Conics in Geometry? - Geometric Constructions of a Parabola, Ellipse, and Hyperbola

Geometry Major Topics: Conic Sections

NCTM Principles and Standards:
     Content Standards

Geometry:

Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships.

Explore relationships (including congruence and similarity) among classes of two- and three-dimensional geometric objects, make conjectures about them, and solve problems involving them.

Apply transformations and use symmetry to analyze mathematical situations.

Objectives: This lessons has students construct the three conic sections through geometric definitions. Using the CABRI Geometry II Application for the TI-89 calculator, students construct a parabola, an ellipse, and a hyperbola. The students are then asked to justify, through transformational geometry properties, the validity of these constructions. This lesson can be used as a teacher directed activity or can be given to the students as a discovery activity. If the students are familiar with the CABRI Geometry II Application, this activity should take between one and two 50 minute class periods to complete.

**Materials:**   TI-89 Graphing Calculator, CABRI Geometry II Application.

Prerequisites: Students should have a working knowledge of the CABRI Geometry II software and be aware of the geometric definitions of the conic sections.

Procedure:

Activity 1: Open the CABRI Geometry II application on the TI-89 by pressing the APPS button and choosing Cabri Geometry. Open a NEW file and choose an appropriate variable name, such as conics. Press F2 and choose **4:Line**. Construct a horizontal line near the bottom of the screen. Press F2 and choose **5:Segment**. Overlay a segment on top of the line. Press F7 and choose **1:Hide / Show**. Hide the line. Press F2 and choose **1:Point**. Construct a point above the segment. Press F2 and choose **1:Point**. Construct a point that is on the segment. Press F2 and choose **5:Segment**. Construct the segment connecting this point on the segment to the point above the segment. Your screens should be similar to the ones below.
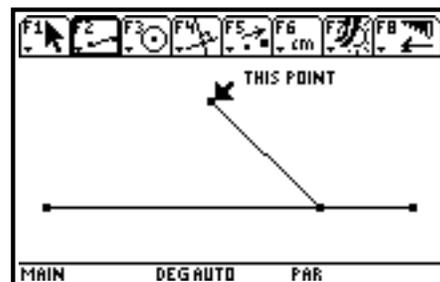
Press F4 and choose **4:Perpendicular Bisector**. Construct the perpendicular bisector of this segment.
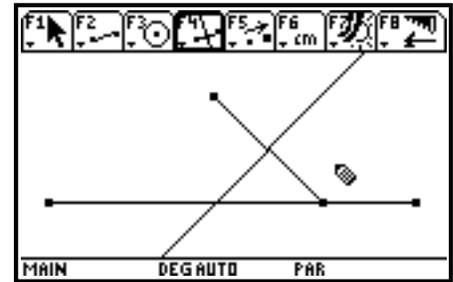
Press F7 and choose **2:Trace On / Off**. Move the cursor to the perpendicular bisector; when the message "this line" appears, press ENTER. This establishes the perpendicular bisector as the object to be traced. Drag the point on the segment, tracing the perpendicular bisector. Observe the shape of the envelope of lines.
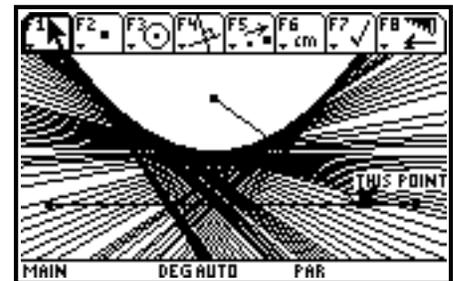
Press the CLEAR button on the calculator to delete the trace. Press F7 and choose **2:Trace On / Off**. Move the cursor to the perpendicular bisector and press ENTER to turn off the trace. Press F4 and choose **A:Locus**. Move the cursor to the perpendicular bisector; when the message "this line" appears, press ENTER. This establishes the perpendicular bisector as the object of the locus. Move the cursor to the point on the horizontal segment; when the message "this point" appears, press ENTER. This establishes the point as the "driver point", the point that creates the locus of points. Observe the shape of this locus and determine geometrically why this locus defines a parabola.
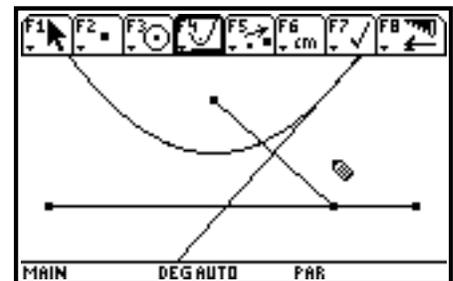
Teacher Note: The geometric definition of a parabola is "the set of points in a plane equidistant from a fixed point, the locus, and a fixed line, the directrix." In the above construction, the horizontal segment models the directrix, and the point above the segment acts as the focus of the parabola. The real questions about the construction are why the locus is a parabola and what "point" actually sketches out this parabola.

Delete the locus temporarily. Press F4 and choose **1:Perpendicular Line**. Move the cursor to the horizontal segment; when the message "perpendicular to this segment" appears, press ENTER. Move the cursor to the point on this segment. When the message "thru this point" appears, press ENTER. This constructs a line perpendicular to the segment through that point. Press F2 and choose **1:Point**. Move the cursor to the intersection of the perpendicular line and the perpendicular bisector; when the message "point at this intersection" appears, press ENTER. Press F2 and choose **5:Segment**. Construct the segment connecting this intersection point to the "focus point."
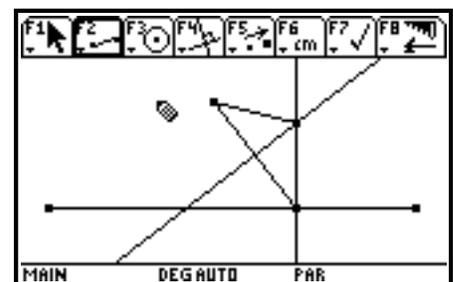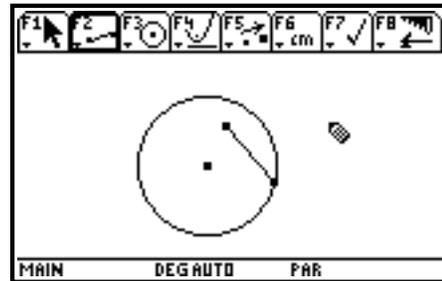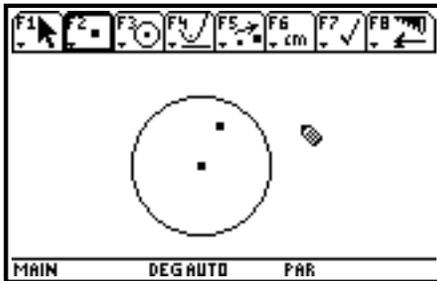
It is now apparent why this intersection point establishes the parabola. Any point on a perpendicular bisector of a segment is equidistant from the two endpoints of the segment. The distance from the intersection point to the "focus" is equal to the distance from the intersection point to the "directrix", thus establishing the geometric definition of a parabola. It should be noted that in geometry, the distance from a point to a line is defined to be the "shortest distance" and thus is always the perpendicular distance. To reestablish the validity of the construction, reconstruct the locus of the perpendicular bisector and drag the "driver point" on the horizontal segment and observe as the point of intersection "traces" the parabola.
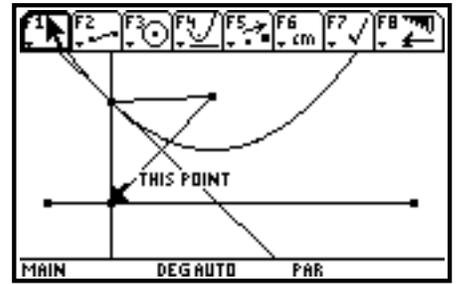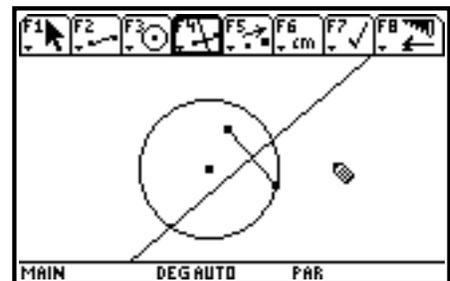
Extension to Calculus: Determine the role the perpendicular bisector models as the "driver point" is dragged along the "directrix".

Teacher Note: The perpendicular bisector is always tangent to the parabola at the intersection point. Thus, its slope models the numeric derivative of the point on the parabola with respect to the parabola.
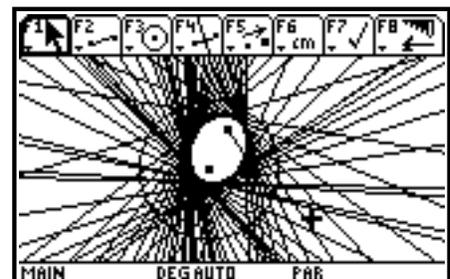
Activity 2: Press F8 and choose **8:Clear All** to clear the screen. Press F3 and choose **1:Circle**. Construct a circle on the screen. Press F2 and choose **1:Point**. Construct a point in the interior of the circle. Press F2 and choose **1:Point**. Construct a point that is on the circle. Press F2 and choose **5:Segment**. Construct a segment from the point on the circle to the point in the interior of the circle. Your screens should be similar to the ones below.

Press F4 and choose **4:Perpendicular Bisector**. Construct the perpendicular bisector of this segment.

Press F7 and choose **2:Trace On / Off**. Move the cursor to the perpendicular bisector; when the message "this line" appears, press ENTER. This establishes the perpendicular bisector as the object to be traced. Drag the point on the circle, tracing the perpendicular bisector. Observe the shape of the envelope of lines.

Press the CLEAR button on the calculator to delete the trace. Press F7 and choose **2:Trace On / Off.** Move the cursor to the perpendicular bisector and press ENTER to turn off the trace. Press F4 and choose **A:Locus.** Move the cursor to the perpendicular bisector; when the message "this line" appears, press ENTER. This establishes the perpendicular bisector as the object of the locus. Move the cursor to the point on the circle; when the message "this point" appears, press ENTER. This establishes the point as the "driver point", the point that creates the locus of points. Observe the shape of this locus and determine geometrically why this locus defines an ellipse.
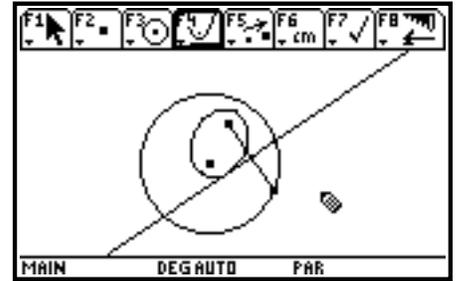
Teacher Note: The geometric definition of an ellipse is "the set of points in a plane such that the sum of the distances from each point to two fixed points, the foci, is a constant." In the above construction, the center of the original circle and the point in the interior of that circle model the foci of the ellipse. The real questions about the construction are why the locus is an ellipse and what "point" actually sketches out this ellipse.
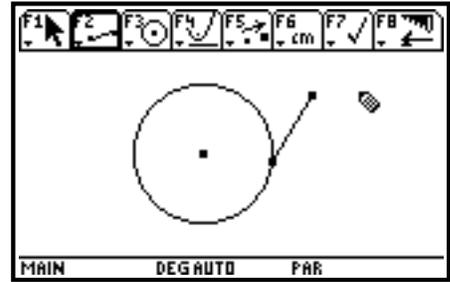
Delete the locus temporarily. Press F2 and choose **4:Line.** Construct a line that contains the center of the original circle and the point on the circle. Press F2 and choose **1:Point.** Move the cursor to the intersection of this line and the perpendicular bisector; when the message "point at this intersection" appears, press ENTER. Press F2 and choose **5:Segment.** Construct the segment connecting this intersection point to the point in the interior of the circle.
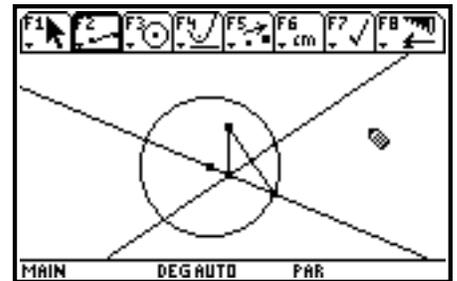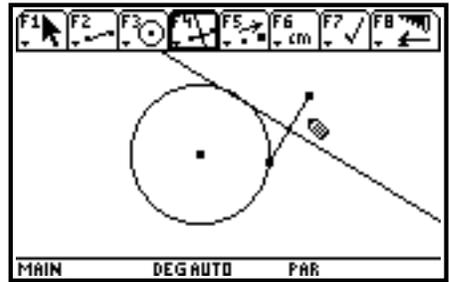
It is now apparent why this intersection point establishes the ellipse; any point on a perpendicular bisector of a segment is equidistant from the two endpoints of the segment. The distance from the intersection point to the point on the circle is the same as the distance from the intersection point to the point in the interior of the circle. The radius of the circle remains constant. This radius is made up of two segments, the segment from the center of the circle to the intersection point and the segment from the intersection point to the point on the circle. Interchanging the two equal distances establishes that the sum of the distance from the center of the circle (one focus) to the intersection point and the distance from this intersection point to the interior point (the other focus) remains constant. To reestablish the validity of the construction, reconstruct the locus of the perpendicular bisector and drag the "driver point" on the circle and observe as the point of intersection "traces" the ellipse.
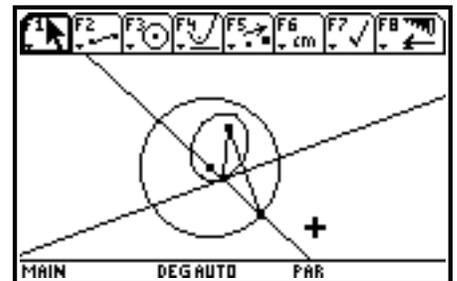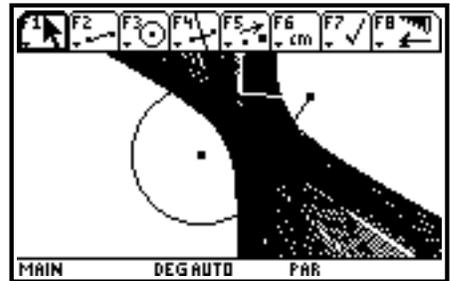
Activity 3: Press F8 and choose **8:Clear All** to clear the screen.
Press F3 and choose **1:Circle**. Construct a circle on the screen.
Press F2 and choose **1:Point**. Construct a point in the exterior of the
circle. Press F2 and choose **1:Point**. Construct a point that is on the
circle. Press F2 and choose **5:Segment**. Construct a segment from
the point on the circle to the point in the exterior of the circle.

Press F4 and choose **4:Perpendicular Bisector**. Construct the
perpendicular bisector of this segment.

Press F7 and choose **2:Trace On / Off**. Move the cursor to the
perpendicular bisector; when the message "this line" appears, press
ENTER. This establishes the perpendicular bisector as the object
to be traced. Drag the point on the circle, tracing the perpendicular
bisector. Observe the shape of the envelope of lines.

Press the CLEAR button on the calculator to delete the trace. Press
F7 and choose **2:Trace On / Off**. Move the cursor to the perpendicular
bisector and press ENTER to turn off the trace. Press F4 and choose
A:Locus. Move the cursor to the perpendicular bisector; when the
message "this line" appears, press ENTER. This establishes the
perpendicular bisector as the object of the locus. Move the cursor to
the point on the circle; when the message "this point" appears, press
ENTER. This establishes the point as the "driver point", the point
that creates the locus of points. Observe the shape of this locus and
determine geometrically why this locus defines a hyperbola.
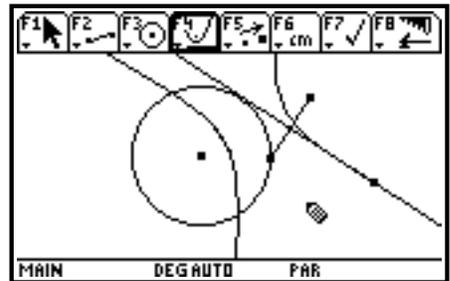
Teacher Note: The geometric definition of a hyperbola is "the set of points in a plane such that the difference
of the distances from each point to two fixed points, the foci, is a constant." In the above construction, the
center of the original circle and the point in the exterior of that circle model the foci of the hyperbola. The real
questions about the construction are why the locus is a hyperbola and what "point" actually sketches out this
hyperbola.

Delete the locus temporarily. Press F2 and choose **4:Line**. Construct a line that contains the center of the original circle and the point on the circle. Press F2 and choose **1:Point**. Move the cursor to the intersection of this line and the perpendicular bisector; when the message "point at this intersection" appears, press ENTER. Press F2 and choose **5:Segment**. Construct the segment connecting this intersection point to the point in the exterior of the circle.
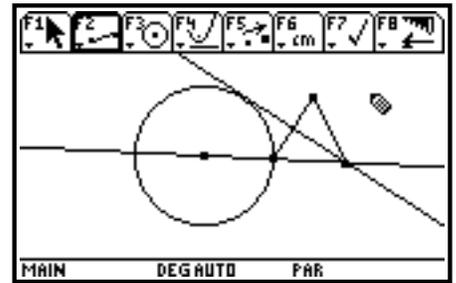
It is now apparent why this intersection point establishes the hyperbola; any point on a perpendicular bisector of a segment is equidistant from the two endpoints of the segment. The distance from the intersection point to the point on the circle is the same as the distance from the intersection point to the point in the exterior of the circle. The segment from the center of the circle to the intersection point is made up of two parts, the segment from the center of the circle to the point on the circle and the segment from the point on the circle to the intersection point. Thus, the distance from the center of the circle to the point on the circle is the difference between the distance from the center to the point in the exterior and the distance from the point on the circle to the point in the exterior. Interchanging the two equal distances establishes that this difference of the distance from the center of the circle (one focus) to the intersection point and the distance from this intersection point to the exterior point (the other focus) remains constant, since this difference is the radius of the circle which remains constant. To reestablish the validity of the construction, reconstruct the locus of the perpendicular bisector and drag the "driver point" on the circle and observe as the point of intersection "traces" the hyperbola.
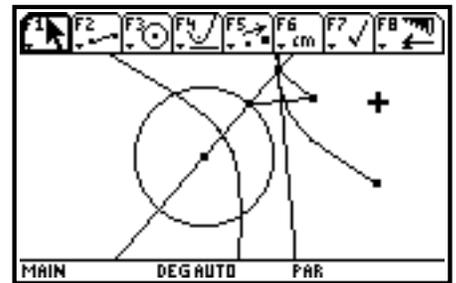
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