Polygons and Vectors

Construct and Investigate:

1. Construct $\triangle ABC$, and place point P anywhere on the screen. Using the **Vector** tool, construct the three vectors *from* point P *to* the three vertices of the triangle. Drag point P around the screen to determine whether it is properly connected to vertices A, B, and C.

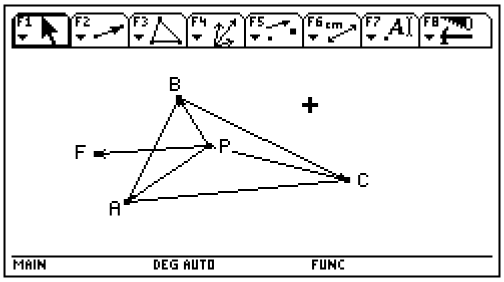

Figure 1

2. Use the **Vector Sum** tool to sum the three vector pairs through point P. Click on any two of the vectors that you constructed in part 1 above, and then click on point P (Figure 1). Repeat this step for the remaining vector pairs. Drag the three vertices of the triangle and point P around the screen to investigate relationships between and among the vectors and $\triangle ABC$. Be creative and follow up on any hunches that may come to you. Make conjectures about relationships that appear to be true and try to explain why they might be true, or show a counterexample.

3. Find the vector sum of the three initial vectors that originate at point P. One way to find this sum is to select one of the initial vectors and the vector that is the sum of the other two initial vectors. This vector sum should have point P as its initial point. Throughout this activity, this vector is referred to as the **resultant vector**. Drag the vertices of $\triangle ABC$ and point P around the screen to see whether you can determine any relationships that are always true between the triangle and the vector sum of the three original vectors.

4. Determine a location for point P so that the terminal end of the resultant vector is at a vertex of $\triangle ABC$. Under what conditions is the terminal end of this vector on a side of the triangle? What region bounds the location of point P such that the entire resultant vector is inside $\triangle ABC$? Where is point P located when the length of the resultant vector is zero? What other property does this location appear to have?

5. On a new screen, repeat constructions in parts 1 and 2 above, summing the vectors in pairs from point P. Hide $\triangle ABC$, and construct a hexagon using the **Polygon** tool to connect the endpoints of the six vectors from point P. Drag point P around the screen and investigate the properties of this hexagon. You may want to show $\triangle ABC$ again to see whether any relationships are true when point P is located at special points of the triangle.

Explore:

1. What relationships appear to be true if you repeat the preceding investigation using a quadrilateral instead of a triangle? Construct a quadrilateral, and place a point P anywhere on the screen. Construct vectors that originate at point P and end at the four vertices of the quadrilateral. Then, use **Vector Sum** on all pairs of these vectors. Are there any apparent patterns that carry over from your triangle investigations? List your conjectures, and explain why these are true, or give a counterexample.

2. Find the resultant vector representing the sum of the four original vectors. See whether any relationships from the triangle investigation carry over to the quadrilateral. Write conjectures that appear to be true regarding the resultant vector from a point P and the original quadrilateral. Are there any special results that have interesting relationships when point P is placed at certain locations relevant to the quadrilateral?

Construct and Investigate:

This construction is relatively simple. Students can benefit from a brief introduction to vectors prior to working on this exploration, if this concept is new to them. A common error is to attach the vectors to the triangle instead of to the vertices of $\triangle \boldsymbol{ABC}$ (Figure 2). By dragging the vertices and $\boldsymbol{P}$ around the screen, students can detect this potential error before attempting more of the constructions in this activity.

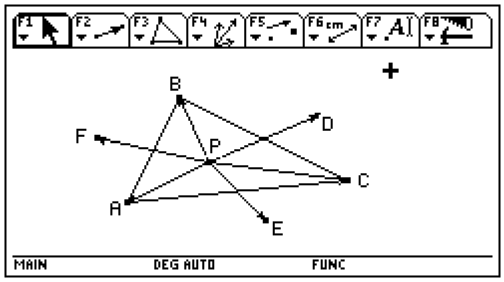

Figure 2

1. The sum of two vectors is always another vector. The three vectors that are the sum of pairs of the original vectors all pass through the midpoints of the sides of $\triangle \boldsymbol{ABC}$. This conjecture can be proved by constructing the quadrilateral using $\boldsymbol{P}$, the two vertices of $\triangle \boldsymbol{ABC}$, and the end point of the resultant vector as vertices.

   In Figure 3, quadrilateral ***BDCP*** can be shown to be a parallelogram by the properties of vectors. The diagonals of the parallelogram are the side of the triangle and the vector. Because the diagonals of a parallelogram bisect each other, the conjecture is proved.

   The endpoints of the three vectors ***D***, ***E***, and ***F*** in Figure 4 are the vertices of $\triangle \boldsymbol{DEF}$. Triangle ***DEF*** is congruent to $\triangle \boldsymbol{ABC}$. You can obtain $\triangle \boldsymbol{DEF}$ by rotating $\triangle \boldsymbol{ABC}$ 180° in either of two ways:

   - around the point of intersection of two segments connecting corresponding points of the two triangles
   - about the midpoint of the resultant vector (Figure 5 and parts 3 and 4).

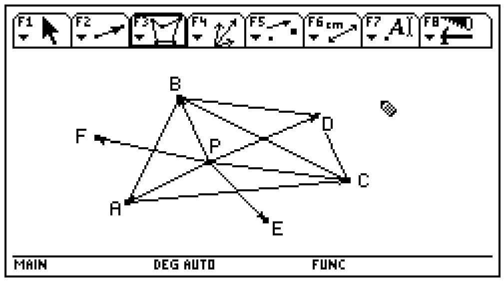

Figure 3

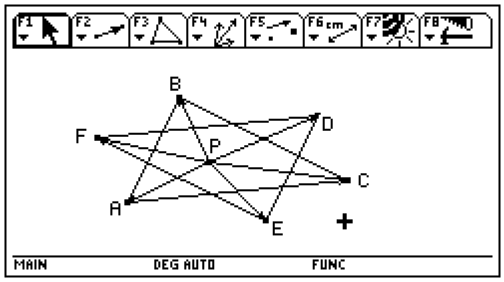

Figure 4

Students may discover that when ***P*** is at a vertex of △***ABC***, the two triangles share a common side and form a parallelogram. When ***P*** is on a side of △***ABC***, then each of the two triangles has a vertex on the side of the other.

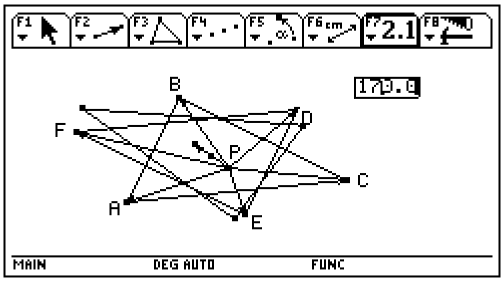

Figure 5

2. The resultant vector of △***ABC*** has several interesting properties. This vector always passes through the centroid of △***ABC*** (Figure 6). If ***P*** is at the centroid, then this vector is a zero vector (Figure 7). If ***P*** is not at the centroid of △***ABC***, then the centroid is always one-third of the distance from ***P*** to the terminal end of the vector (Figure 6).

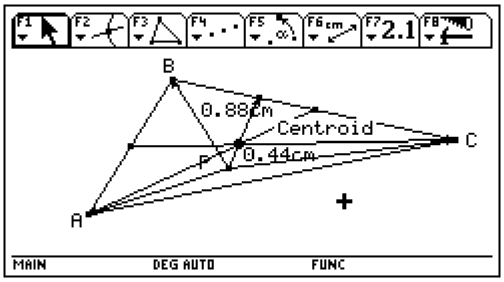

Figure 6

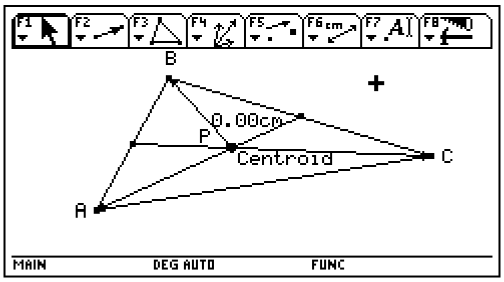

Figure 7

3. The following are some of the many relationships students might discover.

   - If $\boldsymbol{P}$ is at the vertex of the midsegment triangle of $\triangle \boldsymbol{ABC}$, then the terminal end of the resultant vector is at the opposite vertex of $\triangle \boldsymbol{ABC}$ (Figure 8).

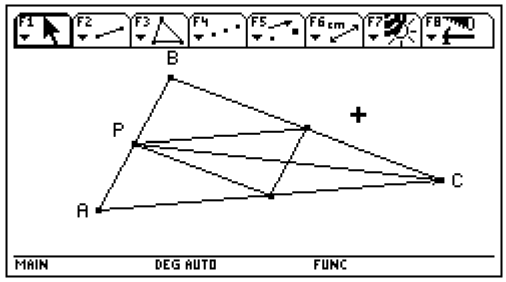

Figure 8

   - If $\boldsymbol{P}$ is on a side of the midsegment triangle of $\triangle \boldsymbol{ABC}$, then the terminal end of the resultant vector is on the side of $\triangle \boldsymbol{ABC}$ that is parallel to this segment (Figure 9).

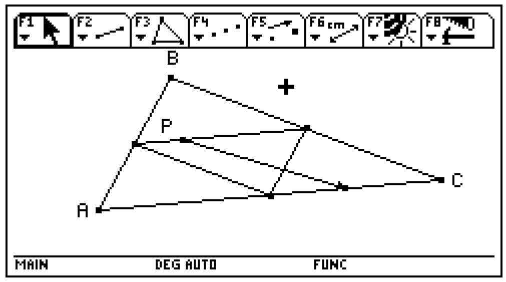

Figure 9

   - If $\boldsymbol{P}$ is in the interior of the midsegment triangle of $\triangle \boldsymbol{ABC}$, then the resultant vector is entirely inside $\triangle \boldsymbol{ABC}$ (Figure 10).

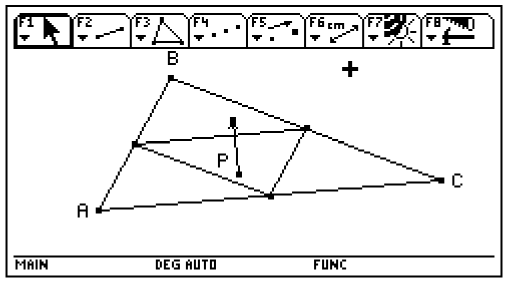

Figure 10

   - If $\boldsymbol{P}$ is exterior to the midsegment triangle of $\triangle \boldsymbol{ABC}$, then the terminal end of the resultant vector is outside of $\triangle \boldsymbol{ABC}$ (Figure 11).

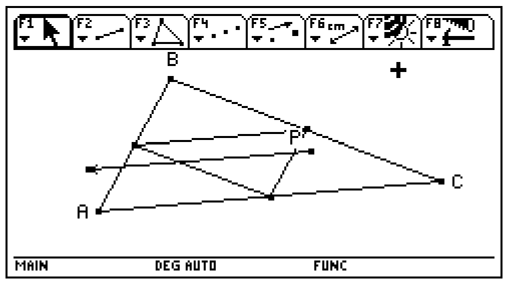

Figure 11

- If $\boldsymbol{P}$ is at a vertex of $\triangle \boldsymbol{ABC}$, then the resultant vector passes through the midpoint of the opposite side of $\triangle \boldsymbol{ABC}$ and is bisected by this side of the triangle (Figure 12).

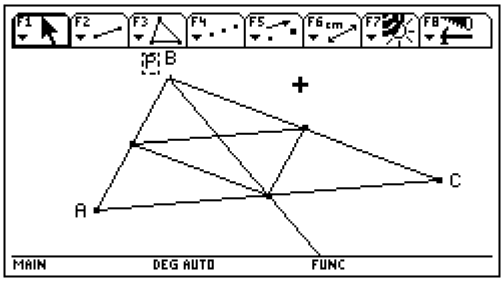

Figure 12

- If $\boldsymbol{P}$ is at the centroid $\boldsymbol{T}$ of $\triangle \boldsymbol{ABC}$, then the resultant vector is a zero vector (Figure 13).

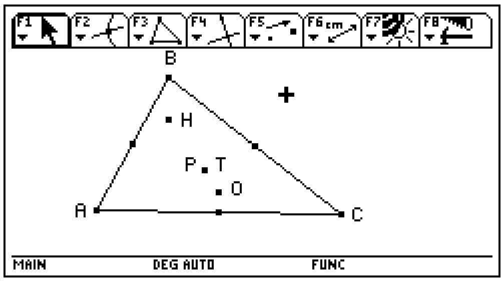

Figure 13

- If $\boldsymbol{P}$ is at the orthocenter $\boldsymbol{H}$ of $\triangle \boldsymbol{ABC}$, then the circumcenter $\boldsymbol{O}$ is the midpoint of the resultant vector (Figure 14).

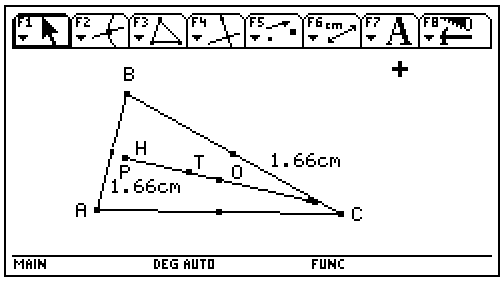

Figure 14

- If $\boldsymbol{P}$ is at the circumcenter $\boldsymbol{O}$ of $\triangle \boldsymbol{ABC}$, then the resultant vector ends at the orthocenter $\boldsymbol{H}$ (Figure 15).

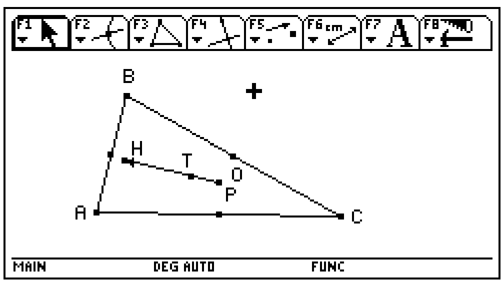

Figure 15

- If $\boldsymbol{P}$ is at $\boldsymbol{N}$, the center of the nine-point circle of $\triangle \boldsymbol{ABC}$, then the resultant vector ends at the circumcenter of $\triangle \boldsymbol{ABC}$ (Figure 16).

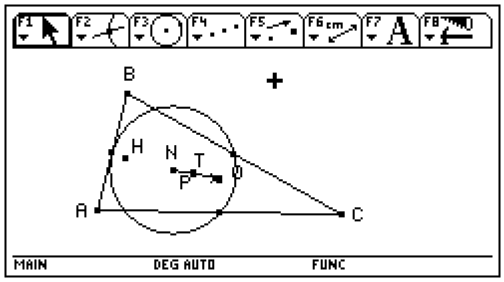

Figure 16

- If $\boldsymbol{P}$ is on the circumcircle of $\triangle \boldsymbol{ABC}$, then the locus of points of the terminal end of the resultant vector is a circle. This circle has four times the area of the circumcircle and is centered at $\boldsymbol{H}$, the orthocenter of $\triangle \boldsymbol{ABC}$ (Figure 17).

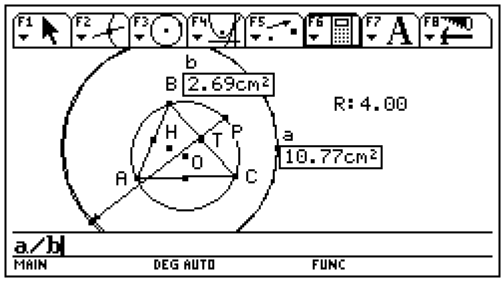

Figure 17

- If $\boldsymbol{P}$ is on the circumcircle of $\triangle \boldsymbol{ABC}$, then the locus of points of the terminal end of the vector sum of any pair of initial vectors is also a circle that is congruent to the circumcircle of $\triangle \boldsymbol{ABC}$. The circles for the vector pair and the circumcircle of $\triangle \boldsymbol{ABC}$ pass through two vertices of the triangle and meet at the orthocenter $\boldsymbol{H}$. These circles are internally tangent to the circle formed by the locus of the terminal end of the resultant vector (Figure 18). The triangle formed by the connecting centers of the three circles is congruent to $\triangle \boldsymbol{ABC}$. The circumcenter of $\triangle \boldsymbol{ABC}$ is the orthocenter of this triangle.

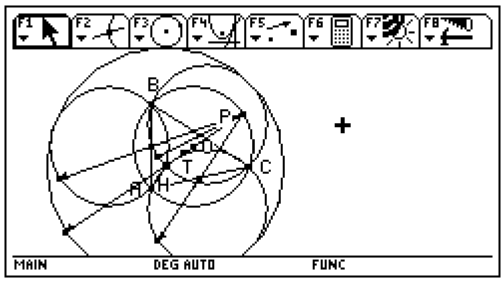

Figure 18

- If P is on the nine-point circle of $\triangle ABC$, then the locus of points of the terminal end of the resultant vector is the circumcircle of $\triangle ABC$ (Figure 19). The locus of points of the midpoint of the resultant vector is the nine-point circle of the triangle formed by connecting the midsegments of $\triangle ABC$ (Figure 20).
- If P is on the nine-point circle of the triangle formed by connecting the midsegments of $\triangle ABC$, then the locus of the terminal end of the resultant vector is the nine-point circle of $\triangle ABC$.

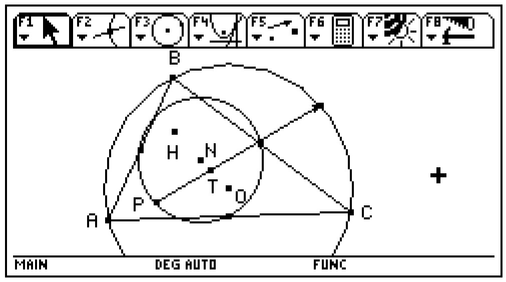

Figure 19

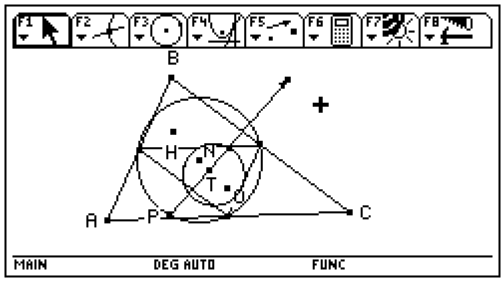

Figure 20

- If P is on $\triangle ABC$, then the locus of points of the terminal end of the resultant vector form a triangle similar to $\triangle ABC$. Triangle ABC becomes the midsegment triangle of the new triangle (Figure 21).

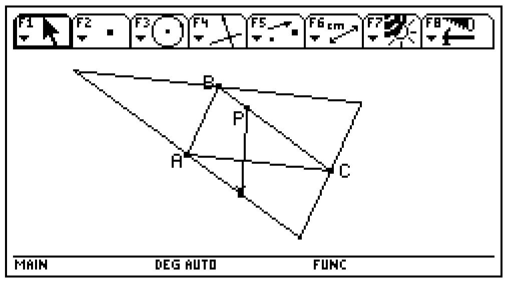

Figure 21

4. The following are some of the many relationships students might discover.

   - For a given $\triangle ABC$, the area of hexagon ***AFBDCE*** is constant for any location of ***P***, even when ***P*** is outside the hexagon and the hexagon is concave (Figures 22 and 23).
   - Opposite sides of hexagon ***AFBDCE*** are congruent and parallel because the hexagon is made up of three parallelograms with mutually parallel sides (Figure 22).

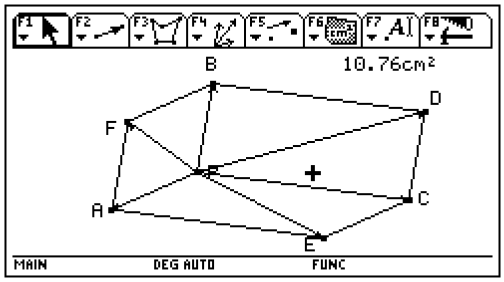

Figure 22

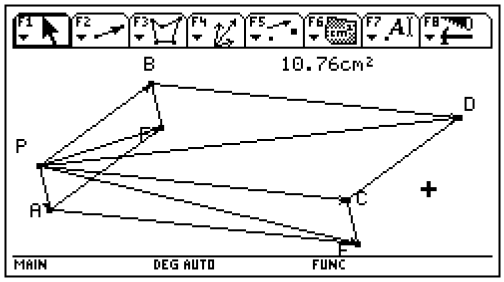

Figure 23

   - Opposite angles of the hexagon are also congruent. The hexagon changes to a parallelogram when point ***P*** lies on a side or at one of the vertices of the hexagon (Figure 24).

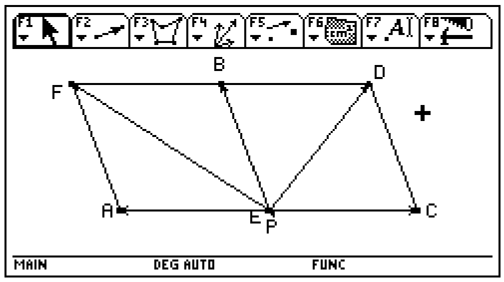

Figure 24

5. Show $\triangle ABC$, and construct its centroid, ***T***. Use the **Redefine Point** tool to redefine point ***P*** onto the centroid ***T*** of $\triangle ABC$. Opposite vectors become the diagonals of the hexagon, and the centroid of $\triangle ABC$ is the midpoint of each of these diagonals (Figure 25).

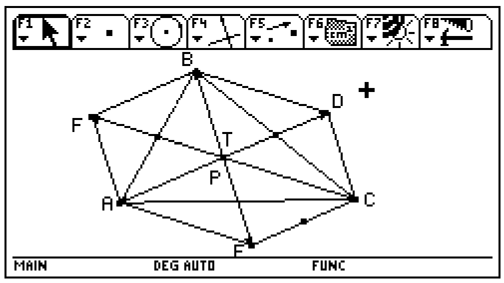

Figure 25

Construct the orthocenter ***H***, the circumcenter ***O***, and the circumcircle of $\triangle$***ABC***. Redefine point ***P*** onto the orthocenter ***H*** of $\triangle$***ABC***. The hexagon becomes cyclic, with all six vertices on the circumcircle of $\triangle$***ABC*** (Figure 26).

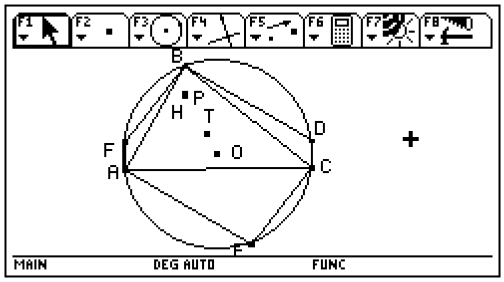

Figure 26

Although the area of the hexagon remains constant for a given $\triangle$***ABC***, the perimeter of the hexagon changes as point ***P*** moves. The minimum perimeter of the hexagon occurs when point ***P*** is located at the **Fermat** point of $\triangle$***ABC***. The **Fermat** point minimizes the sum of the distances from a point to the three vertices of a triangle. Each of the interior angles of the hexagon equals 120° (Figure 27).

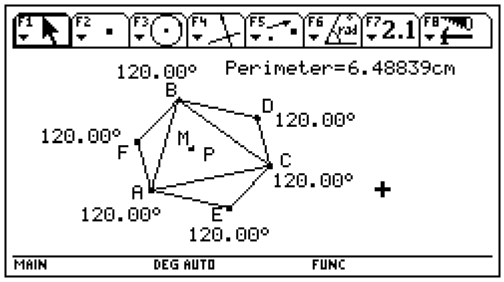

Figure 27

If all the angles of $\triangle$***ABC*** are less than 120°, then the **Fermat** point is inside the triangle. If one of the angles of $\triangle$***ABC*** is greater than or equal to 120°, then the **Fermat** point is at that vertex. When this happens in this construction, the hexagon collapses to a parallelogram.

Explore:

1. The vector sums of consecutive pairs of the original vectors from ***P*** to the vertices of quadrilateral ***ABCD*** pass through the midpoints of the quadrilateral. Quadrilateral ***QRST***, formed by connecting the terminal ends of these vectors, is a parallelogram that has twice the area of quadrilateral ***ABCD*** (Figure 28). Quadrilateral ***QRST*** is similar to and four times the area of quadrilateral ***KLMN*** (Figure 29), which has the midpoints of quadrilateral ***ABCD*** as its vertices. The sides of quadrilateral ***QRST*** are always parallel to and one-half the length of the sides of quadrilateral ***KLMN***.

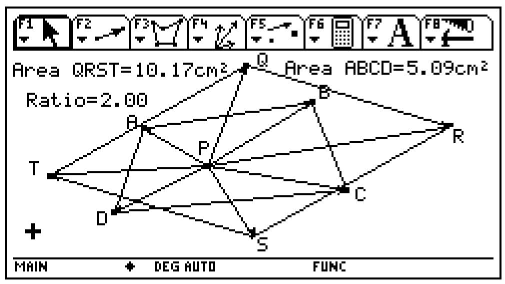

Figure 28

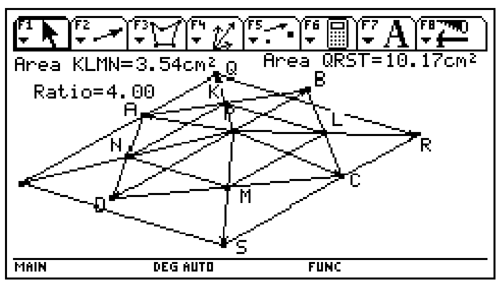

Figure 29

2. The resultant vector that is the sum of the four original vectors of quadrilateral ***ABCD*** passes through centroid ***E*** of quadrilateral ***ABCD***. The distance from ***P*** to the centroid is one-fourth the length of the resultant vector (Figure 30).

Another way to state the distance relationship shown in Figure 30 is that centroid ***E*** of quadrilateral ***ABCD*** divides the resultant vector in the ratio of 3 to 1.

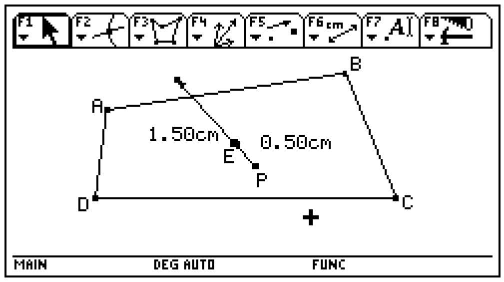

Figure 30

If ***P*** is placed at centroid ***E*** of quadrilateral ***ABCD***, then points ***T***, ***P***, and ***R*** are collinear, as are points ***Q***, ***P***, and ***S***. Segments $\overline{QS}$ and $\overline{TR}$ form the diagonals of the parallelogram ***QRST*** and are bisected by point ***P*** (Figure 31).

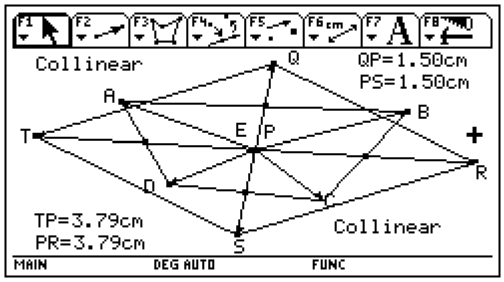

Figure 31

Centroid ***G*** of quadrilateral ***QRST*** is the midpoint of the resultant vector. The resultant vector is the sum of the four original vectors from ***P*** to the vertices of quadrilateral ***ABCD*** (Figure 32).

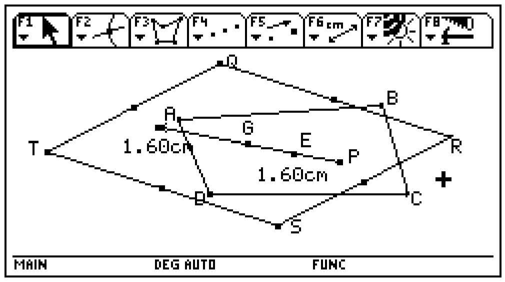

Figure 32

If ***P*** is placed at centroid ***E*** of quadrilateral ***ABCD***, then the resultant vector is zero in length. When this happens, ***P*** is also at the centroid of parallelogram ***QRST***. This is true because the centroid of a parallelogram is located at the intersection of its diagonals as well as at the intersection of the segments connecting the midpoints of its sides (Figure 33).

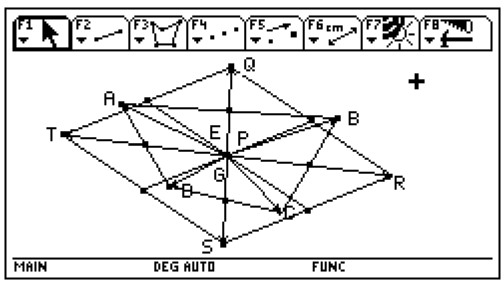

Figure 33

If ***P*** is inside quadrilateral ***KLMN***, then the endpoint of the resultant vector is inside quadrilateral ***QRST*** (Figure 34).

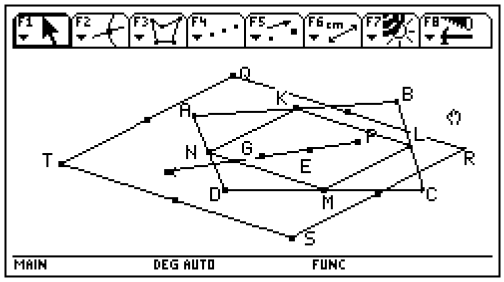

Figure 34

If ***P*** is on a side of quadrilateral ***KLMN***, then

- ***P*** is also on the side of quadrilateral ***QRST***.
- the endpoint of the resultant vector is on a side of quadrilateral ***QRST***.
- centroid ***G*** of ***QRST*** is on the corresponding side of ***KLMN*** (Figure 35).

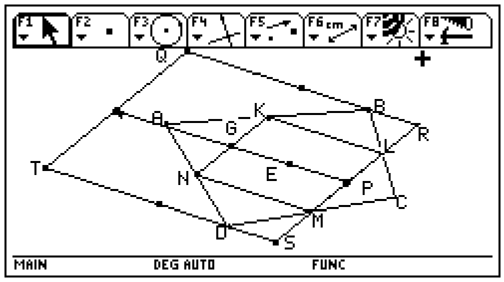

Figure 35

If ***P*** is at a vertex of quadrilateral ***KLMN***, then

- ***P*** is also at a corresponding vertex of quadrilateral ***QRST***.
- the endpoint of the resultant vector is at the opposite vertex of quadrilateral ***QRST***.
- the centroid of ***QRST*** is on the opposite side of ***ABCD*** at a vertex of ***KLMN*** (Figure 36).

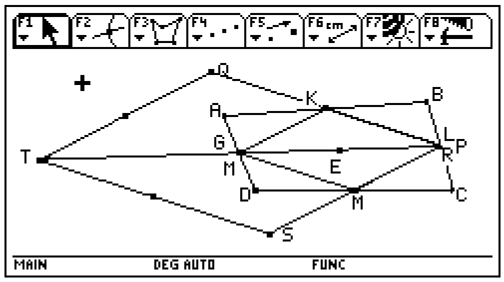

Figure 36

If ***P*** is located at the intersection of the diagonals of quadrilateral ***ABCD***, then quadrilateral ***ABCD*** is inscribed in quadrilateral ***QRST*** (Figure 37).

If ***P*** is outside quadrilateral ***KLMN***, then the endpoint of the resultant vector lies outside quadrilateral ***QRST***.

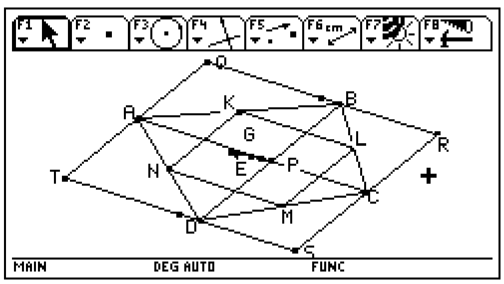

Figure 37

