Ages 15-17 – Parabolas

Parabolas 1

(1) Given $f(x) = x^2 + 2x - 1$:

a) Calculate the slope of the secant line through the points P(-1, -2) and Q(3, 14) on the graph of f.

b) Calculate $f'(1)$.

c) Illustrate the result graphically.

d) Now do the same question with P(1, 2) and Q(4, 23).

e) Calculate $f'(2\tfrac{1}{2})$.

f) Prove that $f'(x) = \dfrac{f(x+h) - f(x-h)}{2h}$ for any x and h.

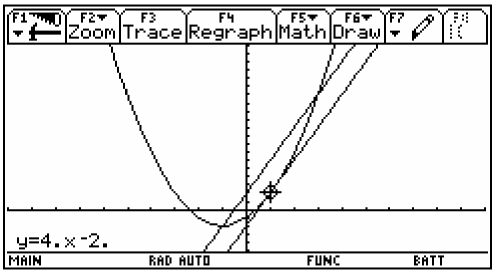
(2) Prove the result in (f) for any polynomial $f(x) = ax^2 + bx + c$.

Solution: (partial)

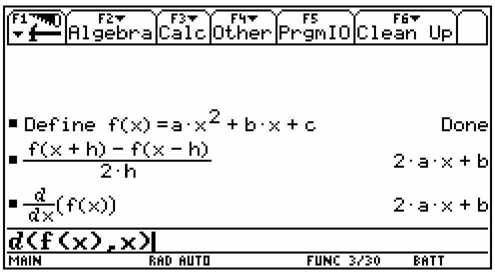
(1) a) $\dfrac{14-(-2)}{3-(-1)} = 4$.

b) $f'(1) = 2 \cdot 1 + 2 = 4$.

c)



(2)



Parabolas 2

(1) Given $f(x) = x^2 + 2x - 1$:

a) Determine the coordinates of the summit S of the parabola.

b) Determine an equation of the tangent line t at the point $R(1, 2)$.

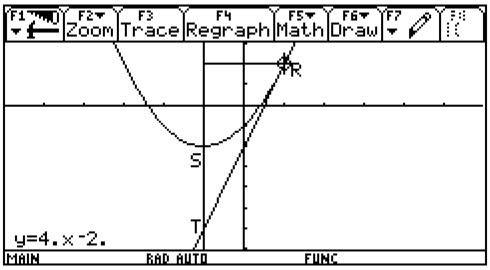
c) Determine the coordinates of the point T of intersection between the tangent line t and the axis of symmetry of the parabola.

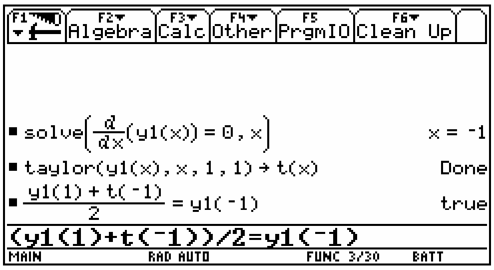
d) Compare the y-coordinates of the points R, S, and T. Conjecture?

(2) For any point R on a parabola $y = ax^2 + bx + c$ prove your conjecture about the vertical position of the points R, S (the summit) and T (the point of intersection between the tangent line at R and the axis of symmetry of the parabola).

Solution: (partial)

(1)

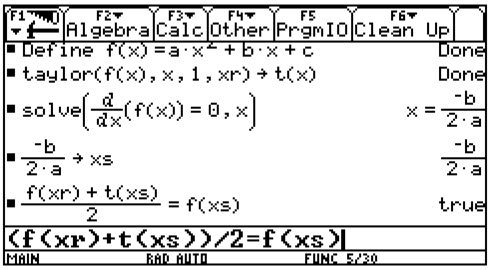




Note: First screen: The equation of the tangent line is found by using a graphic tool (F5).

Second screen: The equation of the tangent line is found by using a CAS tool (Taylor polynomial of degree 1).

(2)



Note: By hand, the algebra is easier if you translate the parabola to the origin. With CAS there is no problem.

Parabolas 3

(1) Given: $f(x) = x^2 + 2x - 1$:

a) Find the abscissa xp of the point of intersection P between the tangent lines at $x_1 = -2$ and $x_2 = 1$. Conjecture?

b) Calculate the area of the region determined by the parabola, the tangent at $x_1 = -2$ and the vertical line $x = xp$.
Calculate the area of the region determined by the parabola, the tangent at $x_2 = 1$ and the vertical line $x = xp$.
Conjecture?

(2) Given: $f(x) = ax^2 + bx + c$:

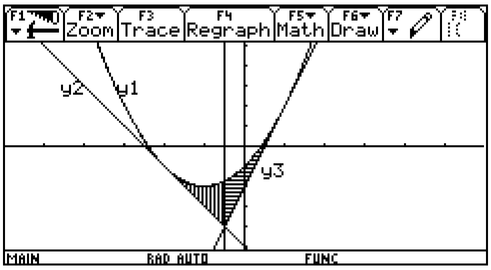
a) Find the abscissa xp of the point P of intersection between the tangent lines at any two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ on the parabola.

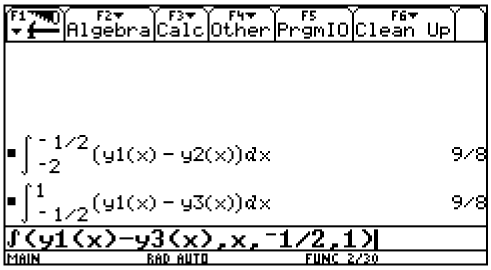
b) Prove that the vertical line $x = xp$ divides the region between the parabola and the tangents into two regions with equal areas.

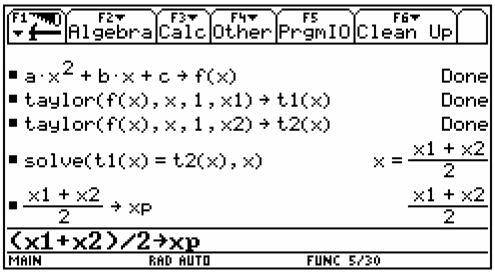
c) Prove that for the parabola $y = x^2$ the coordinates of P are $\left(\dfrac{x_1 + x_2}{2},\ x_1 \cdot x_2\right)$.

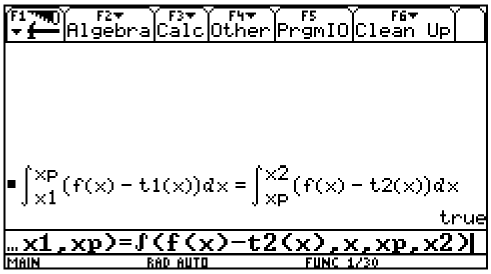
Solution: (partial)

(1) a) $xp = -\frac{1}{2}$.

b) 



(2) 



Exercise

Prove that for any parabola
$y = f(x) = ax^2 + bx + c$
the shaded area does not depend on the values of p and q, but on p-q (and a) only.