

Parametric and Polar Graphing

The ease with which most graphing calculators handle the graphing of parametric equations has brought the topic of parametric equations to a new level of prominence in the study of precalculus and calculus. In this chapter we give an introduction to parametric and polar graphing on the TI-86. The keystroke instructions in this chapter are less detailed since the features discussed in the first two chapters also apply in the parametric and polar graphing modes.

§1 – Initial Mode and Format Settings

Set the mode and graph format settings as shown in (3.1.1) and (3.1.2), respectively.

```
Normal Sci Eng
Float 012345678901
Radian Degree
RectC PolarC
Func Pol Param DfEq
Dec Bin Oct Hex
RectV CylV SphereV
dxDer1 dxNDer
```

(3.1.1)

Clear all functions in the $\langle E(t) = \rangle$ graph editor by using the $\langle \text{DEL} \rangle$ selection.

```
RectGC PolarGC
CoordOn CoordOff
DrawLine DrawDot
SeqG SimulG
GridOff GridOn
AxesOn AxesOff
LabelOff LabelOn
E(t)= WIND ZOOM TRACE GRAPH
```

(3.1.2)

§2 – Graphing a Circle Using Parametric Equations

1. Enter the pair of parametric equations

$$\begin{cases} x = 5 \cos(t) \\ y = 5 \sin(t) \end{cases}$$

as shown in (3.2.1).

```
Plot1 Plot2 Plot3
\xt1=5 cos t
yt1=5 sin t
E(t)= WIND ZOOM TRACE GRAPH
t xt yt DELF SELECT
```

(3.2.1)

2. Set the WINDOW variables as shown in (3.2.2) and (3.2.3), where **tMax** and **tStep** are $3\pi/2$ and $\pi/2$, respectively.

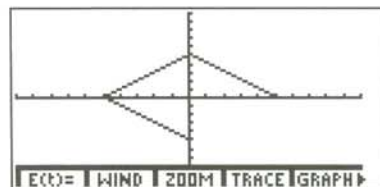
```
WINDOW
tMin=0
tMax=4.71238898038
tStep=1.57079632679...
xMin=-10
xMax=10
↓xScl=1
E(t)= WIND ZOOM TRACE GRAPH
```

(3.2.2)

- Then graph to obtain (3.2.4). The graph is obtained by plotting and then connecting the points obtained with the t values, starting with **tMin** and incremented by the value of **tStep** until a t value larger than **tMax** is obtained. In this case, the graph is obtained by connecting the four points $(5 \cos(t), 5 \sin(t))$, $t = 0, \pi/2, \pi, 3\pi/2$.

```
WINDOW
tMin=-10
xMax=10
xScl=1
yMin=-10
yMax=10
yScl=1
E(t)= WIND ZOOM TRACE GRAPH▶
```

(3.2.3)



(3.2.4)

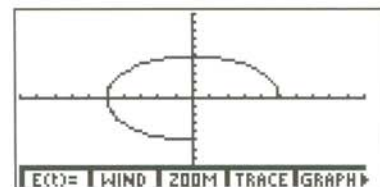
§3 – The Importance of tStep

- Change the setting of **tStep** to 0.05 as shown in (3.3.1).

```
WINDOW
tMin=0
tMax=4.71238898038
tStep=.05
xMin=-10
xMax=10
xScl=1
E(t)= WIND ZOOM TRACE GRAPH▶
```

(3.3.1)

- Regraph to obtain (3.3.2).



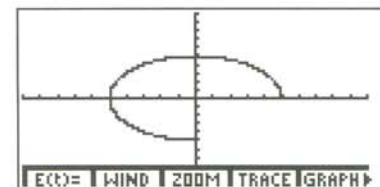
(3.3.2)

- Note that the TI-86 traces out the graph in the counterclockwise sense starting at the point (5, 0). Next change **tStep** to 0.01 as shown in (3.3.3).

```
WINDOW
tMin=0
tMax=4.71238898038
tStep=.01
xMin=-10
xMax=10
xScl=1
E(t)= WIND ZOOM TRACE GRAPH▶
```

(3.3.3)

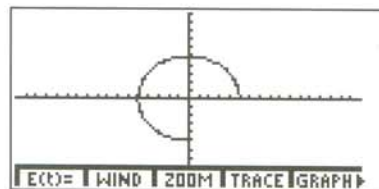
- Regraph to obtain (3.3.4). Notice that it takes considerably longer for the TI-86 to obtain (3.3.4) than it did to obtain (3.3.2). Thus, from (3.2.4), (3.3.2), (3.3.4) and the time required to obtain them, it is clear that careful consideration should be given to the choice of **tStep**.



(3.3.4)

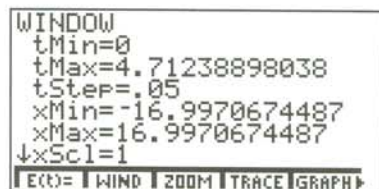
§4 – Using $\langle \text{ZSQR} \rangle$ to Make a Circle Look Like a Circle

1. With the WINDOW settings as shown in (3.2.3) and (3.3.1), select $\langle \text{ZOOM} \rangle$ and then $\langle \text{ZSQR} \rangle$ to obtain (3.4.1), the expected graph of the pair of parametric equations in §2 over the t -interval $[0, 3\pi/2]$.



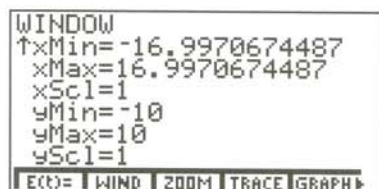
(3.4.1)

2. Select $\langle \text{WIND} \rangle$ to obtain (3.4.2) and (3.4.3).



(3.4.2)

Note that the $\langle \text{ZSQR} \rangle$ option did not change **tMin**, **tMax**, or **tStep**.

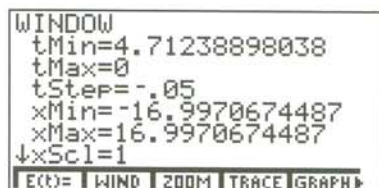


(3.4.3)

§5 – Using Negative **tStep** Values

Negative values of **tStep** are also possible if we interpret the roles of **tMin**, **tMax**, and **tStep** correctly. Actually the names **tMin** and **tMax** are misnomers. Better names would have been **tInitial** and **tFinal**, respectively. With this observation you have probably guessed how we will proceed to graph the parametric equations in §2 over the t -interval $[0, 3\pi/2]$ using a negative **tStep**.

1. Set **tMin** = $3\pi/2$, **tMax** = 0, and **tStep** = -0.05 as shown in (3.5.1). Leave the other WINDOW settings as in (3.4.3). Then regraph to obtain (3.4.1) again, except this time the graph is traced out in the clockwise direction starting from the point $(0, -5)$.



(3.5.1)

§6 – Graphing the Inverse of a Function

The parametric graphing mode on the TI-86 gives a very nice way to obtain the graph of $f^{-1}(x)$ given a one-to-one function $f(x)$. We graph $y = f(x)$ in parametric mode by graphing the pair of parametric equations

$$\begin{cases} x = t \\ y = f(t) \end{cases}$$

The graph of $y = f^{-1}(x)$ results from graphing

$$\begin{cases} x = f(t) \\ y = t \end{cases}$$

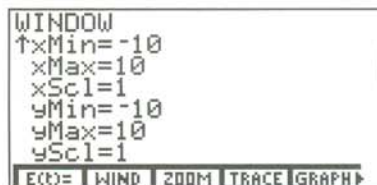
in parametric mode.

- As an example, let's graph $f(x) = (2x+7)/(x+3)$ and its inverse $f^{-1}(x)$ in the $[-10, 10] \times [-10, 10]$ viewing window.

Set the WINDOW as shown in (3.6.1) and (3.6.2).



(3.6.1)



(3.6.2)

- Enter

$$xt1=t, \quad yt1=(2t+7)/(t+3)$$

as shown in (3.6.3).



(3.6.3)

- Then graph to obtain (3.6.4), the graph of $y = f(x)$ in the desired viewing window.



(3.6.4)

- Next enter

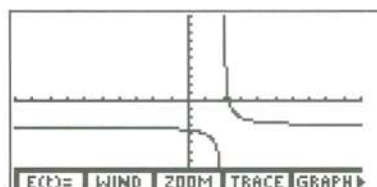
$$xt2=yt1, \quad yt2=xt1$$

as shown in (3.6.5)



(3.6.5)

- Graph only the $(xt2, yt2)$ pair to obtain (3.6.6), the graph of $y = f^{-1}(x)$ in the desired viewing window.



(3.6.6)

§7 – Inverse Functions and Symmetry

It is worth considering a second example of the type done in the previous section in order to emphasize that the viewing window being used can be a significant factor in how we interpret what appears on the screen.

1. In particular, let's graph $f(x) = \sqrt{x}$ and its inverse together in the viewing window $[0, 10, 5] \times [0, 20, 5]$.

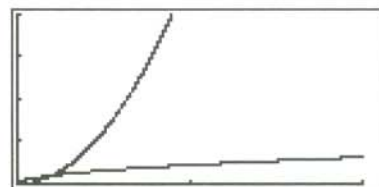
As indicated in (3.7.1), enter $f(x) = \sqrt{x}$ parametrically as the pair $(xt1, yt1)$, and enter $f^{-1}(x)$ as the pair $(xt2 = yt1, yt2 = xt1)$.

(3.7.1)



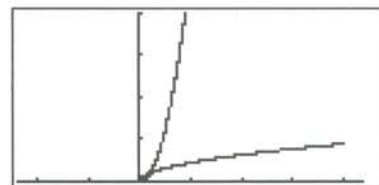
2. With **tMin** = 0, **tMax** = 20, **tStep** = .05, the resulting graph in (3.7.2) seems to violate the notion that the graph of a function and its inverse will be symmetric about the $y = x$ line. This is because the viewing window is such that a line of slope one does not appear to be inclined at 45 degrees.

(3.7.2)



3. From (3.7.2), select **<ZOOM>** **<ZSQR>** to obtain a viewing window in which adjustments have been made for the fact that there are more horizontal pixels than vertical pixels. See (3.7.3).

(3.7.3)



In other words, the **<ZSQR>** option creates a viewing window in which *one unit horizontally is the same distance on the screen as one unit vertically*. Consequently, we see the expected symmetry of $f(x)$ and its inverse with respect to the $y = x$ line.

§8 – Finding Roots of 1

Recall that the n n^{th} roots of 1 are given by $x + iy$, where (x, y) are the n points on the unit circle given by

$$(\cos(t), \sin(t)), t = 0, 2\pi/n, 4\pi/n, 6\pi/n, \dots, (n-1)(2\pi/n).$$

We can use the parametric mode and **<TRACE>** to find these roots of 1.

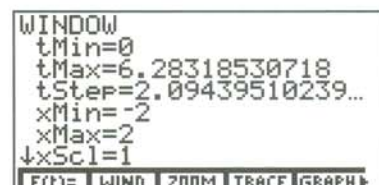
1. As an example, we find the three cube roots of 1. Set $xt1 = \cos(t)$, $yt1 = \sin(t)$ as shown in (3.8.1).

(3.8.1)



2. Set **tMin** = 0, **tMax** = 2π , **tStep** = $2\pi/3$ as shown in (3.8.2).

(3.8.2)



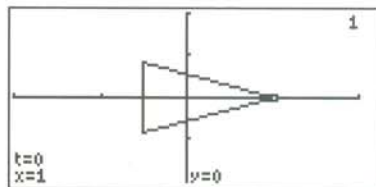
Parametric and Polar Graphing (Continued)

- Set $xMin = yMin = -2$, $xMax = yMax = 2$ as shown in (3.8.3).
- Then graph and use **TRACE** to obtain the results shown in (3.8.4)–(3.8.6). Thus we see that the three cube roots of 1 are approximated by 1, $-0.5 + 0.86602540379i$, and $-0.5 - 0.8660254038i$.

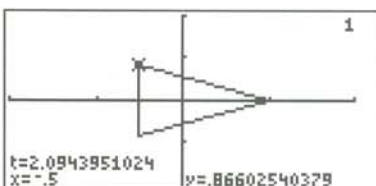
Of course, we know that the exact cube roots are given by 1, $-1/2 + i\sqrt{3}/2$, and $-1/2 - i\sqrt{3}/2$.

```
WINDOW
↑xMin=-2
xMax=2
xScl=1
yMin=-2
yMax=2
yScl=1
E(t)= WIND ZOOM TRACE GRAPH
```

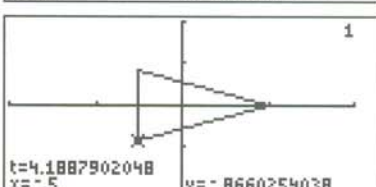
(3.8.3)



(3.8.4)



(3.8.5)



(3.8.6)

Although this procedure is pedagogically appealing and can be generalized to find the n th roots of any non-zero complex number α , a more efficient way to find the roots of α is to solve $x^n - \alpha = 0$ using the POLY feature of the TI-86. See Chapter 4.

§9 – Graphing a Cycloid

There are several famous and significant parametric curves encountered in the study of calculus. One of these is the cycloid given by

$$\begin{cases} x = 2(t - \sin t) \\ y = 2(1 - \cos t) \end{cases}$$

This curve can be thought of as the path traveled by a point on the circumference of a circle of radius 2 as the circle rolls down the positive x -axis if the point was at the origin at time zero.

```
Plot1 Plot2 Plot3
\xt1=2(t-sin t)
yt1=2(1-cos t)
E(t)= WIND ZOOM TRACE GRAPH
t xt yt DELF SELECT
```

(3.9.1)

```
WINDOW
tMin=0
tMax=20
tStep=.1
xMin=-5
xMax=25
xScl=5
yMin=-5
yMax=5
yScl=1
E(t)= WIND ZOOM TRACE GRAPH
```

(3.9.2)

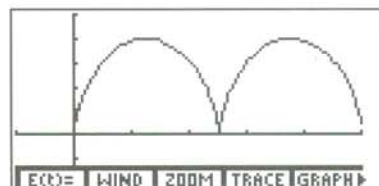
- Enter these equations in the **E(t)=** graph editor and set the WINDOW as indicated in (3.9.1) through (3.9.3).

```
WINDOW
↑xMin=-5
xMax=25
xScl=5
yMin=-2
yMax=5
yScl=1
E(t)= WIND ZOOM TRACE GRAPH
```

(3.9.3)

Parametric and Polar Graphing (Continued)

2. The resulting graph is shown in (3.9.4).



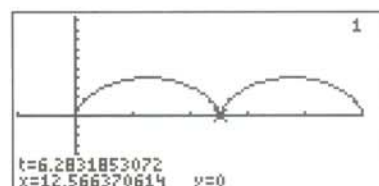
(3.9.4)

3. Since we are dealing with a graph related to some property of a circle, it is appropriate here to perform a **ZSQR** operation to get a better perspective on the shape of this curve. From (3.9.4), select **ZOOM** **ZSQR** to obtain the graph in (3.9.5).



(3.9.5)

4. In figure (3.9.6), we have used the TRACE feature of the TI-86 to see that when $t = 2\pi$, the corresponding x -value is the expected 4π since the circle in this problem has radius 2.



(3.9.6)

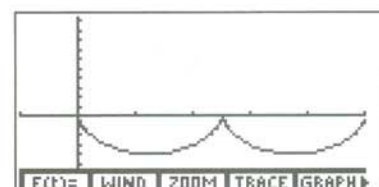
The cycloid curve also provides a solution to the famous brachistochrone problem of finding the path of quickest descent a frictionless bead located at the origin will travel in moving to a lower point not located directly beneath the origin. The curve which provides the quickest descent turns out to be a portion of an inverted cycloid which has a cusp at the origin and which passes through the second point. We can easily generate such a curve by modifying the parametric equations for the standard cycloid.

5. In particular, if we wanted to see an inverted version of the cycloid shown in (3.9.6) we just deselect the $(xt1, yt1)$ pair and enter the following for $xt2$ and $yt2$. Namely, $xt2 = xt1$ and $yt2 = -yt1$ as shown in (3.9.7).



(3.9.7)

6. Then graph to obtain (3.9.8).



(3.9.8)

§10 – Graphing a Line Segment

The problem of graphing a line segment joining two points in the plane also can be handled easily using the parametric graphing features of the TI-86.

- For example, it is not difficult to show that the line segment joining the points (a,b) and (c,d) can be generated by the equations

$$\begin{cases} x = (1-t)a + tc \\ y = (1-t)b + td \end{cases}, 0 \leq t \leq 1. \quad (3.10.1)$$

```

Plot1 Plot2 Plot3
\xt1=1
yt1=2
\xt2=7
yt2=4
\xt3=(1-t)*xt1+t*xt2
E(t)= WIND ZOOM TRACE GRAPH
t xt yt DELF SELECT
    
```

To graph the line segment connecting the points $(1, 2)$ and $(7, 4)$, we enter the equations above as the $(xt3, yt3)$ pair shown in (3.10.1) and (3.10.2).

```

Plot1 Plot2 Plot3
yt1=2
\xt2=7
yt2=4
\xt3=(1-t)*xt1+t*xt2
yt3=(1-t)*yt1+t*yt2
E(t)= WIND ZOOM TRACE GRAPH
t xt yt DELF SELECT
    
```

- Change the viewing window as shown in (3.10.3) and (3.10.4). Note that we use $0 \leq t \leq 1$.

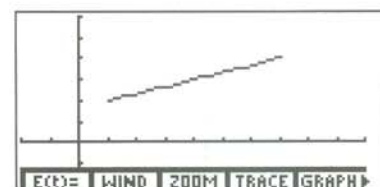
```

WINDOW
tMin=0
tMax=1
tStep=.05
xMin=-2
xMax=10
xScl=1
↓xScl=1
E(t)= WIND ZOOM TRACE GRAPH
    
```

```

WINDOW
↑xMin=-2
xMax=10
xScl=1
yMin=-2
yMax=6
yScl=1
E(t)= WIND ZOOM TRACE GRAPH
    
```

- The resulting graph plotting from $(1, 2)$ to the point $(7, 4)$ is shown in (3.10.5). To plot the graph in such a way that the line segment draws from $(7, 4)$ to $(1, 2)$, we just let **tMin** have value 1 and take **tMax** = 0 with **tStep** = -0.05.



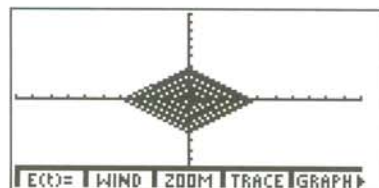
§11 – Further Evidence of the Importance of tStep

The ability of the TI-86 to graph parametric equations can also lead to important mathematical discussions.

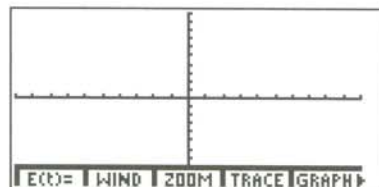
In particular, consider the parametric equations given by

$$\begin{cases} x = t \cos(25\pi t) \\ y = t \sin(25\pi t) \end{cases}$$

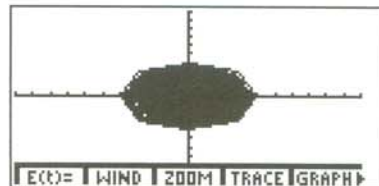
Figures (3.11.1)–(3.11.4) are the respective graphs of this pair of equations in the viewing window $[-10, 10, 1] \times [-10, 10, 1]$ as t ranges from 0 to 4 for **tStep** values of 0.1, 0.2, 0.01, 0.05.



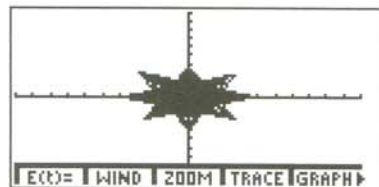
(3.11.1) tStep value .1



(3.11.2) tStep value .2



(3.11.3) tStep value .01



(3.11.4) tStep value .05

We leave as an exercise for the reader to decide which screen image most accurately describes the actual graph of this pair of equations.

§12 – Polar Graphing

Polar coordinate graphing mode principles are very similar to those of the parametric graphing mode. We illustrate by examining the graph of the four-leaved rose $r = 8\sin(2\theta)$ in the viewing window $[-10, 10, 1] \times [-10, 10, 1]$.

1. Set the mode and graph format settings as shown in (3.12.1) and (3.12.2), respectively.



(3.12.1)



(3.12.2)

Parametric and Polar Graphing (Continued)

- Enter the equation $r_1 = 8\sin(2\theta)$ as shown in (3.12.3),



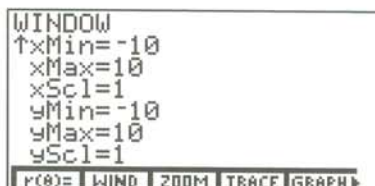
(3.12.3)

and set the WINDOW as indicated in (3.12.4) and (3.12.5) where $\theta_{\text{Max}} = 2\pi$.



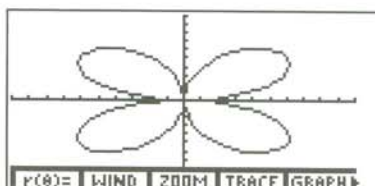
(3.12.4)

The role of θ_{Step} is analogous to that of t_{Step} in the parametric mode.



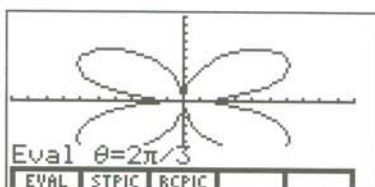
(3.12.5)

- Then graph to obtain (3.12.6).

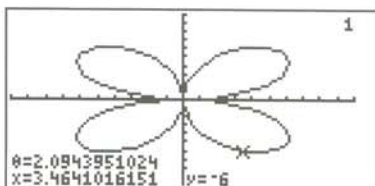


(3.12.6)

- In (3.12.7) and (3.12.8) we use **(EVAL)** to find that when $\theta = 2\pi/3$ the rectangular coordinates of the point on the curve are $x = 3.4641016151$ and $y = -6$.



(3.12.7)



(3.12.8)

§13 – Displaying Polar Coordinates

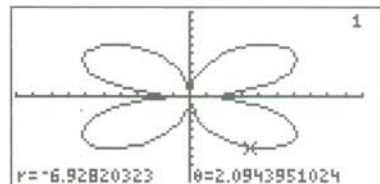
- When graphing in the polar coordinate mode, it is usually desired to give the curve coordinates in polar coordinates rather than rectangular coordinates. We accomplish this by changing the graph format mode to that shown in (3.13.1).



(3.13.1)

Parametric and Polar Graphing (Continued)

- Then when we use **⟨EVAL⟩**, as in (3.12.7), we obtain (3.13.2), which indicates that $r = -6.92820323$ when $\theta = 2\pi/3$. Incidentally, notice from (3.13.2) that the TI-86 handles negative r correctly when graphing in the polar coordinate mode.

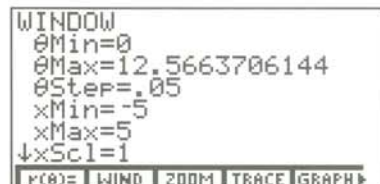


(3.13.2)

§14 – False Asymptotes in Polar Graphing

In §18 of Chapter 2 we considered the phenomenon of the graphing of false asymptotes by the TI-86. In this section we will see that similar things can happen in the polar graphing mode as well.

- Enter $r2 = 2/\theta$ in the **⟨r(θ)⟩** editor after deselecting $r1$ for graphing.



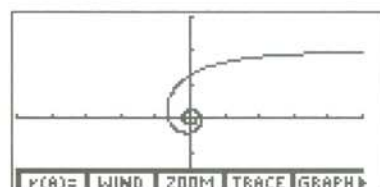
(3.14.1)

- Set the WINDOW variables as indicated in (3.14.1) and (3.14.2), where **θMax** is actually 4π .



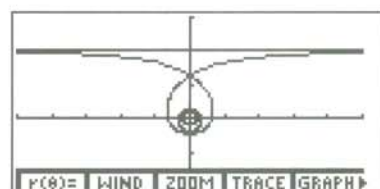
(3.14.2)

- From (3.14.2) select **⟨GRAPH⟩** to obtain the spiral graph shown in (3.14.3).



(3.14.3)

- Return to the WINDOW variables and change the value of **θMin** to -4π , leaving all other values as they were in (3.14.1) and (3.14.2). The resulting graph is shown in (3.14.4).



(3.14.4)

- It seems as though the TI-86 was clever enough to sketch in the $y = 2$ asymptote. But, in fact, this is the polar graphing mode version of a false asymptote. As with graphing in the function mode, we can change the style of the graph of $r2$ to the Dot style as shown in (3.14.5), and regraph to obtain (3.14.6).

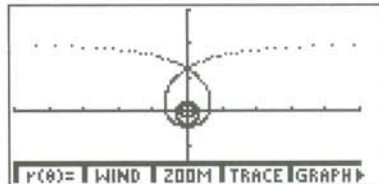


(3.14.5)

Parametric and Polar Graphing (Continued)

This graph confirms that the horizontal line across the top of the screen in (3.14.4) is not really part of the graph of r^2 . Rather it resulted from the calculator connecting a point plotted for a negative θ near to zero (and its correspondingly large in absolute value negative r value) with a point plotted for a positive θ value near to zero (and its correspondingly large positive r value).

(3.14.6)



Exercises

1. Adjust the window in §4 so that the graph is a closed circle.
2. Verify graphically using parametric equations that the inverse of e^x is $\ln x$.
3. Find the five fifth roots of 32 by imitating the technique used in §8 to find the cube roots of 1.
4. Verify graphically using the polar mode that the graphs of $r = \frac{2}{\sin \theta}$ and $r = \frac{2}{\cos \theta}$ are horizontal and vertical lines, respectively. Explain why this is the case.
5. (a) Graph the parametric equations

$$\begin{cases} x = 2t - \sin t \\ y = 2 - \cos t \end{cases}$$

Compare with the graph of the cycloid done in §9.

(b) Do the same for

$$\begin{cases} x = 2t - 2.5 \sin t \\ y = 2 - 2.5 \cos t \end{cases}$$

Use BOX ZOOM to carefully study the lower part of one of the cycles on this graph.

6. Consider the equations

$$\begin{cases} x = \frac{1-t^2}{1+t^2} \\ y = \frac{2t}{1+t^2} \end{cases}$$

Show that the graph of this pair of equations for $-1 \leq t \leq 1$ is the right half of the unit circle.