the dimensions of the rectangle when the area is maximized.

Problem 2—Find the Rectangle with Maximum Area

Step 1: Remind students to <u>slowly</u> drag point B to a new location and to press ctrl . to manually capture the rectangle's width, length, perimeter, and area data for that location of point B. Repeat this process to capture at least 10 different locations for point B.



Each new location of point B captured will generate another row of data in the spreadsheet and plot the (length, area) data in a scatterplot (page 2.2).

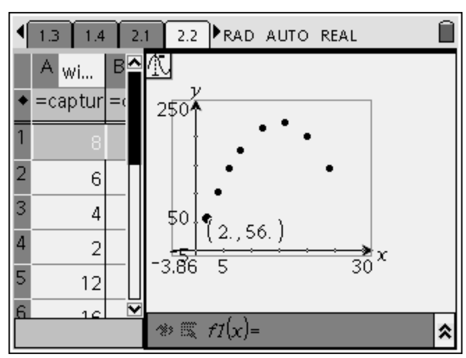


Students are asked to answer the questions on the Student Handout. The teacher should then facilitate a discussion of their findings. This can be done as a large group or small group discussion.

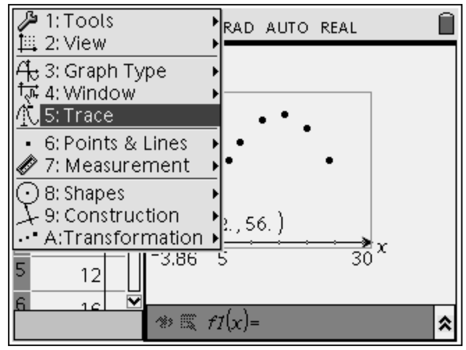
Step 2: During this discussion, students should notice that the scatterplot appears to be quadratic. They should also have used different methods for finding the dimensions of the rectangle with maximum area.

ONE possible method would be to use the trace feature of the scatterplot window on page 2.2.

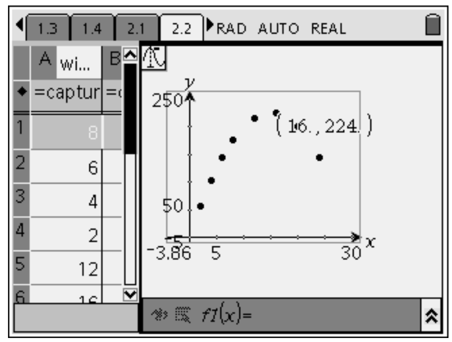
To do this, select the Graphs & Geometry window on page 2.2 *(To move between windows on this page press ctrl tab)*.



Press (menu).
Choose 5: Trace

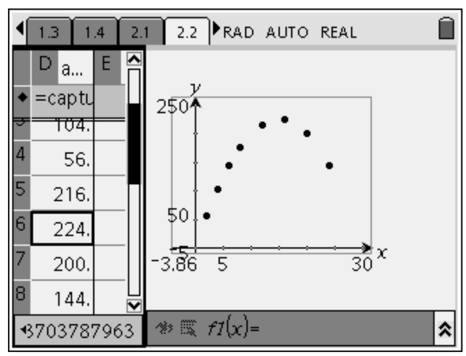


Use the *NavPad* to move between points on the scatterplot until the point with the largest area is highlighted. This will give a good approximation for the width of the rectangle with the maximum area.



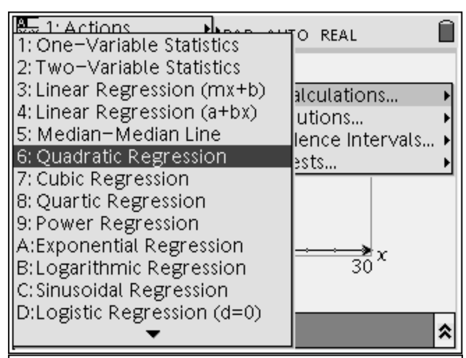
Students may then look in table of collected data for the corresponding length for this rectangle. Or they may use the A=width*length relationship to calculate the length given the area and width.

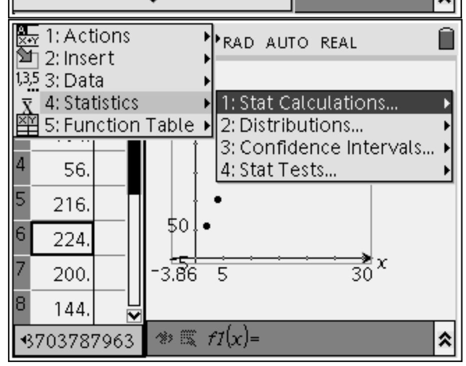
A **SECOND** method might be to simply scroll through the Lists & Spreadsheet window on page 2.2 to find the dimensions that correspond to the largest area collected. This will give them a good approximation for the dimensions of the rectangle and its area.



A **THIRD** method might involve finding the Quadratic Regression for this set of data. This is done in the Lists & Spreadsheet window on page 2.2

Press (menu).
Choose 4: Statistics.
Choose 1: Stat Calculations
Choose 6: Quadratic Regression

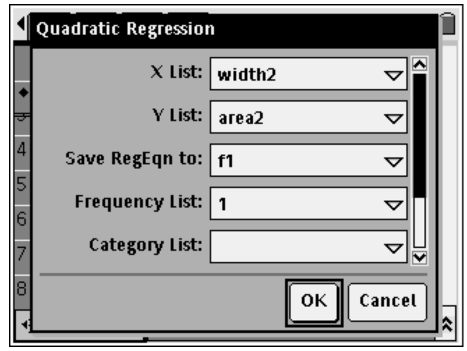




Set the Quadratic Regression window as follows…

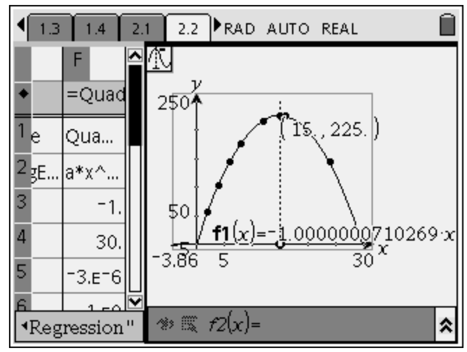
Xlist: width2
Ylist: area2
Save RegEqn to: f1

Press OK.



The quadratic regression equation is now calculated and placed in f1 of the Graphs & Geometry window on page 2.2

Graph the regression equation in this window and trace to find the maximum area.



A possible **FOURTH** method for finding the maximum area would be to use the properties of 1st derivative. This is done by finding the first derivative of the regression equation, setting this new equation equal to zero and then solving for the width. This will be the width value of the local maximum of this parabola (i.e. the width that corresponds to the maximum area).

Note: These are not all the possible methods that your students may come up with. Many of these methods will have some of the elements discussed here. It is important to validate all correct methods and lead the students to use the appropriate method depending on the needs defined by the problem

Step 3: At this point students should be making the following conjecture…

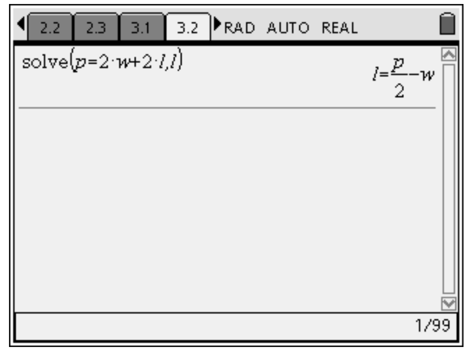
If a rectangle has a fixed perimeter, then the shape that maximizes the rectangle's area is a square.

This conjecture will be proven in **Problem 3—Symbolic Proof**.

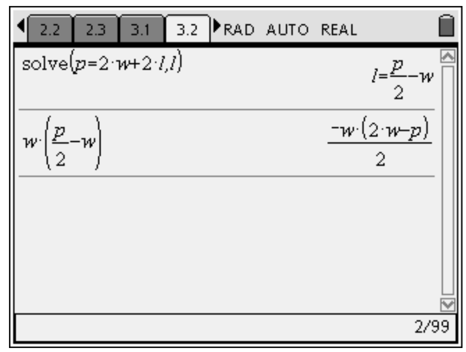
Problem 3—Symbolic Proof

Students are asked to follow the following procedure using the CAS tools of the TI-Nspire CAS.

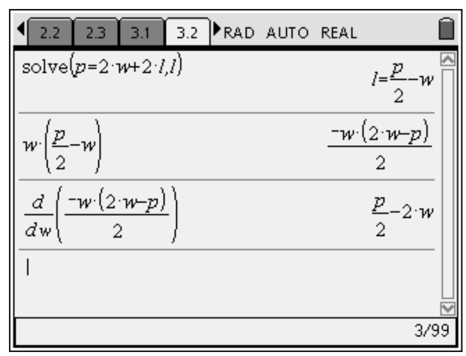
Step 1: a. Solve P=2w+2l for l.



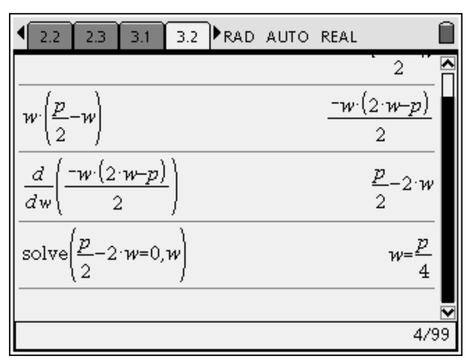
b. Substitute this value of l into A=wl to find an equation for A in terms of the variable (w).



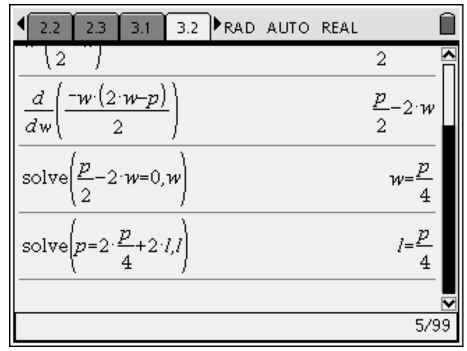
c. Find the 1st derivative of this new equation.



d. Set this new equation equal to zero and solve for w.



e. Substitute this value of w into P=2w+2l and solve for l.



Students are asked to answer the questions on the Student Handout. The teacher should then facilitate a discussion of their findings. This can be done as a large group or small group discussion.

During this discussion, students should be able to explain each step in the proof process. For example they should be able to explain why they took the 1st derivative, set it equal to zero, and then solved it for w. Students should know that this is a method to find local the local minimum or local maximum of a function.

Finally, students should be able to explain that this process proves this conjecture,

If a rectangle has a fixed perimeter, then the shape that maximizes the rectangle's area is a square.

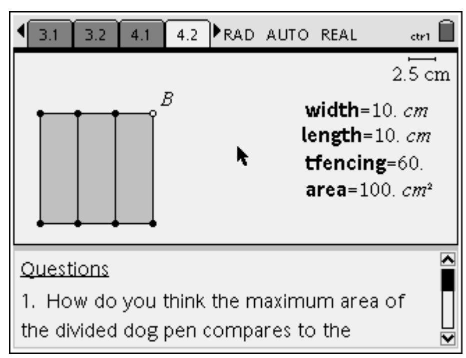
They have shown that both the $w = \frac{p}{4}$ and $l = \frac{p}{4}$ for this rectangle, it follows that all the sides of the rectangle are equal, therefore it must be a square.

Assessment and evaluation

Included in this Activity is **Problem 4—Divided Pen** that can be used as an assessment and evaluation of the students

In this problem you explore the following problem:

The farmer decides he wants to divide his dogs' pen up into three congruent rectangular pens with the fencing running parallel to the width (see page 4.2). He has 60 feet of fencing.



You will use the TI-Nspire CAS to automatically collect data in a spreadsheet, make a scatterplot of the data, and make observations based on these representations.

Activity extensions

Teachers may want to use **Problem 3—Symbolic Proof** as an extension.

Student TI-Nspire CAS Document