8 An application: Linear programming

Linear programming is the branch of applied mathematics that deals with problems like the following example.

8.1 Apples and pears

Suppose you have € 3.6 for which you want to buy apples and pears. The price of one apple is € 0.2 and € 0.3 for a pear. How many apples and pears can I buy if you know that there are only 12 apples and 10 pears in the store?

<u>Solution</u>

Let's x represent the number of apples and y the number of pears.

Obvious the following conditions count: $x \geq 0$ and $y \geq 0$.

And there are the following constraints for x and y:

$20x + 30y \leq 360$, $x \leq 12$ and $y \leq 10$.

To solve the problem we need to find all the points (x, y) that satisfy:

$$\begin{cases} 20x + 30y \leq 360 \\ x \leq 12 \\ y \leq 10 \\ x \geq 0, y \geq 0 \end{cases}$$

We try to solve the problem by a graphical approach by plotting the linear relations $20x + 30y = 360$, $x = 0$, $x = 12$, $y = 10$ and $y = 0$.

Therefore we define the functions: **Y_1=12 - $\frac{2}{3}$X**,

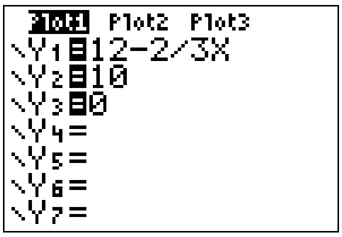
Y_2=10,

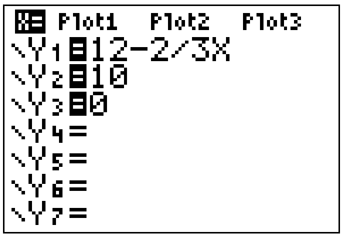
Y_3=0,

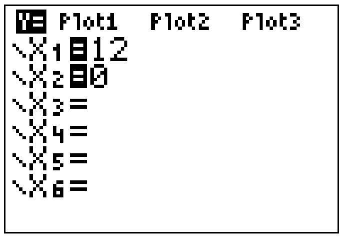
X_1=12,

X_2=0.

To define **X_1=12** and **X_2=0** you first need to activate the application *Inequality Graphing* [1]. And then select **X=**.

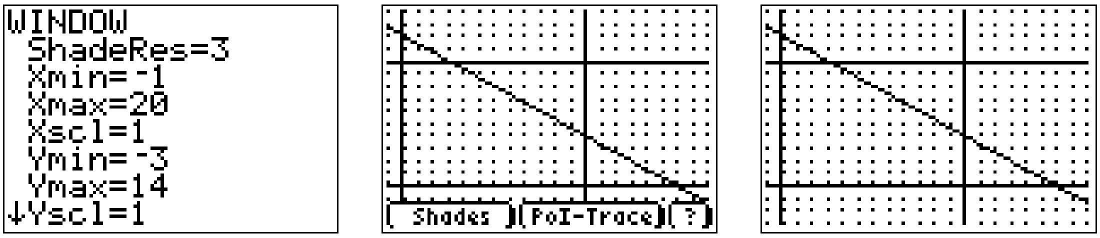




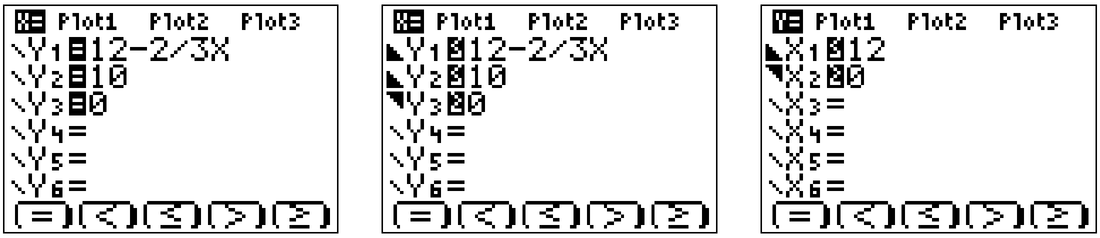


[1] See 9. Appendix.

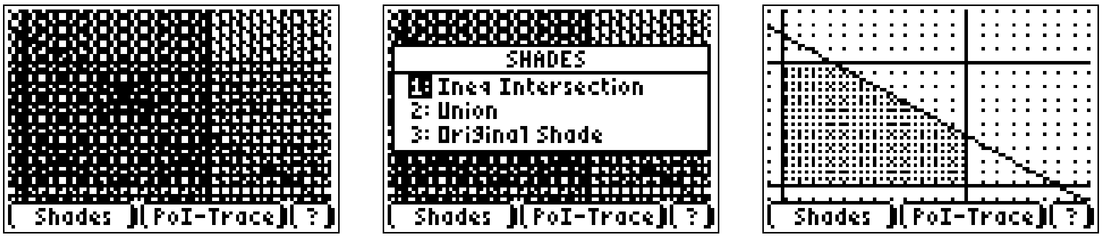
These definitions result into the following graph (press **TRACE ClEAR** to remove the menu at the bottom of the screen):



All the points in the enclosed area are solutions for our problem. It's possible to shade this area and to calculate its vertices. To shade we need to change the equal signs with **F1** through **F6** as follow:

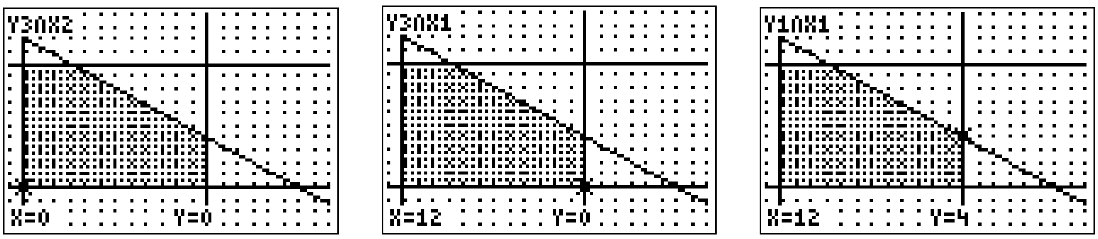


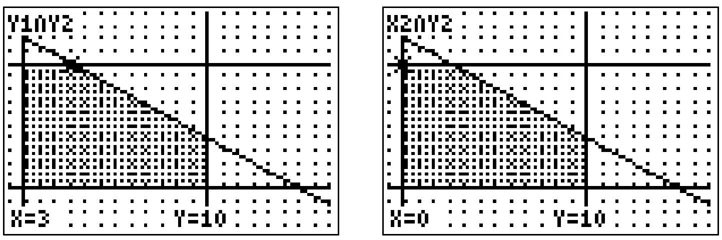
Press **GRAPH**, then select **Shades** and **1: Ineq Intersection**.



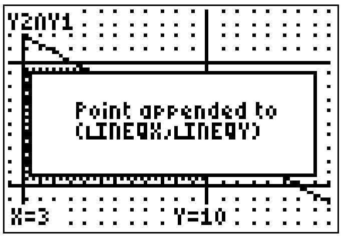
We will now calculate the vertices of shades with **PoI-Trace**:

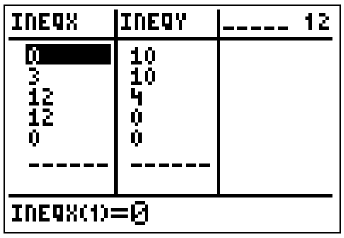
◂ ▸ = change the first function ▴ ▾ = change the second function



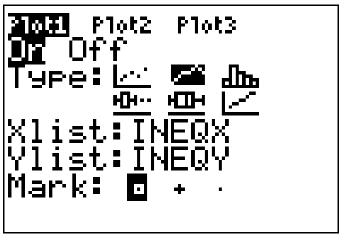


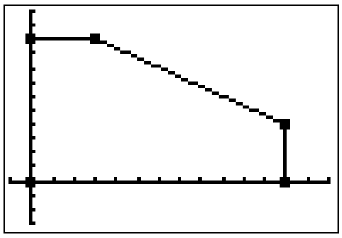
You can store a selected vertex by pressing **STO ▸**. The coordinates of the vertex will automatically be stored in the lists **INEQX** and **INEQY**.





With these lists it's still possible to plot the area even after quitting *Inequality Graphing* and/or deleting the functions. On the graph below the grid is turned off.



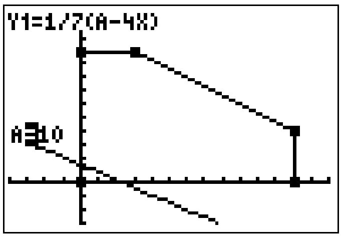


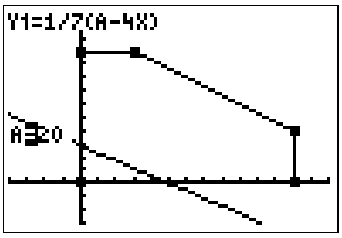
Let's make our example a little bit more complicated. We want to have an as high as possible content of vitamin C in our purchase. Suppose on apple contains 4 gram of vitamin C and a pear 7 gram.

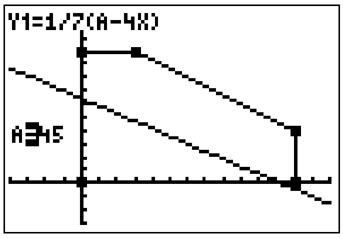
To solve this problem we need to find the maximum value of $4x+7y$ over the area determined above.

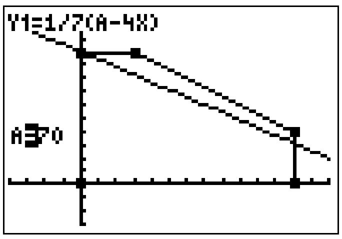
To investigate this problem graphically we define the parameter A as follow $A=4x+7y$ and the function **Y1** $= \frac{1}{7}(A - 4x)$.

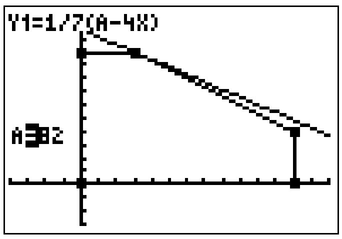
Activate the application *Transformation Graphing* [2] (deactivate first *Inequality Graphing*) and investigate the value of A for several points in the enclosed area.

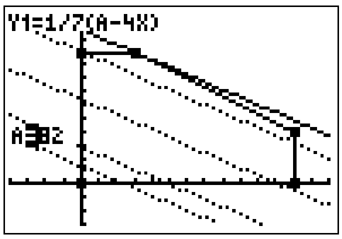










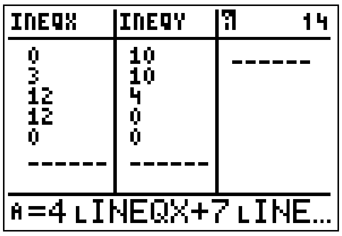


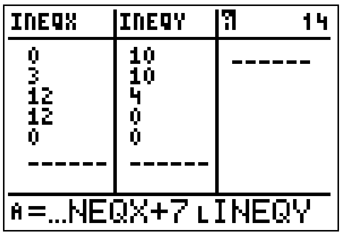
TrailOn – **2nd[FORMAT]**

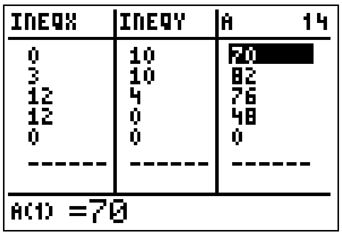
When we study the variation of the parameter A we see that the maximum value of A will be found in one of the vertices.

[2] See 9. Appendix

With **STAT 1:Edit…** we can calculate these values for A as follows:







The maximal amount of vitamin C is 82 gram with a purchase of 3 apples and 10 pears.

8.2 The simplex method

We can write the previous example as follows:

Maximize	$4x+7y$	
Subject to	$20x+30y \le 360$	(1)
	$x \le 12$	
	$y \le 10$	
	$x \ge 0, y \ge 0$	

The simplex method always starts from a feasible solution. For our use we will take the origin $x=0$ and $y=0$. Of course these x and y values aren't the ones that gives us the maximum value for $4x+7y$.

We will rewrite the inequalities into equalities by introducing three new variables u,v,w; called slack variables:

$$u = 360 - 20x - 30y \le 360$$
$$v = 12 - x$$
$$w = 10 - y$$

We define $z = 4x+7y$. The old variables x and y are called the decision variables.

So now we can rewrite our problem as follows:

Maximize	$z = 4x+7y$	
Subject to	$u = 360 - 20x - 30y$	(2)
	$v = 12 - x$	
	$w = 10 - y$	
	$x \ge 0, y \ge 0, u \ge 0, v \ge 0, w \ge 0$	

<u>Note</u>

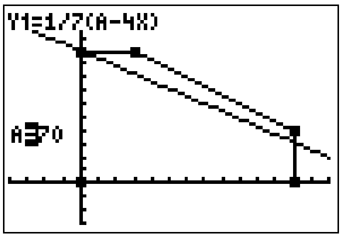
- Each feasible solution of (1) can be extended to a feasible solution of (2).
- Each feasible solution of (2) can be restricted to a feasible solution of (1).
- Each optimal solution of (1) corresponds with an optimal solution of (2).

Our feasible solution to start from is $x=0, y=0, u=360, v=12, w=10$. (3)
This solution gives $z=0$.

We will try to find successive improvements out of this feasible solution x,y,u,v,w to end with a maximal solution. This means that out of x,y,u,v,w we try to deduce a feasible solution $\tilde{x},\tilde{y},\tilde{u},\tilde{v},\tilde{w}$ with $4\tilde{x}+7\tilde{y} \ge 4x+7y$.

If we look at $z = 4x + 7y$ we see that if we increase y, z will increase faster as when we increase x. So we will increase y and keep $x = 0$. How much can we increase y?

For $x = 0, u \ge 0, v \ge 0, w \ge 0$ the following constraints count:



$$\begin{cases} 360 - 30y \ge 0 \\ 12 \ge 0 \\ 10 - y \ge 0 \end{cases} \Leftrightarrow \begin{cases} y \le 12 \\ 12 \ge 0 \\ y \le 10 \end{cases} \Rightarrow \quad y \le 10 .$$

In other words y can increase up to 10. So we become our next solution: $x = 0, y = 10, u = 60, v = 12, w = 0$ which yields $z = 70$.

In our next step we are going for an ever better feasible solution. How can we do this?

We need to manufacture a new system of linear constraint to continue. If we look at (2) we see that it expresses the variables u, v, w that assume positive values in (3) in terms of those variables x, y that assume zero. And also z is expressed in (2) in terms of x, y.

Note that y changed its value from zero to positive and w from positive to zero. So we need to change their position in the system of equations, from the right-hand side to the left-hand side and vice versa. We call y the entering variable and w the leaving variable.

We start with the newcomer y on the left-hand side. With the third equation of (2) we can express y in terms of x, w: $w = 10 - y \Leftrightarrow y = 10 - w$.

Next we express u, v and z in terms of x, w

$$u = 360 - 20x - 30y = 360 - 20x - 30(10 - w) = 60 - 20x + 30w$$
$$v = 12 - x$$
$$z = 4x + 7y = 4x + 7(10 - w) = 70 + 4x - 7w$$

So we can rewrite our problem as follows:

Maximize $\quad z = 70 + 4x - 7w$

Subject to $\quad u = 60 - 20x + 30w$

$v = 12 - x$

$y = 10 - w$

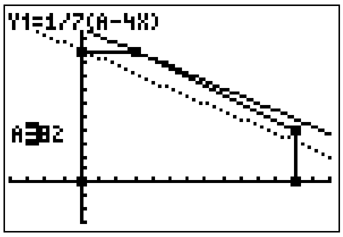
$x \ge 0, y \ge 0, u \ge 0, v \ge 0, w \ge 0$

From our second feasible solution $x = 0, y = 10, u = 60, v = 12, w = 0$ with $z = 70$ we will again try to find an improvement.

If we look at $z = 70 + 4x - 7w$ the only way to let increase z is to increase x.

How much can we increase x?

For $w = 0, y \ge 0, u \ge, v \ge 0$ the following constraints count:



$$\begin{cases} 60 - 20x \ge 0 \\ 12 - x \ge 0 \\ 10 \ge 0 \end{cases} \Leftrightarrow \begin{cases} x \le 3 \\ x \le 12 \\ 10 \ge 0 \end{cases} \Rightarrow \quad x \le 3$$

In other words x can increase up to 3 and our next feasible solution is: $x = 3, y = 10, u = 0, v = 9, w = 0$ with $z = 82$.

Now we express all variables and z in terms of u, w. Again we will start with the newcomer x:

$$u = 60 - 20x + 30w \Leftrightarrow 20x = 60 - u + 30w \Leftrightarrow x = 3 - \frac{1}{20}u + \frac{3}{2}w .$$

It follows that:

$$v = 12 - x = 9 + \frac{1}{20}u - \frac{3}{2}w$$

$$y = 10 - w$$

$$z = 70 + 4x - 7w = 70 + 4(3 - \frac{1}{20}u + \frac{3}{2}w) - 7w = 82 - \frac{1}{5}u - w$$

When we look at z it's clear we can not increase z anymore by increasing u or w.

This means we found an optimal solution $z = 82$ for $x = 3$ and $y = 10$.

The method we just used to find an optimal solution is called the simplex method. In this particular example x and y has to be integers but everything stays the same if we consider x and y as real variables.

8.3 The simplex method using matrices

We rewrite our example into the following modified form.

$$\begin{array}{l} 20x + 30y + u = 360 \\ x + v = 12 \\ y + w = 10 \\ \hline -z + 4x + 7y = 0 \end{array} \quad \text{or} \quad \begin{array}{ccccccccccccc} & & 20x & + & 30y & + & u & & & & & = & 360 \\ & & x & & & & & + & v & & & = & 12 \\ & & & & y & & & & & + & w & = & 10 \\ \hline -z & + & 4x & + & 7y & & & & & & & = & 0 \end{array}$$

Using only the coefficients we can use the following matrix to represent our example.

$$\begin{pmatrix} 20 & 30 & 1 & 0 & 0 & 360 \\ 1 & 0 & 0 & 1 & 0 & 12 \\ 0 & 1 & 0 & 0 & 1 & 10 \\ 4 & 7 & 0 & 0 & 0 & 0 \end{pmatrix}$$

<u>Step 1</u>

Examine the elements of the last raw, except the one in the last column (which represent the present value of -z. If all the elements or negative, the matrix represent an optimal solution. Otherwise select the column associated with the largest positive number. This column is called the pivot column and corresponds with the entering variable.

$$\begin{pmatrix} \boxed{20} & 30 & 1 & 0 & 0 & 360 \\ \boxed{1} & 0 & 0 & 1 & 0 & 12 \\ \boxed{0} & 1 & 0 & 0 & 1 & 10 \\ \boxed{4} & 7 & 0 & 0 & 0 & 0 \end{pmatrix}$$

<u>Step 2</u>

We will calculate the ratios $\frac{p}{q}$ of the elements p of the rightmost column and the positive elements q of the pivot column (except for the last column). If they our all negative the problem is unbounded (see further).

The row with the smallest ratio $\frac{p}{q}$ is called the pivot row and corresponds with the leaving variable.

$$\begin{pmatrix} 20 & 30 & 1 & 0 & 0 & 360 \\ 1 & 0 & 0 & 1 & 0 & 12 \\ 0 & 1 & 0 & 0 & 1 & 10 \\ 4 & 7 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{matrix} \longrightarrow \frac{360}{30}=12 \\ \\ \longrightarrow \frac{10}{1}=10 \\ \\ \end{matrix}$$

<u>Step 3</u>

In this step we divide each element of the pivot row with the pivot (= intersection of the pivot column and the pivot row). In our case (pivot = 1) we don't need to do anything.

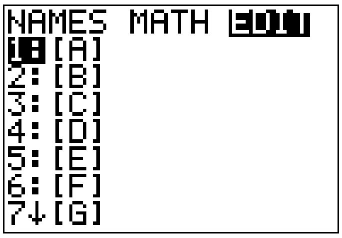
It's not a bad idea to add a column with the positive variables of our present solution.

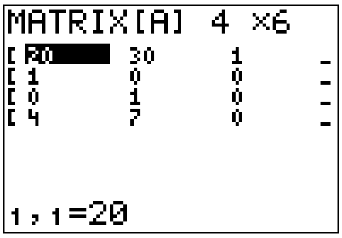
$$\begin{pmatrix} 20 & 30 & 1 & 0 & 0 & 360 \\ 1 & 0 & 0 & 1 & 0 & 12 \\ 0 & 1 & 0 & 0 & 1 & 10 \\ 4 & 7 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} u \\ v \\ w \\ \\ \end{matrix}$$

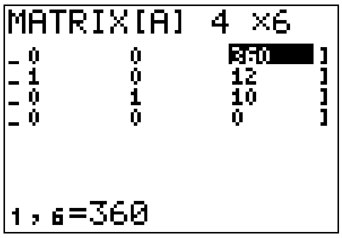
<u>Step 4</u>

Use elementary row operations (**2nd[MATRIX]<MATH>**) to make all the elements of the pivot column, except the pivot, zero.

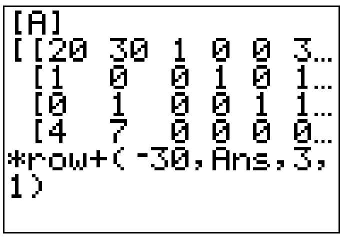
Input of the matrix

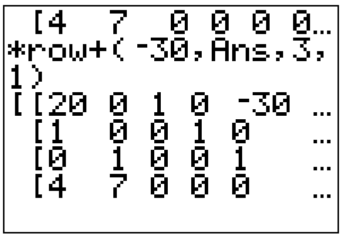


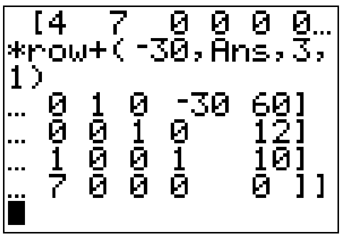




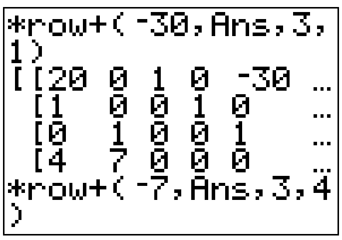
-30 $R_3 + R_1$

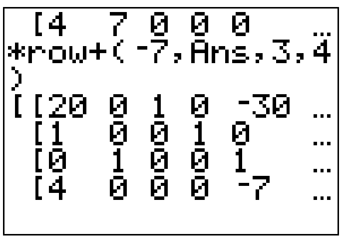


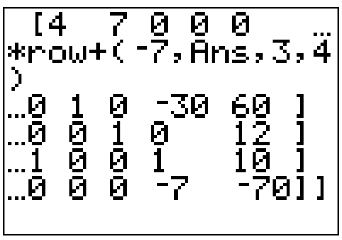




-7 $R_3 + R_4$







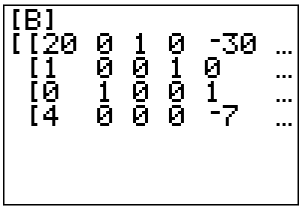
So we become the following new matrix, with $z = 70$ and $x = 0, y = 10, u = 60, v = 12$ and $w = 0$:

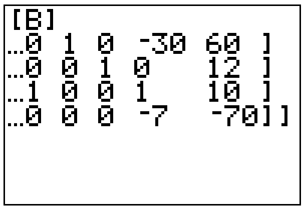
$$\begin{pmatrix} 20 & 0 & 1 & 0 & -30 & 60 \\ 1 & 0 & 0 & 1 & 0 & 12 \\ 0 & 1 & 0 & 0 & 1 & 10 \\ 4 & 0 & 0 & 0 & -7 & -70 \end{pmatrix} \quad \begin{matrix} u \\ v \\ y \\ \ \end{matrix}$$

We need to redo the previous four steps, starting from this matrix, to find a better feasible solution.

<u>Step 1 & 2</u>

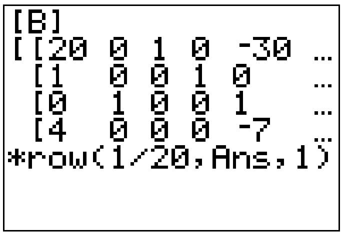
$$\begin{pmatrix} \boxed{20} & 0 & 1 & 0 & -30 & 60 \\ 1 & 0 & 0 & 1 & 0 & 12 \\ 0 & 1 & 0 & 0 & 1 & 10 \\ 4 & 0 & 0 & 0 & -7 & -70 \end{pmatrix} \quad \begin{matrix} u \\ v \\ y \\ \ \end{matrix}$$

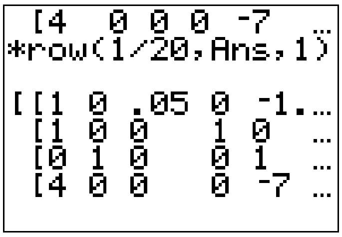


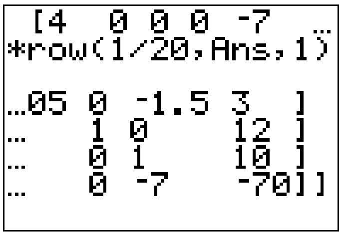


<u>Step 3 & 4</u>

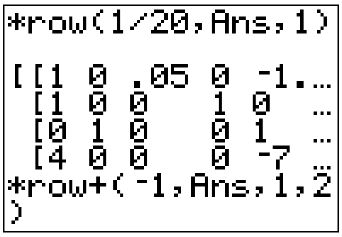
$-\frac{1}{20} R_1$

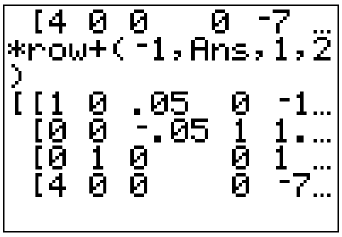


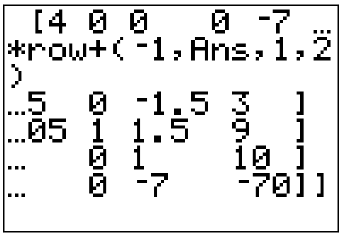




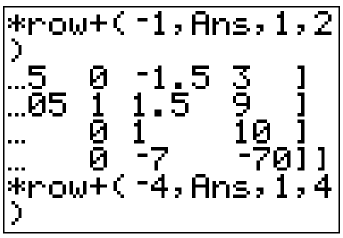
$-R_1 + R_2$

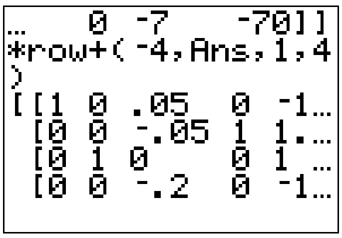


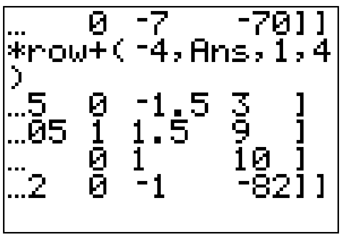


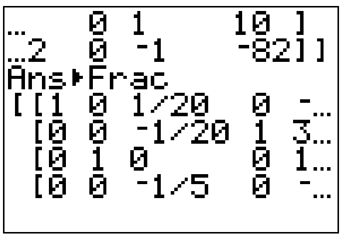


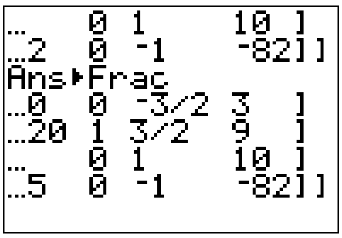
$-4 R_1 + R_4$











Our new matrix is:

$$\begin{pmatrix} 1 & 0 & \frac{1}{20} & 0 & -\frac{3}{2} & 3 \\ 0 & 0 & -\frac{1}{20} & 1 & \frac{3}{2} & 9 \\ 0 & 1 & 0 & 0 & 1 & 10 \\ 0 & 0 & -\frac{1}{5} & 0 & -1 & -82 \end{pmatrix} \quad \begin{matrix} x \\ v \\ y \\ \\ \end{matrix}$$

The last row of our matrix contains only negative numbers which means we reached an optimal solution $x = 3, y = 10, u = 0, v = 9, w = 0$ with $z = 82$.

8.4 Always a unique solution?

Without giving a complete discussion we will end with two examples to show that there is not always a unique solution.

(i) Several solutions – infinite many

Maximize $z = 2x + 4y$ subject to $x - y \le 2$, $x + 2y \le 16$, $x \ge 0, y \ge 0$

or

Maximize $z = 2x + 4y$ subject to $u = 2 - x + y$, $v = 16 - x - 2y$, $x \ge 0, y \ge 0, u \ge 0, v \ge 0$

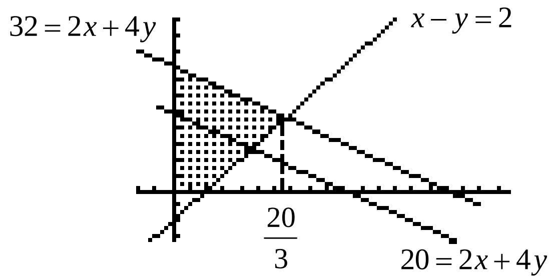
The second constraint give already an indication that the line which represent $2x + 4y - z = 0$ is parallel to one side of the area enclosed by the constaints.
In a following step we become:

Maximize $z = 32 - 2v$
subject to $y = 8 - 0.5x - 0.5v$
$u = 10 - 1.5x - 0.5v$
$x \ge 0, y \ge 0, u \ge 0, v \ge 0$

For each optimal solution ($z = 32$) counts $v = 0$, but not necessary $x = 0$. The condition for x is $10 - 1.5x \ge 0 \Leftrightarrow x \le \frac{20}{3}$.

For all x in $\left[0, \frac{20}{3}\right]$ we find an optimal solution $x, y = 8 - 0.5x, u = 10 - 1.5x, v = 0$.

Note: $y = 8 - 0.5x \Leftrightarrow x + 2y = 16 \Leftrightarrow 2x + 4y = 32$.



(ii) No solution – an unbounded problem

Maximize $z = x + 2y$

subject to $-2x + y \le 4$
$2x - y \le 8$
$x \ge 0, y \ge 0$