NUMB3RS Activity: Tic-Tac-Toe
Episode: "Spree, Part II – Daughters"

Topic: Game Theory **Grade Level:** 9 - 12

Objective: Investigate winning strategies in variations of tic-tac-toe games

Time: 15 minutes

Introduction

Charlie tells David that negotiating with a suspect is similar to playing a game. "It's like two people playing tic-tac-toe. If both play rationally, and neither one makes a mistake, the game will always end in a draw. In order for one side to win, you have to alter the rules of the game."

In this activity, students will explore variations on the game of tic-tac-toe, looking for winning strategies (sets of moves that guarantee a win, regardless of what moves the opponent makes) and other outcomes (such as draws).

Discuss with Students

In the game of tic-tac-toe, players alternate placing markers, usually Xs and Os, on a 3 × 3 board. The first player to place three of his or her own markers in a row (horizontally, vertically, or diagonally) is the winner. If neither player obtains three in a row, the game is a draw. Students will likely be familiar with the game, and may already know that when both players play rationally and neither makes a mistake, the result will always be a draw. Allow students to play and discuss the game until there is agreement on the conditions for a 3 × 3 game to end in a draw. The following questions may help guide the conversation:

1. What do the phrases "play rationally" and "make a mistake" mean?
2. Consider a simpler game of tic-tac-toe played on a 2 × 2 board, where a win is two markers in a row. What is the outcome of this game? Justify your answer.
3. One way to prove that a 3 × 3 game results in a draw is to play all possible games. Why is this more difficult than justifying the 2 × 2 result?

Discuss with Students Answers:

1. *To play rationally means to make moves with the intent to win; to make a mistake means to make a move that leads to an advantage for the opponent, when a different move would have not have resulted in an advantage.* **2.** *The first player always wins; the first move sets up two possible ways to win, and the second player can only block one.* **3.** *Extending from 2 × 2 to 3 × 3 increases the number of possible outcomes. For a 2x2 game, there are 4C2 = 6 boards; for a 3x3 game, there are 9C5 = 126 boards complexity of analysis. As with other problems in mathematics, what seems to be a simple change can greatly alter the complexity of a scenario.*

Student Page Answers:

1. *The first player can ensure a win with an initial move on any interior space (not an end-space).* **2.** *The first player can ensure a win with an initial move on any space in the 2nd or 3rd column. By placing two in a row to force a block on the opponent's second move, the first player can then create an "L" shape, which offers possible wins in two directions.* **3a.** *The second player can guarantee a win with a move on the 3rd space* **3b.** *A move on either end-space will allow the first player to win.* **3c.** *If both play rationally, the game will always be a draw.* **4.** *The first player has a winning strategy with an initial move on the second or fourth square.*

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Name ____ Date ______________

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In the game of tic-tac-toe, players alternate placing markers, usually Xs and Os, on a 3 × 3 board. The first player to place three of his or her own markers in a row (horizontally, vertically, or diagonally) is the winner. If neither player obtains three in a row, the game is a draw. As Charlie states in the episode, if both players are trying to win and neither makes a mistake, the game always ends in a draw. Changing the rules can make the game more interesting and may give a player a winning strategy (a series of moves that will guarantee a win, no matter what the opponent does).

In this activity, you will play variations of the traditional tic-tac-toe game described above. Your goal is to determine when a game will end in a draw or to find a winning strategy.

1. For 2-in-a-row tic-tac-toe, the first player to place two of his or her own markers in a row wins. Play 2-in-a-row on board sizes 1 × 3, 1 × 4, and 1 × 5. Describe the winning strategy for a game played on a 1 × n board.

2. Now play 3-in-a-row tic-tac-toe; this requires three in a row for a win like the traditional game, but uses a 3 × 4 board. Is there a winning strategy, or is a draw guaranteed?

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Another variation is Revenge tic-tac-toe. In this variation, when a player makes a move that would win, the opponent gets one last move; if the opponent makes a move that also gets the correct number in a row, the opponent wins instead.

3. Consider 2-in-a-row Revenge tic-tac-toe on a 1 × 4 board. Because of the symmetry of the board, there are only two possible opening moves.

a. Explain why opening with a marker on an end space is a not a good move.

x			

b. The opening move must be an interior space. What move(s) by the second player would ensure a win for the first player?

	x		

	x		

c. Based on the results of Questions 3a and 3b, determine the outcome of 2-in-a-row Revenge tic-tac-toe on a 1 × 4 board.

4. What is the outcome of 2-in-a-row Revenge tic-tac-toe on a 1×5 board?

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The goal of this activity is to give your students a short and simple snapshot into a very extensive math topic. TI and NCTM encourage you and your students to learn more about this topic using the extensions provided below and through your own independent research.

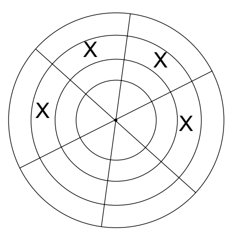
Extensions

For the Student

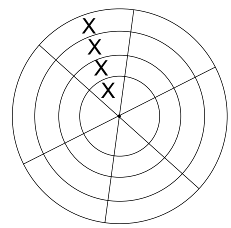
- Play 4-in-a-row tic-tac-toe. Show that playing on a 5 × 5 board results in a draw. Find the winning strategy for the first player on a 5 × 6 board.

- In *Mathematical Carnival*, author Martin Gardner proposes the following game: nine playing cards, ace through nine, are face up on a table. Two players take turns picking a card. The first player to obtain three cards whose sum is 15 is the winner. Determine how this game is related to the commonly played 3-in-a-row tic-tac-toe played on a 3 × 3 board. (Reference: Gardner, Martin. *Mathematical Carnival.* New York: Random House/Vintage Books. 1977.)

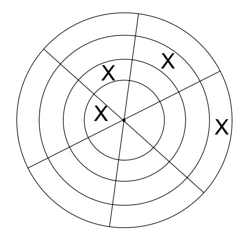
- You can also play tic-tac-toe on different shaped boards. Try the board with 4 concentric circles shown below. You win if you place A) 4 of your marks within the same ring in consecutive wedge-shaped sections, B) 4 of your marks within a single wedge-shaped section, or C) 4 marks in consecutive wedge-shaped sections, spiraling out from the center.



A



B



C

Think of a way that tic-tac-toe can be played on a torus. On the one-dimensional boards, imagine gluing the left edge to the right edge. On the two-dimensional board, imagine gluing the left and right edges together, as well as gluing the top and bottom edges together. How does this change your strategy in these games?

Additional Resources

- For a discussion of the games played in this activity and others, visit the MathWorld site. **http://mathworld.wolfram.com/Tic-Tac-Toe.html**

- Tic-tac-toe can be played in three dimensions (or more!). The game Qubic is 4-in-a-row tic-tac-toe played on a 4 × 4 × 4 board. To play Qubic on your computer, visit either of the Web sites listed below.
  http://home.earthlink.net/~cmalumphy/3d.html
  http://www.4to40.com/games/4fun/html/tictactoe3d/default.htm

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