Vectors and Projectile Motion on the TI-89

David K. Pierce
Tabor Academy
Marion, Massachusetts 02738
dpierce@taboracademy.org
(508) 748-2000 ext. 2243

This paper will investigate various properties of projectile motion using the TI-89 graphing scientific calculator. We begin by showing how the calculator manipulates three dimensional vectors. We then derive the standard equations of motion for projectile motion, and then apply these equations to first and second derivatives with the calculator. Lastly we show aspects of motion for displacement, velocity and acceleration for specific times of flight. The displacement, velocity, and acceleration will be shown on the graphing calculator.

Part I Three-dimensional vector algebra

Let us begin with the following three-dimensional vectors

$$\mathbf{a} = 2\,\mathbf{i} + \mathbf{j} - \mathbf{k} \qquad \mathbf{b} = \mathbf{i} + 4\,\mathbf{j} + 5\,\mathbf{k}$$

can be defined on the calculator by

{2,1,-1} -> a   {1, 4, 5} -> b

here the "->" means "store as" . Make sure all single letter variables are deleted on your calculator by "F1" followed by the number "8" . This clears out of the calculator's memory of single letter variables.

Now let $c = 2a - b = 2\,(2\,\mathbf{i} + \mathbf{j} - \mathbf{k}) - (\mathbf{i} + 4\,\mathbf{j} + 5\,\mathbf{k})$

$$= (4\,\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) - (\mathbf{i} + 4\,\mathbf{j} + 5\,\mathbf{k})$$

$$= 3\,\mathbf{i} - 2\mathbf{j} - 7\mathbf{k}$$

The magnitude of c is given by $c = \sqrt{3^2 + (-2)^2 + (-7)^2} = \sqrt{62} \approx 7.874$

The unit vector of c is defined to be

$$\mathbf{u}_C = \mathbf{c}/c = (3\,\mathbf{i} - 2\mathbf{j} - 7\mathbf{k})/\sqrt{62} = (.381\,\mathbf{i} - .254\,\mathbf{j} - .889\,\mathbf{k})$$

The dot product between **a** and **b** is given by

$$\mathbf{a} \circ \mathbf{b} = a_x b_x + a_y b_y + a_z b_z = a\,b\,\cos|\,\theta$$

$= (2)(1) + (1)(4) + (-1)(5) = 1$

In order to find the angle between a and b, the magnitudes of a and b must first be calculated: $a = \sqrt{(2)^2 + (1)^2 + (-1)^2} = \sqrt{6} \approx 2.54$, $b = \sqrt{(1)^2 + (4)^2 + (5)^2} = \sqrt{42} \approx 6.48$, so the angle θ is given by

$$\theta = \arccos((\mathbf{a} \circ \mathbf{b})/(ab)) = \arccos(1/(\sqrt{6}\sqrt{42})) = 86.4^{o}$$

The cross product of a x b =

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & 4 & 5 \end{vmatrix}$$

$$= \mathbf{i} \begin{vmatrix} 1 & -1 \\ 4 & 5 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 2 & -1 \\ 1 & 5 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix}$$

$$= \mathbf{i}\,(4 + 5) - \mathbf{j}\,(10 + 1) + \mathbf{k}\,(8 - 1)$$

$$= 9\,\mathbf{i} - 11\,\mathbf{j} + 7\,\mathbf{k}$$

The since **a** and **b** are both perpendicular to the cross product **a** x **b**, **a** o (**a** x **b**) and **b** o (**a** x **b**) are zero.

On the TI-89 calculator these operations are:
the vector c

2 a - b -> c
c {3, -2, -7}

the magnitude of c

√(dotp(c,c)) -> d
d √62
"green diamond", "Enter" d 7.874

the unit vector of c

c/d $\{3, -2, -7\}/\sqrt{62}$
"green diamond", "Enter" e {.381 -.254 -.889}

the dot product of a and b

dotp(a,b) 1

the magnitude of a and b

√(dotp(a,a)) -> f
√(dotp(b,b)) -> g

the angle between a and b

$\cos^{-1}$(dotp(a,b)/(f*g)) 86.4^{o}

the cross product of a and b

crossp(a,b) -> h
h {9. -11. 7.}

the dot product of a and a x b

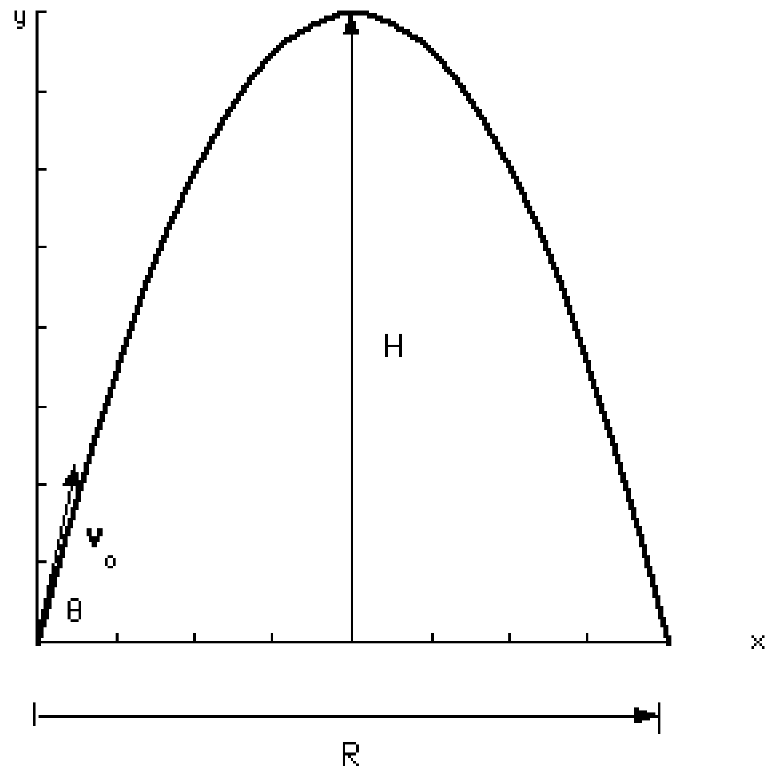
dotp(a,h) 0

the dot product of b and a x b

dotp(b,h) 0

Part II The algebraic equations of projectile motion

A projectile is fired with a speed of v_o at an angle of θ upward from the ground. The object reaches maximum height H and lands a distance R down-range.



The equations of motion are given by

x-direction equations

$a_x = 0$

$v_x = v_{xo}$

$x = v_{xo}\, t$

y-direction equations

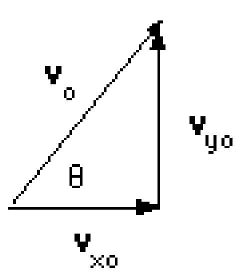
$a_y = -g$

$v_y = v_{yo} - g\, t$

$y = v_{yo}\, t - (1/2)\, g\, t^2$

The initial velocity $\mathbf{v}_o$ can be resolved into the x and y components

$\mathbf{v}_{xo}$ and $\mathbf{v}_{yo}$ in the following manner



$v_{yo} = v_o \sin\theta$

$v_{xo} = v_o \cos\theta$

At maximum height $v_y = 0$ so from $v_y = v_{yo} - g\,t = 0$, so the time t_{up} is found from $t_{up} = v_{yo}/g = v_o \sin\theta/g$. In order to find the maximum height H this time t_{up} is substituted into the equation

$$y_{max} = H = v_{yo}\,t - (1/2)\,g\,t^2 = v_o \sin\theta\,(v_o \sin\theta/g) - (1/2)\,g\,(v_o \sin\theta/g)^2$$

$$H = v_o^2 (\sin\theta)^2/g - (1/2)\,g\,(v_o^2 (\sin\theta)^2/g^2) = v_o^2 (\sin\theta)^2/2g$$

The total trip time is twice the time up for a parabolic flight

$$T = 2\,t_{up} = 2\,vo \sin\theta/g.$$

The range is given by $R = v_{xo}\,T = v_{xo}\,2\,t_{up} = (v_o \cos\theta)\,2\,v_o \sin\theta/g$

$$R = (v_o)^2\,2 \cos\theta\,\sin\theta/g = (v_o)^2 \sin(2\theta)/g$$

From the equations of motion $x = v_{xo}\,t$ and $y = v_{yo}\,t - (1/2)\,g\,t^2$
we can eliminate the time variable by substituting $t = x/v_o \cos\theta$ into the y equation:

$$y = v_{yo}\,t - (1/2)\,g\,t^2 = v_o \sin\theta\,(x/v_o \cos\theta) - (1/2)\,g\,(x/v_o \cos\theta)^2$$

$$= (\tan\theta)\,(x) - (1/2)\,g\,x^2/(v_o^2 \cos(\theta)^2)$$

This equation expresses the height of the object as a function of the initial speed v_o, angle θ, and x position. Let us find the range R by letting this new equation become 0, for when $y = 0$, the object is on the ground.

$$y = (\tan\theta)\,(x) - (1/2)\,g\,x^2/(v_o^2 \cos(\theta)^2)$$
$$y = (\tan\theta)\,(x) - (1/2)\,g\,x^2/(v_o^2 \cos(\theta)^2) = 0$$

$$(\tan\theta) - (1/2)\,g\,x/(v_o^2 \cos(\theta)^2) = 0$$

$$(\sin\theta)/(\cos\theta) = (1/2)\,g\,x/(v_o^2 \cos(\theta)^2)$$

$$2\,(\sin\theta) = g\,x/(v_o^2 \cos(\theta))$$

$$x = 2\,(\sin\theta)\,(v_o^2 \cos(\theta))/g = (\sin 2\theta)\,v_o^2/g$$

and this confirms our result that we derived earlier.
We plot some of the trajectories by going to the window and putting in the following conditions:

xmin = 0
xmax = 100000
xscl = 10000
ymin = 0
ymax = 100000
yscl = 10000
xres = 2

and the following equations of generated by using the copy and paste routine while changing the subscript of the angle θ_ι.

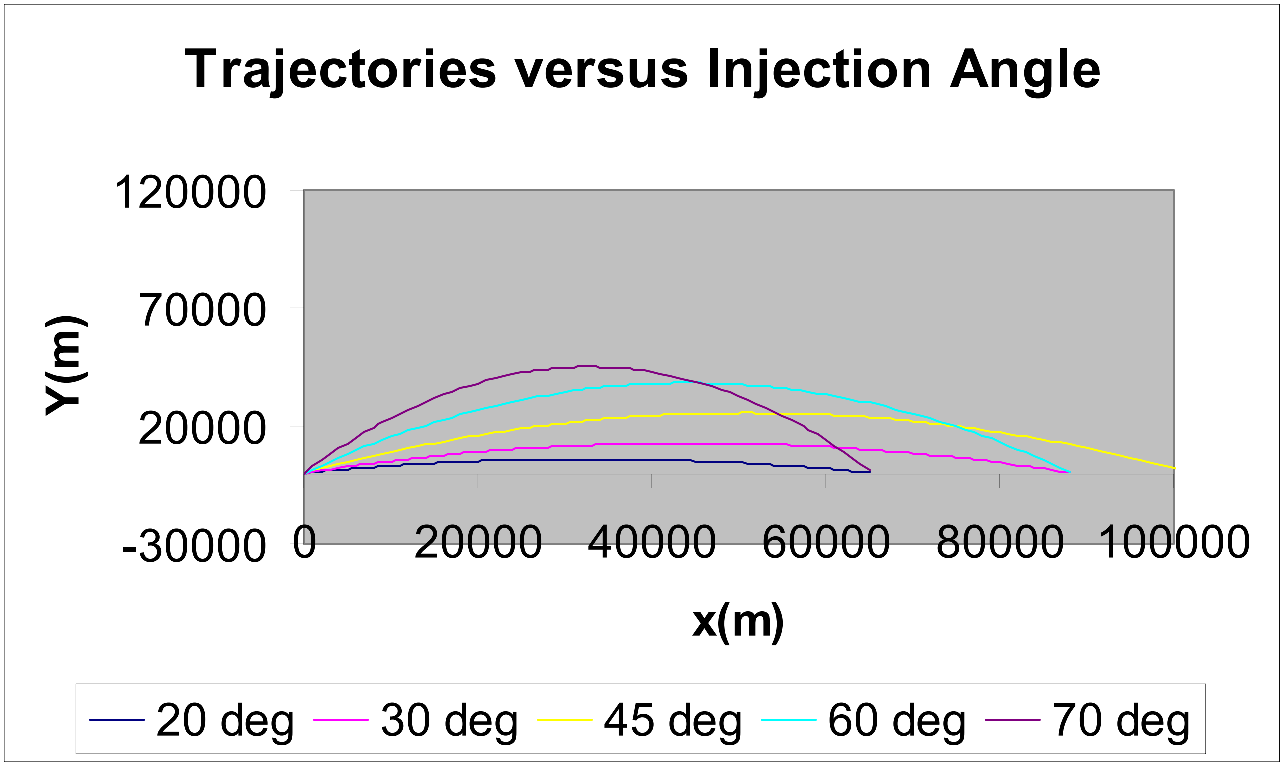
Y1 = tan(θ1)* x - .5 *g*x^2/(vo^2*(cos(θ1))^2)

Y2 = tan(θ2)* x - .5 *g*x^2/(vo^2*(cos(θ2))^2)
.
.
.
Y5 = tan(θ5)* x - .5 *g*x^2/(vo^2*(cos(θ5))^2)

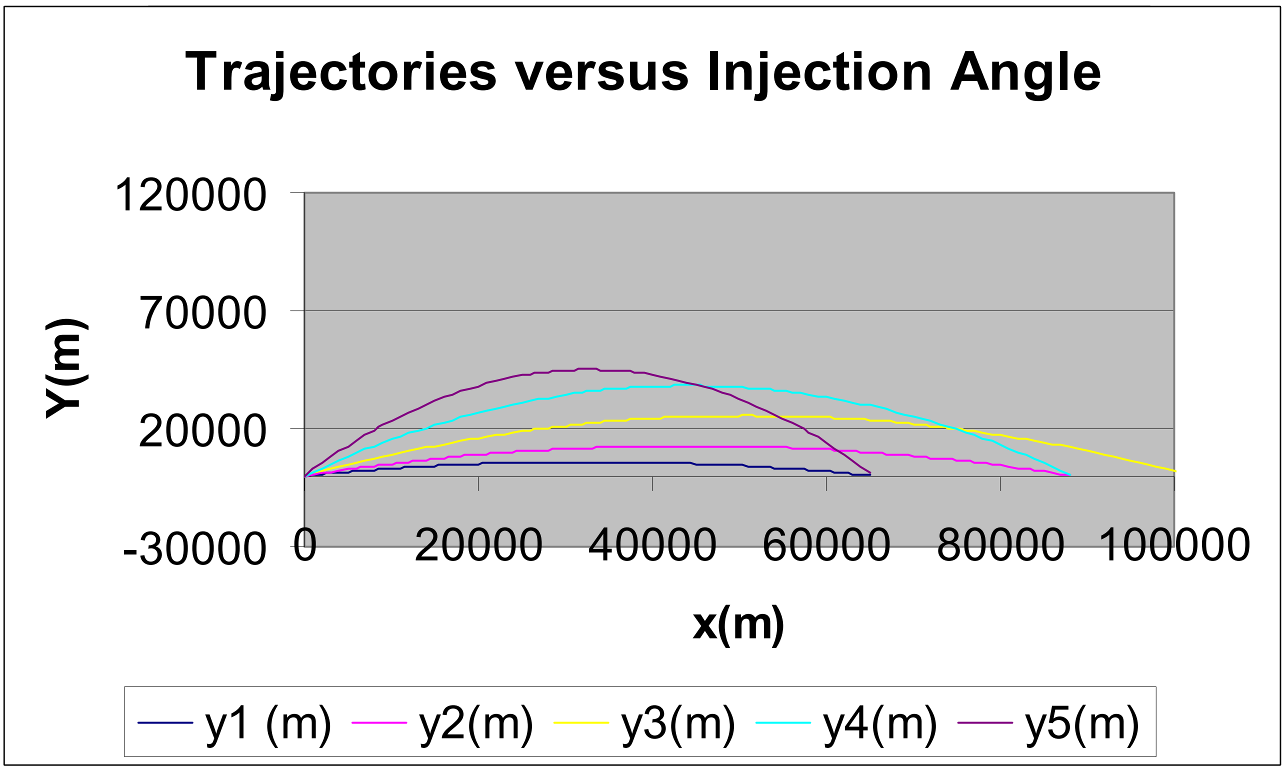
And now the values are stored in the calculator:

1000 -> vo
9.8 -> g
20 -> θ1
30 -> θ2
45 -> θ3
60 -> θ4
70 -> θ5

the following trajectories are plotted



x (m)	y1 (m)	y2(m)	y3(m)	y4(m)	y5(m)
0	0	0	0	0	0
1000	358.42	570.82	990.20	1712.45	2705.59
2000	705.74	1128.57	1960.80	3385.70	5327.40
3000	1041.97	1673.25	2911.80	5019.75	7865.44
4000	1367.09	2204.87	3843.20	6614.60	10319.70
5000	1681.12	2723.42	4755.00	8170.25	12690.18
6000	1984.05	3228.90	5647.20	9686.70	14976.89
7000	2275.88	3721.32	6519.80	11163.96	17179.82
8000	2556.62	4200.67	7372.80	12602.01	19298.97
9000	2826.25	4666.95	8206.20	14000.86	21334.34
10000	3084.79	5120.17	9020.00	15360.51	23285.94



We now take into account air resistance, where we assume that the dissipative force is proportional to the velocity of the object traveling through the air

$$\mathbf{F}_{fr} = - b\,\mathbf{v}$$

where $\mathbf{F}_{fr}$ is the frictional force of air resistance, b is the frictional coefficient of air resistance with units of N/(m/s), and v is the velocity of the object.

We begin with the example of a mass m dropped from rest at a height of

5000 m. We assume (incorrectly) that the gravitational acceleration is constant during the fall. Initially, the mass is at rest, so there is no frictional force present. As time goes by and the object picks up speed in the downward direction, the frictional force $\mathbf{F}_{fr}$ begins to cause the acceleration increases to decrease until the terminal velocity.

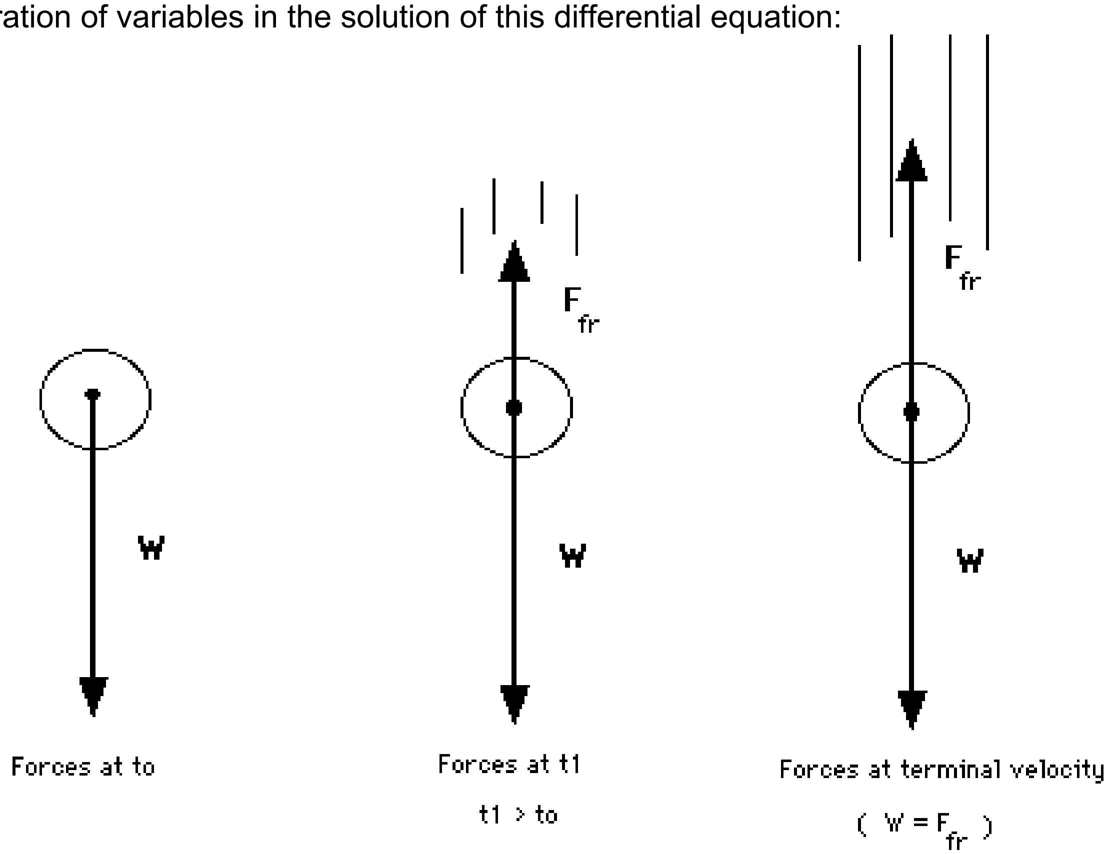
It is assumed that the positive sign is used for the "upward" direction of motion. This means that the net velocities and accelerations will be negative.

$$\mathbf{W} + \mathbf{F}_{fr} = \mathbf{F}_{net} = m\,\mathbf{a}$$

$$- mg - b\,v = m\,a$$

note that the first term is the weight of the falling object, the second term is the frictional force of air resistance. The direction is upward because the velocity is downward and is negative. The term on the right is the net force and from Newton's Second Law of Motion, $\mathbf{F} = m\,\mathbf{a}$.

Since the weight will always have a larger magnitude that the frictional force, the direction of acceleration will always be negative (downward). We employ separation of variables in the solution of this differential equation:



$$- (mg + b v) = m a = m dv/dt$$

$$-(b/m) dt = dv (1/((mg/b) + v))$$

$$-\int_0^t (b/m) dt = \int_0^v dv (1/((mg/b) + v))$$

$$- (b/m) t = \ln (mg/b + v) \Big|_0^v$$

$$- (b/m) t = \ln (mg/b + v) - \ln (mg/b) = \ln ((mg/b + v)/(mg/b))$$

$$e^{-(bt/m)} = ((mg/b) + v)/(mg/b)$$

$$(mg/b) e^{-(bm/t)} = (mg/b) + v$$

$$v = (mg/b) e^{-(bt/m)} - (mg/b)$$

$$v = (mg/b)(e^{-(bt/m)} - 1)$$

We now show how this velocity varies with time with two practical examples. The first assumes a constant value of mass m = 4 kg while the coefficients of air resistance vary (b = .1, .2, and .5 N/(m/s)), and the second set of formulas will be given by a constant coefficient of air resistance b = .1 N/(m/s). Pull up the y equations (green diamond F1) and select the following function assignments:

y1 = (4*9.8/.1)*(exp^(-(.1*x/4) - 1)

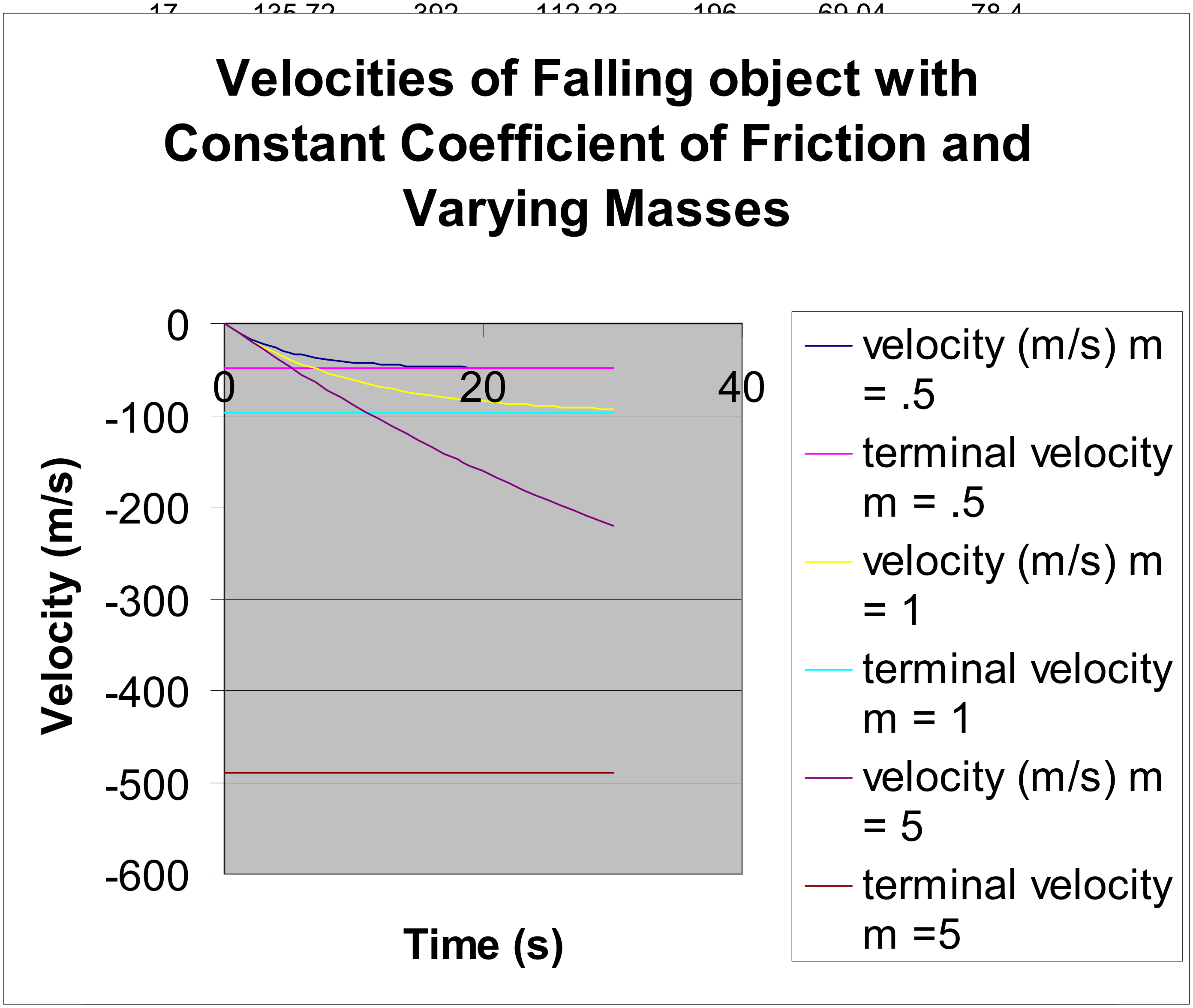
y2 = (4*9.8/.2)*(exp^(-(.2*x/4) - 1)

y3 = (4*9.8/.5)*(exp^(-(.5*x/4) - 1)

and the window assignments are:

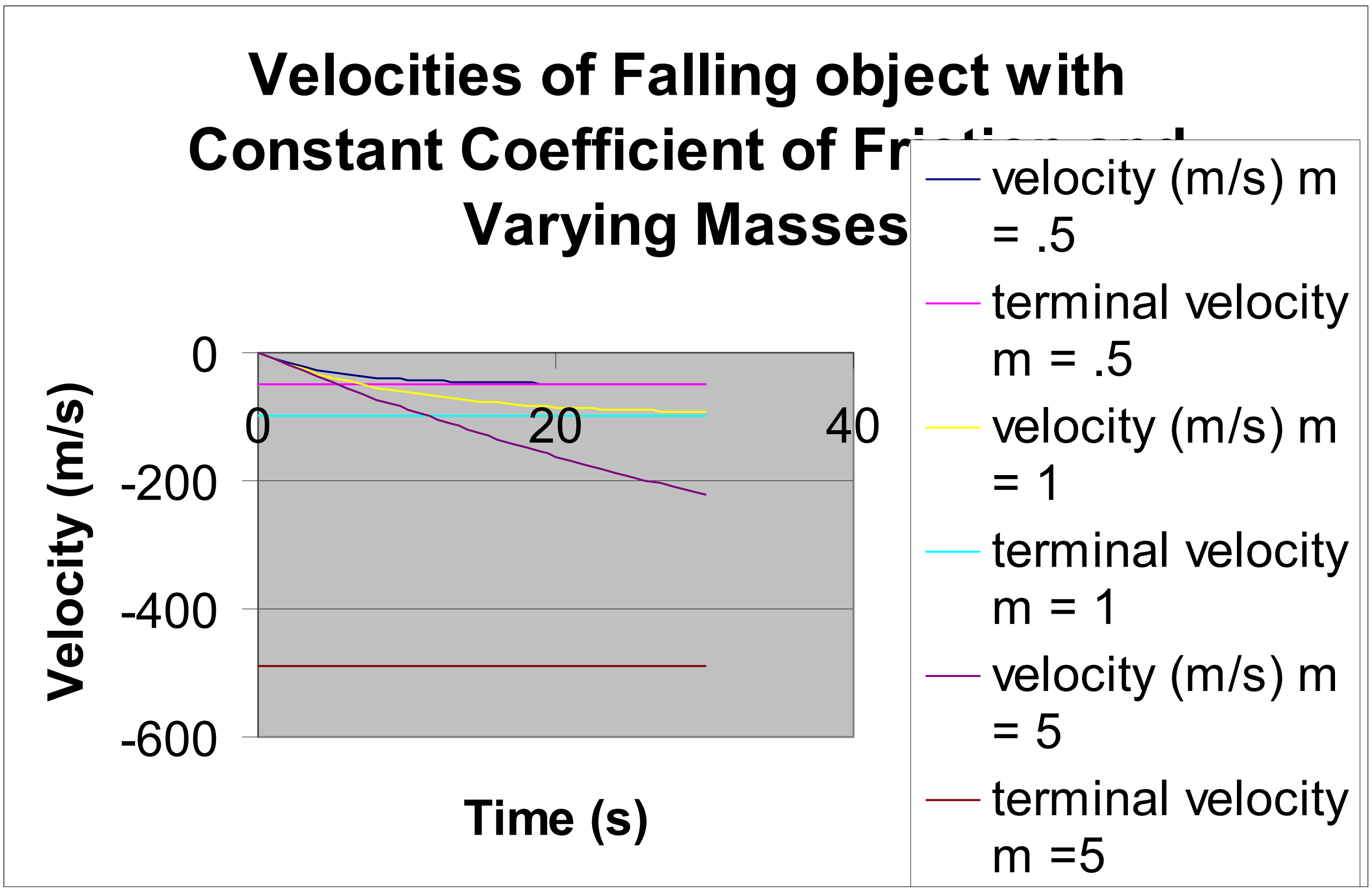
xmin = .1
xmax = 50
xscl = 10
ymin = -500
ymax = 0
yscl = 10
xres = 2

Time (s)	v (m/s)	term v (m/s)	v (m/s)	term v (m/s)	v (m/s)	term v (m/s)
m = 4 kg	b = .1	b = .1	b =.2	b =.2	b= .5	b= .5
0	0	v (m/s)	0	-196	0	-78.4
1	-9.68	-392	-9.56	-196	-9.21	-78.4
2	-19.12	-392	-18.65	-196	-17.34	-78.4
3	-28.32	-392	-27.30	-196	-24.52	-78.4
4	-37.30	-392	-35.53	-196	-30.85	-78.4
5	-46.06	-392	-43.36	-196	-36.44	-78.4
6	-54.60	-392	-50.80	-196	-41.37	-78.4
7	-62.93	-392	-57.88	-196	-45.72	-78.4
8	-71.06	-392	-64.62	-196	-49.56	-78.4
9	-78.98	-392	-71.02	-196	-52.95	-78.4
10	-86.71	-392	-77.12	-196	-55.94	-78.4
11	-94.25	-392	-82.92	-196	-58.58	-78.4
12	-101.60	-392	-88.43	-196	-60.91	-78.4
13	-108.77	-392	-93.68	-196	-62.96	-78.4
14	-115.76	-392	-98.67	-196	-64.78	-78.4
15	-122.58	-392	-103.42	-196	-66.38	-78.4
16	-129.23	-392	-107.93	-196	-67.79	-78.4
17	-135.72	-392	-112.23	-196	-69.04	-78.4



We can see from the graph of Velocities of Falling Objects with different Coefficients of Friction (air resistance) that the smaller values of b take much longer to reach terminal velocity than for larger values. Even for the small assignment of b = .5, the 4.0 kg object arrives within 1 m/s of the -78.4 m/s in 35 seconds, while at b = .2 it takes 106 seconds, and for b = .1 m/s it takes 239 seconds.

With the graph of Velocities of Falling object with Constant Coefficient of Friction and Varying Masses, b = .1 N/(m/s) while m = .50, 1.0, and 5.0 kg. The 5.0 kg mass reaches within 1 m/s of terminal velocity at 275 seconds, a 1.0 kg mass in 46 seconds, and .5 kg mass in 20 seconds



If we return to the general velocity equation

$$v = (mg/b)(e^{-(bt/m)} - 1)$$

we notice that this expression for velocity gives v = 0 when t = 0 and v = -mg/b when t -> ∞. This, of course, is the terminal velocity.

Time (s) b = .1	velocity (m/s) m = .5	terminal velocity m = .5	velocity (m/s) m = 1	terminal velocity m = 1	velocity (m/s) m = 5	terminal velocity m =5
0	0	-49	0	-98	0	-490
1	-8.88	-49	-9.33	-98	-9.70	-490
2	-16.15	-49	-17.76	-98	-19.21	-490
3	-22.11	-49	-25.40	-98	-28.54	-490
4	-26.98	-49	-32.31	-98	-37.67	-490
5	-30.97	-49	-38.56	-98	-46.63	-490
6	-34.24	-49	-44.22	-98	-55.41	-490
7	-36.92	-49	-49.33	-98	-64.01	-490
8	-39.11	-49	-53.97	-98	-72.45	-490
9	-40.90	-49	-58.16	-98	-80.72	-490
10	-42.37	-49	-61.95	-98	-88.82	-490
11	-43.57	-49	-65.38	-98	-96.77	-490
12	-44.55	-49	-68.48	-98	-104.55	-490
13	-45.36	-49	-71.29	-98	-112.18	-490
14	-46.02	-49	-73.83	-98	-119.67	-490
15	-46.56	-49	-76.13	-98	-127.00	-490
16	-47.00	-49	-78.21	-98	-134.19	-490
17	-47.36	-49	-80.10	-98	-141.23	-490
18	-47.66	-49	-81.80	-98	-148.14	-490
19	-47.90	-49	-83.34	-98	-154.91	-490
20	-48.10	-49	-84.74	-98	-161.54	-490
21	-48.27	-49	-86.00	-98	-168.05	-490
22	-48.40	-49	-87.14	-98	-174.42	-490
23	-48.51	-49	-88.17	-98	-180.67	-490
24	-48.60	-49	-89.11	-98	-186.80	-490
25	-48.67	-49	-89.96	-98	-192.80	-490
26	-48.73	-49	-90.72	-98	-198.68	-490
27	-48.78	-49	-91.41	-98	-204.45	-490
28	-48.82	-49	-92.04	-98	-210.11	-490
29	-48.85	-49	-92.61	-98	-215.65	-490
30	-48.88	-49	-93.12	-98	-221.08	-490

We now return to the problem of finding height as a function of time from this equation.

$$v = dy/dt = (mg/b)(e^{-(bt/m)} - 1)$$

$$dy = (mg/b)(e^{-(bt/m)} - 1)\, dt$$

$$\int_{y_o}^{y} dy = \int_{0}^{t} (mg/b)(e^{-(bt/m)} - 1)\, dt$$

$$y = y_o - (m^2g/b^2)\,((e^{-(bt/m)} - 1) - (mg/b)\, t$$

This equation bears little resemblance to the familiar frictionless equation

$$y = y_o + v_{yo}\, t - (1/2)\, g\, t^2.$$

Since the object falls from rest, $v_{yo} = 0$ and the equation becomes

$$y = y_o - (1/2)\, g\, t^2.$$

We now show why it is that the more general equation reduces to the familiar result. The Taylor Series expansion for the function $f(x) = e^x$ is

$$f(x) = e^x = 1 + x + x^2/2 + x^3/3 + ... + x^n/n + ...$$

in this formula, $x = -(bt/m)$ so if we take the first three terms of this expansion,

$$e^{-(bt/m)} = 1 - (bt/m) + (1/2)\,(bt/m)^2 \text{ so}$$

$$y = y_o - (m^2g/b^2)\,((e^{-(bt/m)} - 1) - (mg/b)\, t$$

$$= y_o - (m^2g/b^2)\,((1 - (bt/m) + (1/2)\,(bt/m)^2 - 1) - (mg/b)\, t$$

$$= y_o - (m^2g/b^2)\,((- (bt/m) + (1/2)\,(bt/m)^2\,) - (mg/b)\, t$$

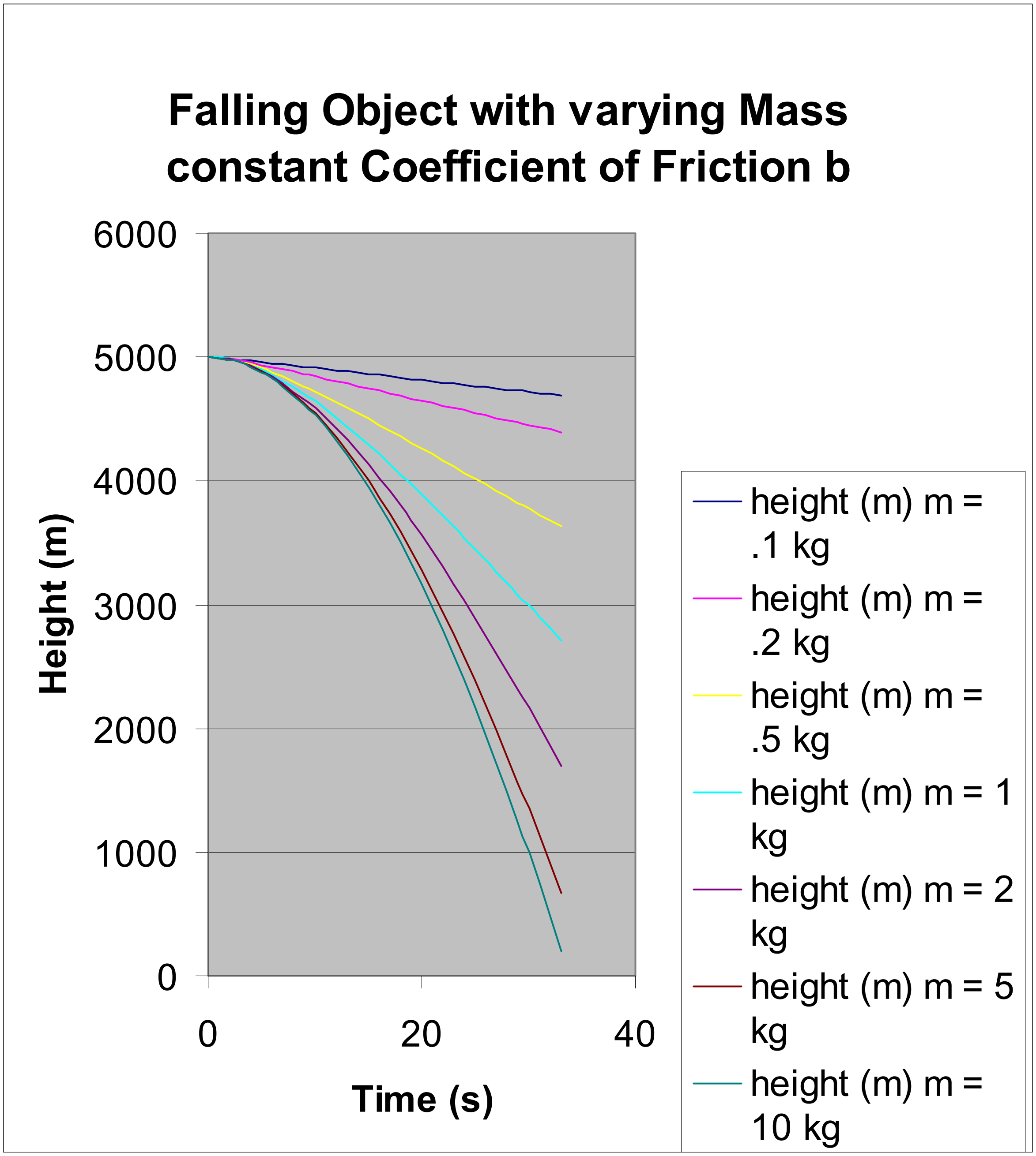
$$= y_o + (mg/b)\, t + (1/2)\,(- (m^2g/b^2))(bt/m)^2\,) - (mg/b)\, t$$

$$y = y_o - (1/2)\, g\, t^2\ .$$

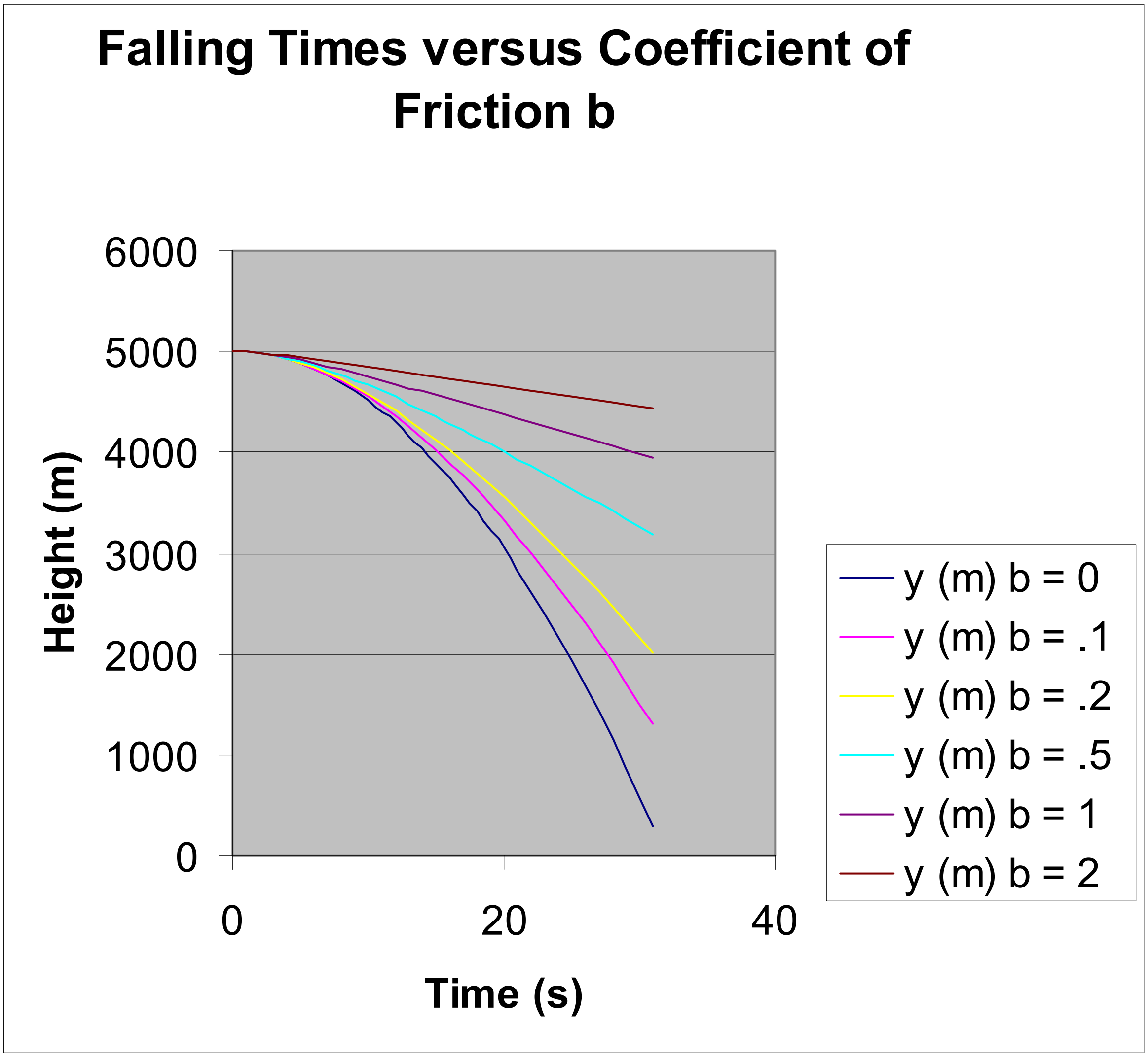
The equation $y = y_o - (m^2g/b^2)\,((e^{-(bt/m)} - 1) - (mg/b)\, t$ gives the position of the dropped object at a height $y = y_o$, mass m, $g = 9.8\ m/s^2$, and a coefficient of friction b in units of N/(m/s). In our problem, the 4.0 kg mass is dropped from rest at a height of 5000 m, and the values for b = 0, .1, .2, .5, 1 N/(m/s). Using the solve option with the TI-89 calculator, the trip times are t = 31.9 s, 36.8 s, 43.2 s, 71.8 s, and 131.6 s. The position versus time graph for m = 4 kg and b = 0, .1, .2, .5, and 1 N/(m/s) along with the data table for this graph are given below.

Similar computations can be found for an object of varying mass dropped from 5000 m with a constant coefficient of air resistance b = .1 N/(m/s) for m = .1, .2, .5, 1, 2, 5, and 10 kg. The trip times for these different masses are 511.2, 257.1, 107.0, 61.0, 43.2, 35.7, and 33.7 seconds. As the mass tends toward infinity, the falling time approaches the frictionless trip time of 31.9 seconds. The graph of Position versus Time for b = .1 N/(m/s) and b = .1, .2, .5, 1, 2, .5, 1, 2, 5 and 10 kg follows below.

This search for falling times can be performed on the TI-89 calculator. Solve can be found by F2 option 1, solve(5000 = (m^2g/b^2) $((e^{-(bt/m)} - 1) + (mg/b)$ t,t) and by using .1 ->b, and then .1 -> m and select the "Enter" button. Several seconds will go by before the calculator gives you the answer. Repeat this procedure each time replacing a new mass from the list provided.



Time (s) b = .1 N/(m/s)	height (m) m = .1 kg	height (m) m = .2 kg	height (m) m = .5 kg	height (m) m = 1 kg	height (m) m = 2 kg	height (m) m = 5 kg	height (m) m = 10 kg
0	5000	5000	5000	5000	5000	5000	5000
1	4996.39	4995.82	4995.41	4995.26	4995.18	4995.13	4995.12
2	4988.87	4985.58	4982.77	4981.64	4981.04	4980.66	4980.53
3	4979.91	4971.65	4963.54	4960.00	4958.02	4956.77	4956.34
4	4970.42	4955.49	4938.91	4931.09	4926.58	4923.65	4922.63
5	4960.73	4937.98	4909.87	4895.60	4887.10	4881.48	4879.52
6	4950.98	4919.65	4877.21	4854.16	4839.99	4830.45	4827.08
7	4941.19	4900.82	4841.58	4807.35	4785.62	4770.72	4765.41
8	4931.40	4881.68	4803.54	4755.66	4724.35	4702.48	4694.60
9	4921.60	4862.36	4763.50	4699.56	4656.50	4625.88	4614.74
10	4911.80	4842.94	4721.84	4639.48	4582.40	4541.10	4525.93
11	4902.00	4823.44	4678.85	4575.79	4502.36	4448.29	4428.25
12	4892.20	4803.90	4634.77	4508.83	4416.66	4347.62	4321.80
13	4882.40	4784.34	4589.80	4438.92	4325.58	4239.24	4206.65
14	4872.60	4764.76	4544.10	4366.33	4229.39	4123.30	4082.89
15	4862.80	4745.18	4497.80	4291.33	4128.32	3999.95	3950.62
16	4853.00	4725.59	4451.01	4214.14	4022.63	3869.35	3809.91
17	4843.20	4705.99	4403.82	4134.97	3912.53	3731.63	3660.85
18	4833.40	4686.40	4356.31	4054.01	3798.25	3586.93	3503.52
19	4823.60	4666.80	4308.52	3971.42	3679.98	3435.40	3338.00
20	4813.80	4647.20	4260.51	3887.37	3557.91	3277.16	3164.39
21	4804.00	4627.60	4212.33	3801.99	3432.24	3112.35	2982.74
22	4794.20	4608.00	4163.99	3715.41	3303.15	2941.11	2793.16
23	4784.40	4588.40	4115.54	3627.75	3170.78	2763.55	2595.71
24	4774.60	4568.80	4066.98	3539.10	3035.32	2579.81	2390.47
25	4764.80	4549.20	4018.35	3449.56	2896.90	2390.00	2177.52
26	4755.00	4529.60	3969.65	3359.21	2755.68	2194.25	1956.94
27	4745.20	4510.00	3920.89	3268.14	2611.78	1992.67	1728.81
28	4735.40	4490.40	3872.09	3176.41	2465.34	1785.38	1493.19
29	4725.60	4470.80	3823.26	3084.08	2316.48	1572.49	1250.17
30	4715.80	4451.20	3774.39	2991.21	2165.33	1354.11	999.81
31	4706.00	4431.60	3725.50	2897.85	2011.99	1130.36	742.20
32	4696.20	4412.00	3676.59	2804.05	1856.57	901.34	477.39
33	4686.40	4392.40	3627.67	2709.85	1699.16	667.14	205.47
34	4676.60	4372.80	3578.73	2615.29	1539.88	427.88	
35	4666.80	4353.20	3529.78	2520.41	1378.81	183.66	
36	4657.00	4333.60	3480.82	2425.22	1216.03		
37	4647.20	4314.00	3431.85	2329.77	1051.63		
38	4637.40	4294.40	3382.88	2234.08	885.69		
39	4627.60	4274.80	3333.90	2138.16	718.29		
40	4617.80	4255.20	3284.92	2042.05	549.49		
41	4608.00	4235.60	3235.93	1945.76	379.36		
42	4598.20	4216.00	3186.94	1849.30	207.97		
43	4588.40	4196.40	3137.95	1752.70	35.38		
44	4578.60	4176.80	3088.96	1655.97			
45	4568.80	4157.20	3039.97	1559.11			


Falling Times versus Coefficient of Friction b
6000
5000
4000
3000
2000
1000
0
Height (m)
0
20
40
Time (s)
y (m) b = 0
y (m) b = .1
y (m) b = .2
y (m) b = .5
y (m) b = 1
y (m) b = 2

Time (s) m = 4.0 kg	y (m) b = 0	y (m) b = .1	y (m) b = .2	y (m) b = .5	y (m) b = 1	y (m) b = 2
0	5000	5000	5000	5000	5000	5000
1	4995.1	4995.14	4995.18	4995.30	4995.48	4995.82
2	4980.4	4980.72	4981.04	4981.94	4983.30	4985.58
3	4955.9	4956.98	4958.02	4960.93	4965.13	4971.65
4	4921.6	4924.15	4926.58	4933.18	4942.32	4955.49
5	4877.5	4882.45	4887.10	4899.48	4915.88	4937.98
6	4823.6	4832.10	4839.99	4860.53	4886.61	4919.65
7	4759.9	4773.31	4785.62	4816.94	4855.15	4900.82
8	4686.4	4706.30	4724.35	4769.27	4821.98	4881.68
9	4603.1	4631.27	4656.50	4717.98	4787.47	4862.36
10	4510	4548.40	4582.40	4663.50	4751.93	4842.94
11	4407.1	4457.91	4502.36	4606.22	4715.58	4823.44
12	4294.4	4359.97	4416.66	4546.45	4678.59	4803.90
13	4171.9	4254.77	4325.58	4484.50	4641.12	4784.34
14	4039.6	4142.49	4229.39	4420.61	4603.27	4764.76
15	3897.5	4023.30	4128.32	4355.02	4565.11	4745.18
16	3745.6	3897.38	4022.63	4287.92	4526.73	4725.59
17	3583.9	3764.89	3912.53	4219.49	4488.16	4705.99
18	3412.4	3625.99	3798.25	4149.89	4449.46	4686.40
19	3231.1	3480.84	3679.98	4079.26	4410.64	4666.80
20	3040	3329.60	3557.91	4007.72	4371.74	4647.20
21	2839.1	3172.41	3432.24	3935.37	4332.78	4627.60
22	2628.4	3009.43	3303.15	3862.30	4293.76	4608.00
23	2407.9	2840.79	3170.78	3788.62	4254.70	4588.40
24	2177.6	2666.63	3035.32	3714.37	4215.61	4568.80
25	1937.5	2487.10	2896.90	3639.64	4176.50	4549.20
26	1687.6	2302.32	2755.68	3564.48	4137.36	4529.60
27	1427.9	2112.43	2611.78	3488.94	4098.22	4510.00
28	1158.4	1917.54	2465.34	3413.06	4059.06	4490.40
29	879.1	1717.79	2316.48	3336.89	4019.89	4470.80
30	590	1513.29	2165.33	3260.45	3980.71	4451.20
31	291.1	1304.16	2011.99	3183.78	3941.53	4431.60
32		1090.52	1856.57	3106.91	3902.35	4412.00
33		872.48	1699.16	3029.86	3863.16	4392.40
34		650.13	1539.88	2952.65	3823.97	4372.80
35		423.60	1378.81	2875.30	3784.78	4353.20
36		192.99	1216.03	2797.83	3745.58	4333.60
37			1051.63	2720.25	3706.38	4314.00
38			885.69	2642.57	3667.19	4294.40
39			718.29	2564.81	3627.99	4274.80
40			549.49	2486.97	3588.79	4255.20

When we first encountered projectile motion the motion can be broken up into two sets of mutually exclusive equations

equations in the x-direction

$a_x = 0$

$v_x = v_{xo}$

$x = xo + vxo\ t$

equations in the y-direction

$a_y = -g = -9.8\ m/s^2$

$v_y = v_{yo} - g\ t$

$y = y_o + v_{yo}\ t - (1/2)\ g\ t^2$

Simply by specifying the firing speed v_o and the injection angle θ, we know where the object is, its velocity and its acceleration at any time t.

Things get more complicated when the force of air resistance $\mathbf{F}_{fr} = -b\ \mathbf{v}$ is introduced. Now both the x and y directions of motion have to be considered since $\mathbf{v} = v_x\ \mathbf{i} + v_y\ \mathbf{j}$. The net force

$$\mathbf{F}_{net} = -b\ \mathbf{v} - m\mathbf{g}$$

$$\mathbf{F}_{net} = m\ \mathbf{a} = m\ (a_x\ \mathbf{i} + a_y\ \mathbf{j}) = -b\ (v_x\ \mathbf{i} + v_y\ \mathbf{j}) - mg\ \mathbf{j}$$

equations in the x-direction

$a_x = -(b/m)\ v_x$

$v_x = v_{xo}$

$x = xo + vxo\ t$

equations in the y-direction

$a_y = -(b/m)\ v_y - g$

$v_y = v_{yo} - g\ t$

$y = y_o + v_{yo}\ t - (1/2)\ g\ t^2$

$$a_x = dv_x/dt = -(b/m)\ v_x$$

$$\int_{v_{xo}}^{vx} dv_x/v_x = -(b/m) \int_0^t dt$$

$$\ln\ (vx/vxo) = -b\ t/m$$

$$v_x\ (t) = v_{xo}\ e^{-bt/m}$$

$$v_x(t) = dx/dt = v_{xo}\ e^{-bt/m}$$

x t

$$a_y = dv_y/dt = -(b/m)\ v_y - g$$

$$a_y = dv_y/dt = -(b/m)\ v_y - g$$

$$\int_{v_{yo}}^{vy} dv_y\ /(v_y + mg/b) = -(b/m) \int_0^t dt$$

$$v_y\ (t) = (vyo + mg/b)e^{-bt/m} - mg/b$$

$$v_y\ (t) = dy/dt = (vyo + mg/b)e^{-bt/m} - mg/b$$

y t t

$$\int_o dx = v_{xo} \int_o e^{-bt/m}\, dt \qquad \int_o dy = (vyo + mg/b) \int_o e^{-bt/m}\, dt - mg/b \int_o dt$$

$$x(t) = (m\, v_{xo}/b)(1 - e^{-bt/m}) \qquad y(t) = (m^2 g/b^2 + m\, v_{yo}/b)(1 - e^{-bt/m}) - m\, g\, t/b$$

These are parametric equations for the independent parameter time t. The y equation may be expressed as a function of x instead through direct substitution.

$$y = ((m\, g / b\, v_{xo}) + (v_{yo}/v_{xo}))\, x - (m^2 g/b^2) \ln(m\, v_{xo} /(m\, v_{xo} - b\, x)).$$ [1]

We will use the TI-89 to show five trajectories: when b = 0, .05, .1, .2, and .5 with the object's mass m = 4 kg, g = 9.8 m/s^2, the injection speed v_o = 1000 m/s, and the injection angle of $\theta = 60^0$.

We begin by placing the above lengthy equation for y into the calculator by calling up y1(x) and putting in the above equation. Then store the following constants:

vo cos (θ) -> vxo
vo sin (θ) -> vyo
4 -> m
9.8 -> g
60 -> θ
vo -> 1000

The window should be set at

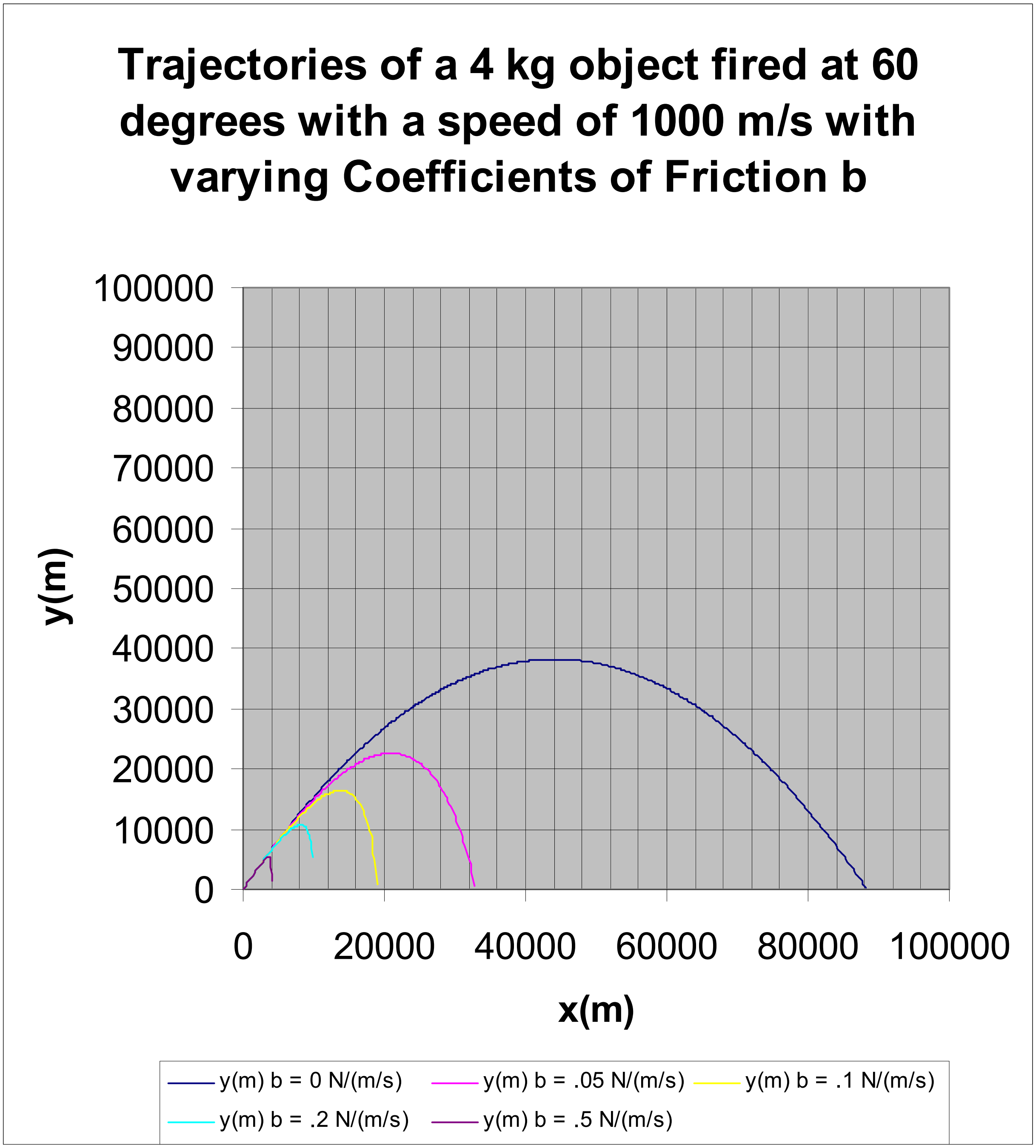
xmin = 0
xmax = 70000
xscl = 10000
ymin = 0
ymax = 40000
yscl = 10000

1. Mechanics, Keith Symon, Addison-Wesley, third edition, p. 113.

An interesting calculation to be done is to graph the trajectories such that maximum range is given for velocity. As it turns out, the larger the object's velocity, the smaller the injection angle in order to generate the maximum range. This is not an easy thing to demonstrate. We begin by starting with the equation for y(t) where b = .1 N/(m/s), m = 2 kg

$$y(t) = (m^2 g/b^2 + m\, v_{yo}/b)(1 - e^{-bt/m}) - m\, g\, t/b$$

and we also use the equation for x(t) as well



$$x(t) = (m\, v_{xo}/b)\,(1 - e^{-bt/m})$$

The x position is assigned to $y1 = (m\, v_{xo}/b)\,(1 - e^{-bx/m})$ where the variable x is replaced by time t. Then $y2 = (m^2\, g/b^2 + m\, v_{yo}/b)(1 - e^{-bx/m}) - m\, g\, x/b$. A search for the maximum value to y1 is begun as v_o = 500 m/s. Different values of injection angle θ where tried and from this rather tedious procedure, the maximum range is found. For v_o = 500 m/s, θ = 30º; for v_o = 1000 m/s, θ =22º; and finally for v_o = 2000 m/s, θ = 16º. With this information in hand, we can plot the height y as a function of x,

$$y = \left(\left(m g / b v_{xo} \right) + \left(v_{yo}/v_{xo} \right) \right) x - \left(m^2 g/b^2 \right) \ln\left(m v_{xo} / \left(m v_{xo} - b x \right) \right).$$

where three equations are plotted.