Geometric Series

ID: 10236

Time required
35 minutes

Activity Overview

Students will use the TI-89 to explore infinite geometric series and the partial sums of geometric series. The students will determine the limits of these sequences and series using tables and graphs.

Topic: Series

- *Derive and apply a formula for the sum of an infinite convergent geometric series.*
- *Use the* $\sum$ *template to verify the formula for the sum of an infinite series in specific cases.*
- *Prove and apply the ratio (of consecutive terms) test to prove a series convergent or divergent.*
- *Prove that a necessary condition that a geometric series converges is that* $|r| < 1$ *where r is the common ratio.*

Teacher Preparation and Notes

- *Students will use the TI-89 to explore infinite geometric series and partial sums of geometric series. Students will also explore convergence and divergence of a geometric series. The students will use tables and graphs that display the limits of series, and then determine the limit by observation.*
- *Students should already be familiar with sequences, partial sums, and the definition of convergent and divergent series.*
- *This activity is designed to be* ***teacher-led.*** *You may use the following pages to present the material to the class and encourage discussion. Students will follow along using their calculators. Although the majority of the ideas and concepts are only presented in this document, be sure to cover all the material necessary for students' comprehension.*
- *Before starting this activity, students should go to the home screen and select* ***F6:Clean Up > 2:NewProb****, then press* ENTER*. This will clear any stored variables, turn off any functions and plots, and clear the drawing and home screens.*
- ***To download the student worksheet, go to education.ti.com/exchange and enter "10236" in the quick search box.***

Associated Materials

- *GeometricSeries_Student.doc*

Problem 1 – Infinite Series

An infinite series can be defined as $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$, where a_1, a_2, and a_3 are terms of the series. It is beneficial to students if they are already introduced to sequences.

1. Find the next three terms of the infinite series.

a. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$ **b.** $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\ldots$ **c.** $2+\frac{3}{2}+\frac{9}{8}+\frac{27}{32}\ldots$

$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}$ $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\frac{5}{6}+\frac{6}{7}$ $2+\frac{3}{2}+\frac{9}{8}+\frac{27}{32}+\frac{81}{128}+\frac{243}{512}$

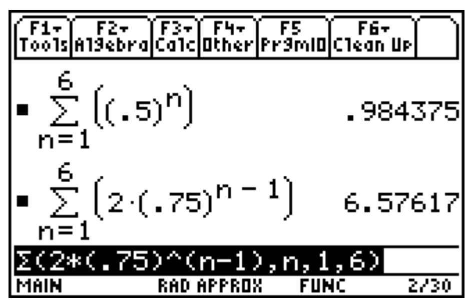
Students should see for question **1.c.** that the first term 2 is the same as *a* in a geometric series, meaning $a + ar + ar^2 + \ \ldots \ + ar^n + \ \ldots$

2. Write an expression in terms of *n* that describe each of the above series using sigma notation.

a. $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$ or $\sum_{n=1}^{\infty}\frac{1}{2^n}$ **b.** $\sum_{n=1}^{\infty}\frac{n}{n+1}$ **c.** $\sum_{n=1}^{\infty} 2\left(\frac{3}{4}\right)^{n-1}$

Problem 2 – Finding the sum of a geometric series

Students can find the partial sum of a geometric series. In this problem, students will find the sixth partial sum of two geometric series. To find the sum of a series, go to **F3:Calc** and select **∑(sum**.



3. $\sum_{n=1}^{6}\left(\frac{1}{2}\right)^n = \frac{63}{64} = 0.984375$

4. $\sum_{n=1}^{6} 2\left(\frac{3}{4}\right)^{n-1} = \frac{3367}{512} = 6.57617$

Problem 3 – Convergence and divergence of geometric series

A geometric series with first term *a* and common ratio *r* is given by

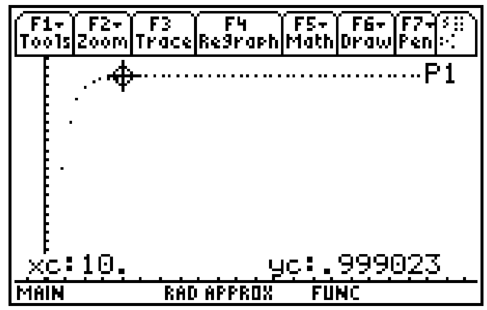
$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \ \ldots \ + ar^n + \ \ldots, \ a \neq 0.$$

A geometric series diverges if $|r| \geq 1$. It converges to the sum $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ if $0 < |r| < 1$.

These conditions must be used in determining whether a series diverges or converges. It is also worth noting that a series may diverge, but will not necessarily diverge to infinity. A value r that is less than –1 will result in a series that diverges and has terms whose signs alternate from positive to negative, not diverging to infinity.

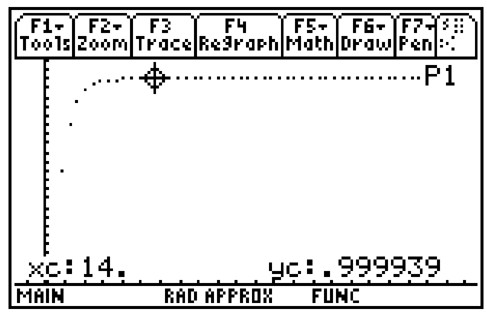
Another important note to students is that a series converges or diverges if the sequence of the partial sums converges to its sum or diverges.

To find the sum of a geometric series, students must take the limit of the nth sum. The series $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$, is a special case of the geometric series. One way of attempting this problem is to list the partial sums of the series then determine the nth sum.



The partial sums for this series would be

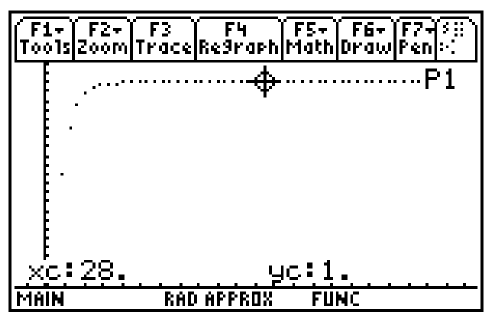
$S_1 = \frac{1}{2},\ S_2 = \frac{3}{4},\ S_3 = \frac{7}{8},\ S_4 = \frac{15}{16},\ . . .$



Guide students so they see that the denominator is 2 raised to the nth power and the numerator is always 1 less than the denominator. This gives $\frac{2^n - 1}{2^n}$.

Take the limit of the sum:

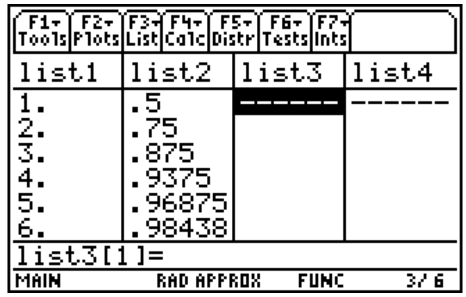
$$\lim_{n\to\infty}\frac{2^n - 1}{2^n} = 1$$



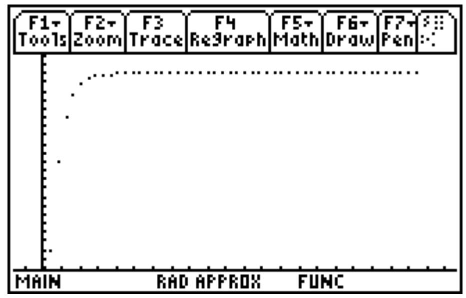
By the geometric series test, the series converges because $0 < \frac{1}{2} < 1$, and the sum is 1, by finding the limit.

Another way is to use the geometric series test that can be stated $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ or $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$. It is important to note to students the index of the series.

The graphs to the right represent the series $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$. Notice when traced, the values of the series approach 1.



To obtain the graph above, instruct students to select **Stats/List Editor** from the *Apps desktop*. If the **Stats/List Editor** is not available, students can use the **Data/Matrix Editor** to create lists. In **list1**, students should enter **seq(x,x,1,50)**, this function can be found by pressing [CATALOG]. In **list2,** students use **Σ((.5)^x,x,1,list1)**. This will list the first 50 partial sums of the series. Direct students to create a scatter plot to represent the data from **List Editor**. Enter **Plot Setup**, press **F1** to define the graph. Have students select *x* as **list1** and *y* as **list2**, then press [ENTER].



They repeat the same steps to find the series $\sum_{n=1}^{\infty} 2\left(\frac{3}{4}\right)^{n-1}$ and $\sum_{n=1}^{\infty} \frac{2}{3}\left(\frac{3}{2}\right)^{n-1}$; adjust the window accordingly.

Have students sketch the graphs in the space provided on their worksheet. (*Optional:* Encourage students to participate in a class discussion.)