Exponential Growth

ID: 8254

Time required
45 minutes

Activity Overview

The main objective of the activity is to find an approximation for the value of the mathematical constant e and to apply it to exponential growth and decay problems. To accomplish this, students are asked to search for the base, b, that defines a function $f(x) = b^x$ with the property that at any point on the graph, the slope of the tangent line (instantaneous rate of change) is equal to f(x). The result is approximating the value of e—Euler's number and the base of the natural logarithms.

Topic: Exponential & Logarithmic Functions & Equations

- *Graph exponential functions of the form $f(x) = b^x$.*
- *Evaluate the exponential function $f(x) = b^x$ for any value of x.*
- *Calculate the doubling time or half-life in a problem involving exponential growth or decay.*

Teacher Preparation and Notes

- *Students encounter the exponential constant e at various levels in their mathematics schooling. It may happen well before they reach calculus, and it is often used without an appreciation for where it originates (or why it is important). A good time to use this activity is when students first encounter e, but it is also appropriate for Precalculus and Calculus students when they are studying derivatives and instantaneous rate of change.*
- *Prerequisites for the students are: familiarity with the handheld (grabbing and dragging points); an understanding of functions and function notation (both "y =" and "f(x) ="); and an intuitive understanding of rate of change.*
- ***To download the student and solution TI-Nspire documents (.tns files) and student worksheet, go to education.ti.com/exchange and enter "8254" in the search box.***

Associated Materials

- *ExpGrowth_Student.doc*
- *ExpGrowth.tns*
- *ExpGrowth_Soln.tns*

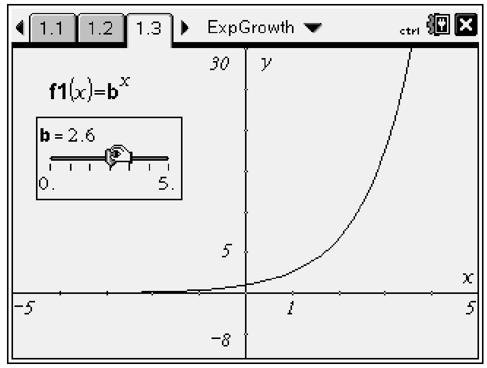
Suggested Related Activities

To download any activity listed, go to education.ti.com/exchange and enter the number in the quick search box.

- *NUMB3RS – Season 1 – "Identity Crisis" – Exponential Growth (TI-84 Plus and TI-Navigator) — 7727*
- *Half-Life (TI-Nspire technology) — 9288*

Problem 1

The activity begins with an investigation of how the value of b affects the shape of the graph of **f1**(x) = b^x shown on page 1.3. Students control the value of b by changing the value of the slider and are asked to make observations and draw conclusions.



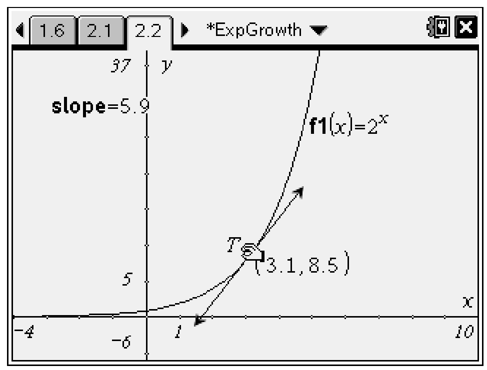
The last question posed in this problem asks students to explain why the value of b cannot be negative. It may be worthwhile to discuss this in a whole-class setting.

Solutions

- Answers may vary. Possible observations:
  graph gets "steeper" as b increases and "flatter" as b decreases; always passes through the point (0, 1); increasing when $b > 1$ and decreasing when $0 < b < 1$
- $b = 1$
- Answers may vary. Possible explanation: Even roots of negative numbers are not real numbers. Consider, for example, $(-1)^{0.5} = \sqrt{-1}$, which is not a real number.

Problem 2

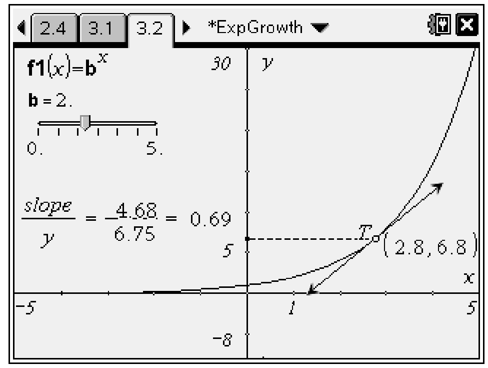
In this problem, students work specifically with the graph of **f1**(x) = 2^x with a tangent line to the curve at a point T. As they drag T along the curve, they will explore the relationship between the slope of the tangent line at point T and the value of the function at point T.



Solutions

- the slope is less than **f1**(x)
- Answers may vary. Possible observations: slope is always positive; as x increases, the slope increases; curve never reaches the x-axis

Problem 3



Problem 3 seeks to combine the two diagrams from Problems 1 and 2 into one dynamic exploration. Here, students can first change the value of the slider to set the value of *b* and then drag point *T* along the curve to explore the slope of the tangent line. They are again asked to observe the changing values of the slope of the tangent line and the value of the function—and how they are related.

Students will discover that there is *exactly* one value of *b* for which the slope of the tangent and value of the function are equal—and that this value is a very interesting number!

Solutions

- $b \approx 2.718$
- Answers will vary.

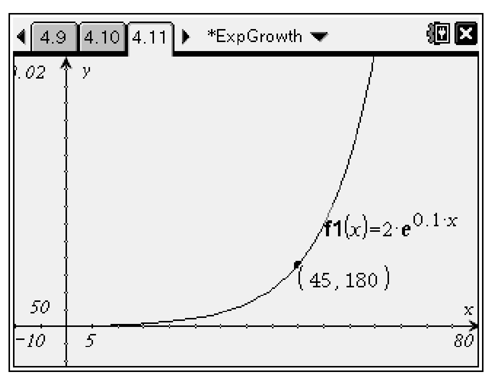
Applications

Students are given a series of application problems. A *Calculator* application page is given at the bottom of each of the question page. The last exercise asks students to use a graph to aid them.

Solutions

1. Modeling equation: $P = 1{,}000e^{0.05t}$ (where *P* is the value and *t* is the time in years); one year: \$1,051.27; two years: \$1,105.17; five years: \$1,284.03
2. Modeling equation: $P = 500e^{0.5 \cdot 24}$ (where *P* is population); about 81,000,000
3. Modeling equation: $P = 1{,}000{,}000e^{-0.15 \cdot 10}$ (where *P* is the volume); about 22.3%
4. Modeling equation for growing snowball: $P = 2e^{0.1t}$ (where *P* is the weight and *t* is the time in seconds); 10 seconds: 5.43 pounds; 20 seconds: 14.78 pounds; 45 seconds: 180.03 pounds; 1 minute: 806.86 pounds

   Possible limitations: the modeling equation might not be appropriate after too long a period of time, for example—the snowball may break apart if it gets too big, or it might reach the end of the hill.

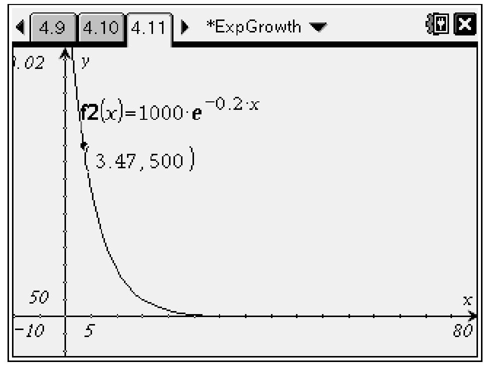
   

   To use the graph of the function to find answers to the questions, you can use the **Point On** feature to place a point on the curve, then edit the *x*-coordinate of the point to find the corresponding *y*-coordinate.

4. (cont.)

Modeling equation for melting snowball:
$P = 1{,}000e^{-0.2t}$ (where *P* is the weight and *t* is the time in hours); half its weight (500 lb): 3.5 hours; its original weight (2 lb): 31 hours

(On the graph, students can use the **Point On** tool and then edit the *y*-coordinate of the point to find the corresponding *x*-coordinate.)



After the snowball has completely melted, its weight would be 0 pounds. But the value an exponential function is strictly greater than 0, never equal to it. Theoretically, the snowball will never completely melt, but in practicality, it would probably melt in about 47–50 hours, when the snowball's weight is less than 1 ounce. (Accept reasonable estimates.)

Extension Possibilities

- Have students write their own exponential growth or decay problems and exchange with a classmate to solve.
- Students can search for the "mathematical constant *e*" on the internet and prepare a short paper or oral report of the history of this remarkable number.