Teacher Notes

Activity 19

Taylor Polynomials

Objectives

- Define a Taylor polynomial approximation to a function f of degree n about a point $x = a$
- Graph convergence of Taylor polynomials
- Use Taylor polynomials to approximate function values

Materials

- TI-84 Plus / TI-83 Plus

Teaching Time

- 45 minutes

Abstract

Taylor polynomial approximations are introduced as generalizations of tangent line approximations. The graphing handheld is used as a tool to graph Taylor polynomial approximations of functions. Taylor polynomials are also used to approximate specific function values.

Management Tips and Hints

Prerequisites

Students should:

- be familiar with tangent line approximations.
- be familiar with higher order derivatives.

Evidence of Learning

Given a function f, students should be able to:

- find the Taylor polynomial approximation of degree n about a point $x = a$.
- approximate specific function values using a Taylor polynomial.
- approximate the graph of a function using a Taylor polynomial.

Common Student Errors/Misconceptions

- Students sometimes neglect $n!$ in determining the coefficients of a Taylor polynomial.

Teaching Hints

If you graph several Taylor polynomial approximations of increasing degree on the same screen with an overhead projection panel, it is most effective visually if you choose **Sequential MODE** (rather than **Simul**).

Extensions

The Lagrange error bound for Taylor polynomial approximations would be a possible follow-up to this activity.

If $\left|f^{(n+1)}(x)\right|$ is bounded by M over the interval [a, b], then for any x between a and b, the error in using $P_n(x)$ to approximate $f(x)$ is no larger than

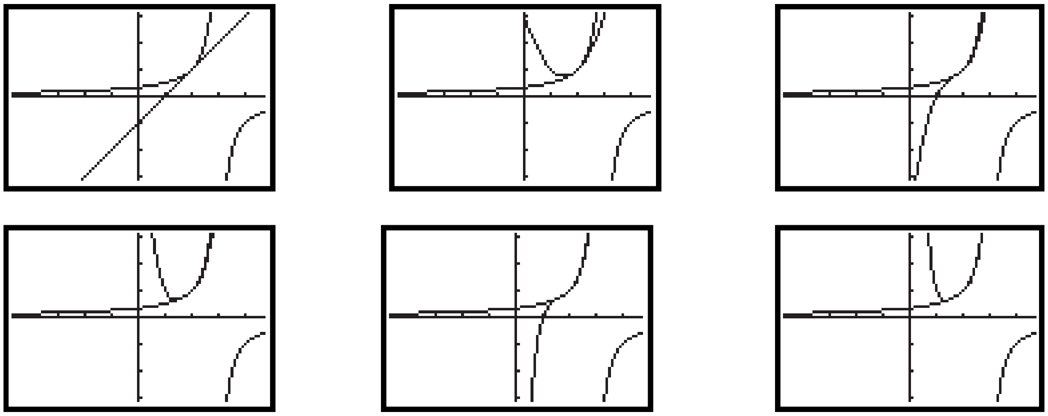
$$\frac{M|b-a|^{n+1}}{(n+1)!}$$

One way to make use of the Lagrange error bound with a graphing handheld would be to graph $\left|f^{(n+1)}(x)\right|$ over the interval [a, b] as a means of finding an appropriate bound, M. The actual error in the Taylor polynomial approximation could then be compared with the “worst case” guaranteed by the Lagrange error bound.

Taylor series would be another natural follow-up topic to this activity, provided that students have been introduced to series of constants, such as geometric series.

Activity Solutions

1.



$f(x) = \sin(x)$ $\quad f(3) = 0.1411200081$

$P_1(x) = x$ $\quad P_1(3) = 3$

$P_3(x) = x - \frac{x^3}{3!}$ $\quad P_3(3) = {}^{-}1.5$

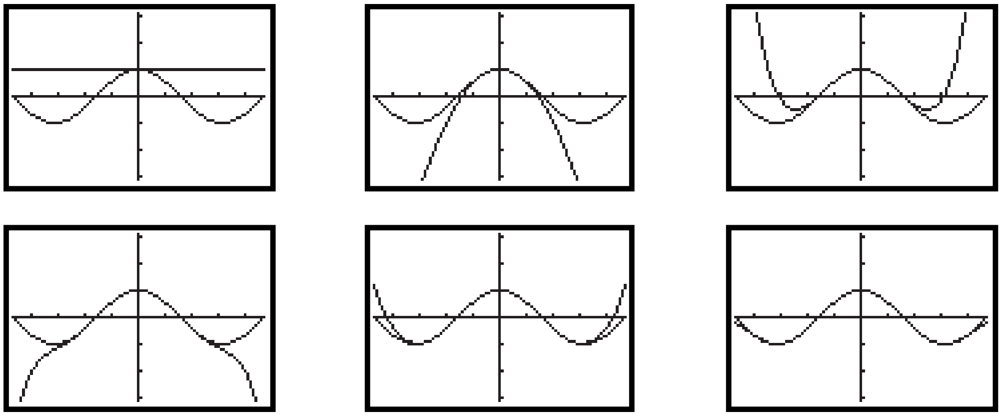
$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$ $\quad P_5(3) = 0.525$

$P_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$ $\quad P_7(3) = 0.0910714286$

$P_9(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$ $\quad P_9(3) = 0.1453125$

$P_{11}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!}$ $\quad P_{11}(3) = 0.1408745942$

2.



$f(x) = \cos(x)$ $f(3) = -0.9899924966$

$P_0(x) = 1$ $P_0(3) = 1$

$P_2(x) = 1 - \frac{x^2}{2!}$ $P_2(3) = -3.5$

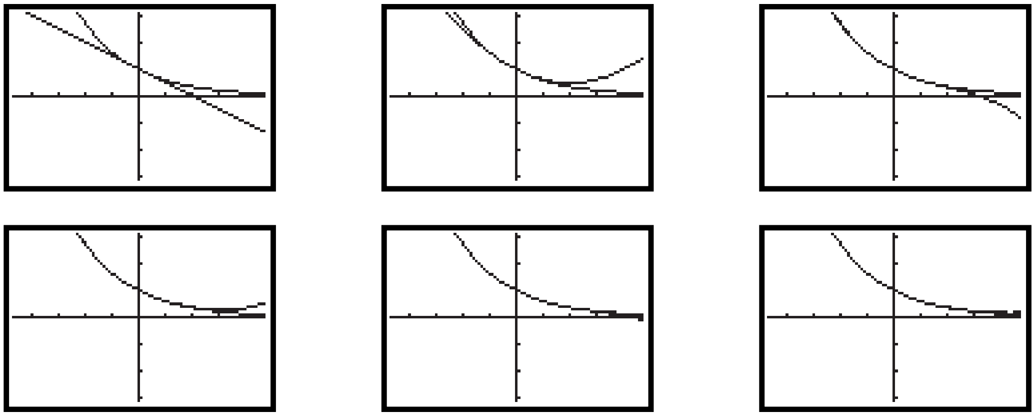
$P_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ $P_4(3) = -0.125$

$P_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$ $P_6(3) = -1.1375$

$P_8(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$ $P_8(3) = -0.9747767857$

$P_{10}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!}$ $P_{10}(3) = -0.9910491071$

3.



$f(x) = e^{-x/2}$ $\qquad$ $f(3) = 0.2231301601$

$P_1(x) = 1 - \frac{x}{2}$ $\qquad$ $P_1(3) = -0.5$

$P_2(x) = 1 - \frac{x}{2} + \frac{x^2}{2^2 \cdot 2!}$ $\qquad$ $P_2(3) = 0.625$

$P_3(x) = 1 - \frac{x}{2} + \frac{x^2}{2^2 \cdot 2!} - \frac{x^3}{2^3 \cdot 3!}$ $\qquad$ $P_3(3) = 0.0625$

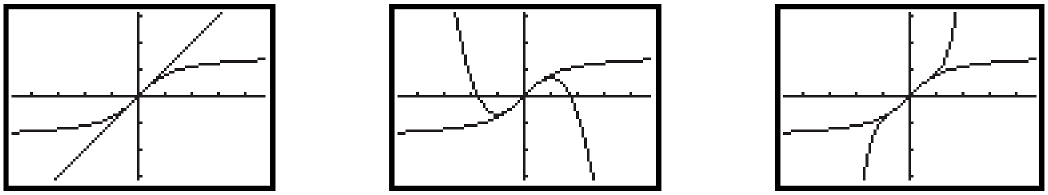
$P_4(x) = 1 - \frac{x}{2} + \frac{x^2}{2^2 \cdot 2!} - \frac{x^3}{2^3 \cdot 3!} + \frac{x^4}{2^4 \cdot 4!}$ $\qquad$ $P_4(3) = 0.2734375$

$P_5(x) = 1 - \frac{x}{2} + \frac{x^2}{2^2 \cdot 2!} - \frac{x^3}{2^3 \cdot 3!} + \frac{x^4}{2^4 \cdot 4!} - \frac{x^5}{2^5 \cdot 5!}$ $\qquad$ $P_5(3) = 0.21015625$

$P_6(x) = 1 - \frac{x}{2} + \frac{x^2}{2^2 \cdot 2!} - \frac{x^3}{2^3 \cdot 3!} + \frac{x^4}{2^4 \cdot 4!} - \frac{x^5}{2^5 \cdot 5!} + \frac{x^6}{2^6 \cdot 6!}$ $\qquad$ $P_6(3) = 0.2259765625$

Note: *The Taylor polynomials for* $f(x) = e^{-x/2}$ *could also be obtained by substituting* $\frac{-x}{2}$ *in place of* x *in the Taylor polynomials for* $f(x) = e^x$ *discussed in the activity.*

4.



$f(x) = \arctan(x)$ $\quad$ $f(3) = 1.249045772$

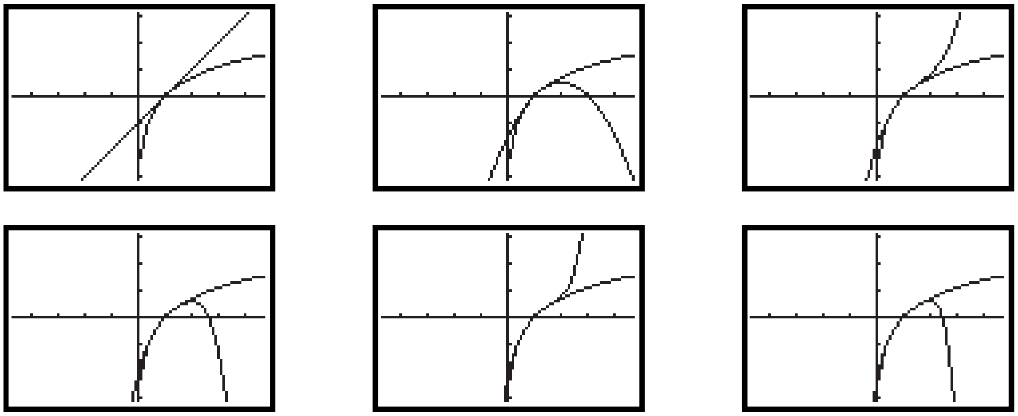
$P_1(x) = x$ $\quad$ $P_1(3) = 3$

$P_3(x) = x - \frac{x^3}{3}$ $\quad$ $P_3(3) = {}^-6$

$P_5(x) = x - \frac{x^3}{3} + \frac{x^5}{5}$ $\quad$ $P_5(3) = 42.6$

Note: *The Taylor series for arctan(*x*) is very slow in converging.*

5.



$f(x) = \ln(x)$ $\qquad$ $f(3) = 1.098612289$

$P_1(x) = x - 1$ $\qquad$ $P_1(3) = 2$

$$P_2(x) = (x-1) - \frac{(x-1)^2}{2}$$ $\qquad$ $P_2(3) = 0$

$$P_3(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$$ $\qquad$ $P_3(3) = 2.666666667$

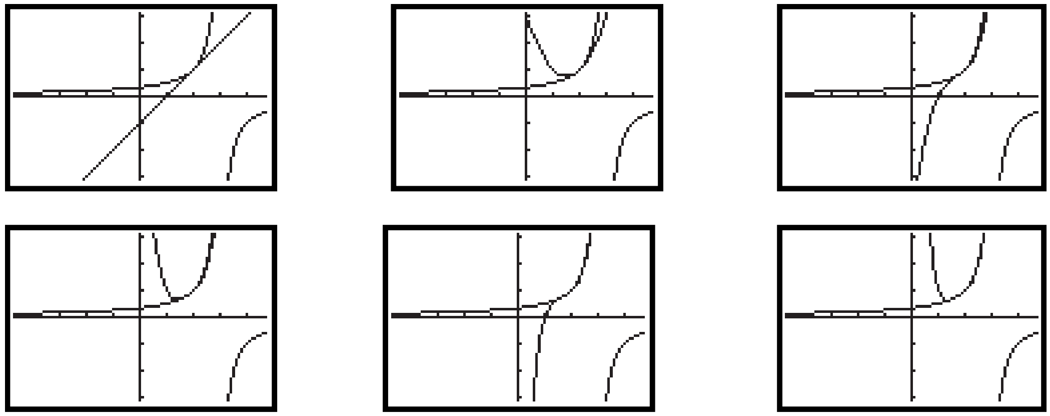
$$P_4(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}$$ $\qquad$ $P_4(3) = -1.333333333$

$$P_5(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5}$$ $\qquad$ $P_5(3) = 5.066666667$

$$P_6(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \frac{(x-1)^6}{6}$$ $\qquad$ $P_6(3) = -5.6$

Note: *The value* x = *3 lies outside the interval of convergence for these Taylor polynomials for ln(*x*) (the interval of convergence is $0 < x \le 2$). You could compare the numerical results obtained for approximating another value of* x *that lies within this interval of convergence (such as $x = \frac{3}{2}$).*

6.



$f(x) = \dfrac{1}{3-x}$ $f(3)$ is undefined.

$P_1(x) = 1 + (x-2)$ $P_1(3) = 2$

$P_2(x) = 1 + (x-2) + (x-2)^2$ $P_2(3) = 3$

$P_3(x) = 1 + (x-2) + (x-2)^2 + (x-2)^3$ $P_3(3) = 4$

$P_4(x) = 1 + (x-2) + (x-2)^2 + (x-2)^3 + (x-2)^4$ $P_4(3) = 5$

$P_5(x) = 1 + (x-2) + (x-2)^2 + (x-2)^3 + (x-2)^4 + (x-2)^5$ $P_5(3) = 6$

$P_6(x) = 1 + (x-2) + (x-2)^2 + (x-2)^3 + (x-2)^4 + (x-2)^5 + (x-2)^6$ $P_6(3) = 7$

Note: *The value* $x = 3$ *lies just outside the interval of convergence for these Taylor polynomials for* $f(x) = \frac{1}{3-x}$ *(the interval of convergence is* $1 < x < 3$*). You could compare the numerical results obtained for approximating another value of* x *that lies within this interval of convergence (such as* $x = \frac{3}{2}$ *). Indeed, it does not make sense to use these Taylor polynomials to approximate the value of a function not even defined at* $x = 3$*.*

Also, the Taylor polynomials for $f(x) = \frac{1}{3-x}$ *represent a sequence of geometric sums and can be used to make connections with geometric series. The interval of convergence for these Taylor polynomials corresponds exactly to the values of* x *for which the corresponding geometric series converges.*