NUMB3RS Activity: Pack It In
Episode: "The Janus List"

Topic: Knapsack problems **Grade Level:** 9 - 12
Objective: To explore a type of bin-packing problem called "knapsack" problems
Time: 15 minutes

Introduction

In "The Janus List," a bomber is threatening to blow up a bridge. The bomber calls for Charlie and asks him a series of seemingly meaningless questions. Charlie later realizes that the questions are clues to decode a secret message. Charlie compares this to the knapsack problem—every clue contributes something to the end result.

Discuss with Students

In the classic knapsack problem, a knapsack with a certain capacity has to be filled by a set of items. Each item accounts for part of the capacity, usually weight or volume. In some versions of the problem, the items also carry a value, so that the value of the items must be maximized while staying within the capacity limitations. In this activity, students will see simple examples where the only concern is getting as much weight as possible into a knapsack. After understanding the basic idea, students then attempt to maximize the value of the items while staying within the capacity limitations.

The problems can be adapted to other mathematics that students are studying. For example, a geometry class could be given a problem of packing a "box" (perhaps a rectangle) with non-overlapping items from a set of shapes that they are studying.

Student Page Answers:

1. *Selecting 21, 8, 3, and 1 would fill the 33 pounds. One approach is as follows: 34 cannot be selected because it is already over the limit. If 21 is not selected, the total of the remaining items is only 32, so to get to 33, 21 must be selected. That leaves 12 pounds to select from 1, 2, 3, 5, 8, and 13. Because 13 > 12, it cannot be selected. If 8 is not selected, because 1 + 2 + 3 + 5 = 11, the maximum would again be 32. So 21 and 8 must be selected which gives a total of 29, and items 1, 2, 3, and 5 remain. The only way to get the 4 remaining pounds from these items is to select 3 and 1.* **2.** *There is no number in the range that cannot be attained. Note to teacher: The weights are powers of 2 from 2^0 to 2^4. These are the position values for each digit in the binary system, and numbers from 1 to 31 can be achieved by writing the weight in binary and choosing the weights corresponding to the 1s. For example, if a pack can hold 25 pounds, $25_{10} = 11001_2$ (i.e., $1(2^4) + 1(2^3) + 0(2^2) + 0(2^1) + 1(2^0)$) so the selection is 16, 8, and 1. Because of the uniqueness of binary representation, this is the only way to get 25. This is the motivation behind calling the type of knapsack problem with only one item of each size, a "1/0" type of problem.* **3.** *Start by putting in item 9 to get the maximum value of 8. That leaves only 10 pounds—even though items 7 and 8 have high values, they weigh too much. The next choice is item 1 with value 6. That means 41 pounds have been used for a total value of 14. Item 2 is the next choice giving totals of 44 pounds and 19. Item 4 adds 5 pounds for as total of 49 pounds and a value of 4 for a total of 23. Because there are no items of weight 1 left, this is the best that this method will do.* **4.** *Answers will vary. One possible combination of items is items 7, 6, 1, and 2. Giving up the "greedy" choice of item 9 (the value of 8) makes room to select items 6 and 7 which have a total value of 13. Dropping item 3 but keeping items 1 and 2, gives a total value of 24 and still weighs only 49 pounds. A better choice would be items 1, 2, 3, 4, and 7 which weighs 44 and has a value of 25. This eliminates item 6 and its 20 pounds of weight and replaces it with items 3 and 4 with a combined weight of 15 and value of 7 which lowers the weight to 44 and raises the value to 25.* **5.** *Let R be the capacity of the knapsack. The bills are the items and the weight of each item is its cost. The value is not as objective, but should represent the value and priorities that the community assigns to each bill.*

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Name: ________________________________ Date: ________________

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In the simplest case of the knapsack problem, a person has items of varying weights and a knapsack that can only hold a certain amount of weight. The knapsack problem asks what selection of items will maximize the weight in the bag. It is only an interesting problem when the sum of the weights of the items is greater than the capacity of the knapsack. The knapsack problem is part of a larger set called *bin-packing* problems.

1. Suppose you have a set of items weighing 1, 2, 3, 5, 8, 13, 21, and 34 pounds and a knapsack that can hold 33 pounds. Which items should be selected to make the knapsack as close to capacity as possible? Explain your reasoning.

2. Suppose a knapsack can hold y pounds (where $y \leq 25$) and a set of items has weights of 1, 2, 4, 8, and 16. Is there any value of y that cannot be achieved with these items? Why or why not?

In the actual case that Charlie mentions, each item has both a weight and a value. The challenge is to select those items that will make the sum of the values as large as possible while keeping the sum of the weights at or below a given bound.

3. Consider the set of items shown in the following table. There is only one of each item.

Item	1	2	3	4	5	6	7	8	9
Weight	1	3	5	10	16	20	25	30	40
Value	6	5	4	3	2	6	7	1	8

You have to place the items in a knapsack. The knapsack can only hold 50 pounds. One strategy to maximize the weight is to pick the item with the highest value that will keep the weight within the limit. Repeat until it is no longer possible. Which items would be picked and what is the final value and weight?

4. Use a different approach and see if you get better results. Explain your reasoning.

Although this is traditionally called a knapsack problem, it has little to do with knapsacks and most of its applications are in different fields.

5. Suppose a community government requires a balanced budget, and the revenues are known for the next year (for example, there will be R dollars for discretionary spending). The bills from the legislature have a total cost that is more than R. Reword the situation as a knapsack problem.

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The goal of this activity is to give your students a short and simple snapshot into a very extensive mathematical topic. TI and NCTM encourage you and your students to learn more about this topic using the extensions provided below and through your own independent research.

Extensions

Introduction

As the final problem illustrates, knapsack problems are not always about knapsacks. Instead there are a whole range of problems which have structures and strategies similar to those needed to pack a knapsack. While many forms of mathematics have domains with no limits, the constraint placed by the size of the knapsack is very similar to the limitations on many real world problems—especially in the field of discrete mathematics.

For the Student

There are many applications of the knapsack problem that can be applied to other items in the high school curriculum. For example, a physical education class might appreciate a discussion of how this is used in professional sports. Before the age of salary caps, there was a different dynamic—the salary cap effectively introduced the knapsack problem to the sports world. Research salary caps (cap = knapsack size) and how that relates to a player's salary and the value the team places upon that player.

The knapsack problem can be used in other subjects. For example, in social studies classes, the knapsack problem can be used in budgeting processes. Research how other school subjects can use knapsack problems (e.g., storing wood in a shop class, scheduling processes in a business class, etc.).

For simplification, this activity considers only the weight of the items. However, in other situations, like packing the trunk of a car, there are other constraints, such as the sizes and shapes of the objects. Do some research to find out how these considerations are factored in during a particular process (e.g., a local manufacturer might use this for packing and shipping).

Additional Resources

The story of three thieves (a greedy thief, a slow thief, and a smart thief) and their knapsacks is given at the Web site below. This gives a very understandable explanation and results of three different algorithms for this problem.
http://webspace.ship.edu/thbrig/DynamicProgramming/Knapsack%20Program

Another version of the problem allows an unlimited supply of each item. There is an applet with whimsical items at the Web site below.
http://www-cse.uta.edu/~holder/courses/cse2320/lectures/applets/knapsack/knapsack.html

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