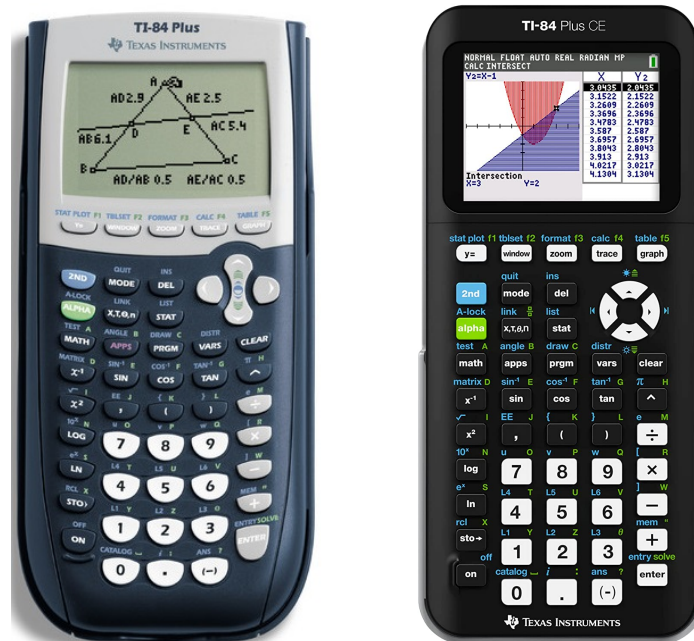


Mathematics on a TI-84/CE

Volume 3: Advanced

Chapters 17 – 24



Peter McIntyre
School of Science
UNSW Canberra
2022

Contents

17 Sequences and Series	2
17.1 Introduction	2
17.2 Sequences	3
17.2.1 Using the sequence grapher	4
17.2.2 Using LIST commands	8
17.2.3 Using the SEQUENCE/SEQNCECE program	9
17.3 Series	10
17.3.1 Using the sequence grapher	11
17.3.2 Using LIST commands	12
17.3.3 Using the SERIES/SERIESCE program	13
17.4 Questions	14
17.5 Further investigations	17
17.5.1 Fibonacci numbers	17
17.5.2 The logistic sequence	19
17.5.3 The harmonic series	22
17.6 Solutions to exercises, questions and activities	23
18 Probability and Statistics 2	35
18.1 Introduction	35
18.1.1 Australian Curriculum	35
18.1.2 Calculator note	35
18.2 Discrete probability distributions	37
18.2.1 Counting methods	37
18.2.2 Random numbers	37
18.2.3 Expected value and variance	38
18.2.4 The Binomial Distribution	39
18.2.5 The Geometric Distribution	41
18.2.6 The Hypergeometric Distribution	43
18.2.7 The Poisson Distribution	45
18.2.8 Discrete-distribution spinner	46
18.3 Continuous probability distributions	47
18.3.1 The Normal Distribution	47
18.3.2 The Exponential Distribution	52
18.4 Statistical inference	53

18.4.1	Hypothesis testing & confidence intervals for means	53
18.4.2	Hypothesis testing for population proportion	56
18.5	Probability and gambling schemes	60
18.5.1	Throwing dice	60
18.5.2	Random walks	60
18.5.3	Basic betting	61
18.5.4	Gambling systems	61
18.5.5	Tossing coins	63
18.6	Solutions to exercises	64
19	Matrix and Vector Operations	70
19.1	Basic matrix operations	70
19.1.1	Putting matrices in your calculator	70
19.1.2	Displaying matrices	70
19.1.3	Adding and subtracting matrices	71
19.1.4	Multiplying matrices	71
19.1.5	Squaring matrices	72
19.1.6	More-involved expressions	72
19.1.7	Storing matrices	72
19.1.8	Illegal operations	73
19.1.9	Other matrix operations	73
19.2	Systematic Gauss Elimination	74
19.2.1	The method	74
19.2.2	Using the calculator	75
19.2.3	Exercises	77
19.3	Eigenvalues and eigenvectors	79
19.3.1	Theory	79
19.3.2	Example	79
19.3.3	EIGENV5/EIGENV5CE — finds real eigenvalues and eigenvectors . . .	82
19.3.4	Exercises	84
19.4	Vector operations	85
19.4.1	Introduction	85
19.4.2	Operations	85
19.4.3	Exercises	86
19.4.4	Solutions	87

20 Population Modelling 3	101
20.1 Population-projection matrices	101
20.1.1 Leslie matrices	101
20.1.2 Populations and oscillations	103
20.1.3 Exercise: Killer whales	104
20.1.4 Exercise: Age distribution of trees in a forest	106
20.2 Matrix-modelling programs	107
20.3 Solutions	108
21 Fitting Curves to Data 2	114
21.1 Introduction	114
21.2 Exact fitting of polynomials to data	114
21.2.1 Manual method	114
21.2.2 Using a program	115
21.3 Using the CURVEFIT/CRVFITCE program	117
21.3.1 Linear: $Y=A+BX$	118
21.3.2 Reciprocal: $Y=1/(A+BX)$	118
21.3.3 Linear-hyperbolic: $Y=A+BX+C/X$	118
21.3.4 Hyperbolic: $Y=A+B/X$	119
21.3.5 Reciprocal hyperbolic: $Y=X/(AX+B)$	119
21.3.6 Second-order hyperbolic: $Y=A+B/X+C/X^2$	119
21.3.7 Parabolic: $Y=A+BX+CX^2$	120
21.3.8 Cauchy Distribution: $Y=1/(A(X+B)^2+C)$	120
21.3.9 Logarithmic: $Y=A+B\ln(X)$	120
21.3.10 Reciprocal logarithmic: $Y=1/(A+B\ln(X))$	121
21.3.11 Power: $Y=AX^B$	121
21.3.12 Super exponential: $Y=AX^{(BX)}$	121
21.3.13 Modified exponential: $Y=AX^{(B/X)}$	122
21.3.14 Hoerl Function: $Y=AB^X X^C$	122
21.3.15 Modified Hoerl: $Y=AB^{(1/X)} X^C$	122
21.3.16 Log-Normal: $Y=Ae^{((B-\ln(X))^2/C)}$	123
21.3.17 Exponential: $Y=AB^X$	123
21.3.18 Root: $Y=AB^{(1/X)}$	123
21.3.19 Normal Distribution: $Y=Ae^{((X-B)^2/C)}$	124
21.3.20 Best fit	124

21.4	Time averaging of data	125
22	Financial Mathematics 2	127
22.1	Introduction	127
22.2	Using the TVM Solver	128
22.2.1	Selecting the TVM Solver	128
22.2.2	Compound interest	129
22.2.3	Discounting	130
22.2.4	Effective rate of interest	131
22.2.5	Analysing a loan	132
22.2.6	Annuities	134
22.2.7	Sinking funds	134
22.2.8	To lease or to buy?	135
22.2.9	Bonds	136
22.2.10	A cautionary note	137
22.3	Appendix: Amortisation tables	137
22.4	TVM exercises and solutions	139
22.5	TVM activities and solutions	143
23	Complex Numbers	151
23.1	Setting complex mode	151
23.2	Basic operations	151
23.2.1	Addition and subtraction	151
23.2.2	Multiplication and division	151
23.2.3	Conjugation	152
23.2.4	Real part	152
23.2.5	Imaginary part	152
23.2.6	Modulus	152
23.3	Polar form	152
23.3.1	Modulus and angle	152
23.3.2	Forms of output	153
23.3.3	Forms of input	154
23.3.4	Conversion between forms	154
23.4	Powers and roots	154
23.4.1	Powers	154
23.4.2	Roots	154

23.5 Exercises	155
23.6 Appendix	155
23.6.1 Input/output of complex numbers	155
23.6.2 CMPXROOT/CXROOTCE program	156
23.7 Answers to Exercises	157
24 Programming	158
24.1 Introduction	158
24.2 Basics	158
24.2.1 Writing the basic program	158
24.2.2 Making corrections	159
24.2.3 Making the program more user-friendly	159
24.3 Looping	160
24.3.1 Looping for input	160
24.3.2 Looping to check input	160
24.3.3 Including an error message	161
24.3.4 Looping through a range of input values	162
24.4 Displaying results as a table	163
24.5 Displaying results as a graph	164
24.5.1 Graphing data points from a program	164
24.5.2 Graphing functions from a program	165
24.5.3 Putting text on a graph	167
24.6 Simple animation	169
24.7 Extensions	170
24.8 Exercises	173
24.9 Possible solutions to the exercises	174
24.10 Keystrokes used in the programs	176

Mathematics on a TI-84/CE

Volume 1 of this book contains the basic topics: Graphics Calculators and Mathematics; Getting Started; Coordinate Geometry; Inequalities and Linear Programming; Fitting Curves to Data 1 – Calculator Functions; Population Modelling 1 – Exponential Growth; Financial Mathematics 1 – Compound Interest; and Probability and Statistics 1 – Descriptive Statistics. The Volume 1 Supplement contains extra activities for Coordinate Geometry and Probability and Statistics 1.

Volume 2 of this book contains the Calculus topics: Functions and their Graphs; Graph and Calculus Operations; Numerical Integration; Taylor Series; Differential Equations; Population Modelling 2 – Logistic and Epidemic Models; and Multivariable Calculus.

The last chapter of Volume 2 gives a list of the programs, and information on copying and using the programs in all volumes of the book.

Volume 4 of this book contains 28 Mathematics labs or projects, together with a Lab Manual for teachers/instructors.

Calculator versions

Currently (early 2022), TI-84 calculators come in two versions: the TI-84Plus and the more recent TI-84CE. The main difference is that the CE screen has much higher resolution. It also has colour but I have done most of the screens in black and white to avoid the need for colour printers or photocopiers. Calculations, screenshots and figures were done on a TI-84CE in CLASSIC mode.

Some programs have had to be changed for the CE because of the different screen: I usually append ‘CE’ to the program name to indicate this.

All the programs here are available at *www.XXX*.

17 Sequences and Series

17.1 Introduction

These notes provide a comprehensive review of generating, displaying and graphing sequences and series on a TI-84/CE graphics calculator. An arithmetic progression, a geometric progression and the Fibonacci numbers are used as examples. A number of questions (with solutions) illustrate the use of the calculator. Finally there are three topics that could be used as a basis for group investigation or a small project.

The TI-84/CE can generate sequences, sum series, and display sequence terms in a table or graph. However, we should first ask whether it makes sense to use a graphics calculator at all for sequences and series.

Certainly the first few lessons on sequences should be pencil and paper, until some of the concepts and calculations are understood, although a class activity such as that on page 10 can add variety to the early learning stages. However, having to work out terms of a sequence or series by hand eventually becomes tedious, especially those terms that are not very simple and require a calculator anyway. This becomes an impediment to further learning and exploration.

The calculator automates the process of calculating terms in a sequence or series *once it is given an appropriate definition*. It is in finding an appropriate definition that most of the thought goes — the calculator can't do this. With automatic calculation comes the ability to explore particular sequences and series, to conjecture and test, and to look at ideas such as the convergence of an infinite sequence or series.¹

Some of the questions and investigations in Sections 17.4 and 17.5 demonstrate this extra capability when using a graphics calculator.

At this stage, it is perhaps useful for the reader to review the use of the TI-84/CE graphics keys, the top row, by graphing, say $y = x^3$ and generating a table of function values.

- Set a suitable `window` manually first and graph $Y_1 = X^3$.
- Then use ZoomFit (`zoom` `0`) to carry out the process of finding a suitable Y scale automatically. Use both `graph` and `trace` to generate the graph.
- Review line types (to the left of the function definition in `y=`), particularly solid and dotted lines. On a TI-84CE, you can choose a colour for the graph here too.
- For the table, use `tblset` to specify the starting value and increment for X. Generate a table with `table`.

¹These are not esoteric beasts — the humble AP and GP continue on indefinitely.

17.2 Sequences

A sequence is an ordered set of numbers, usually with the numbers or terms in the sequence determined by some sort of formula.² For example,

$$1, 3, 5, 7, 9, 11, \dots \qquad 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots \qquad 1, 1, 2, 3, 5, 8, 13, \dots$$

are sequences for which we can find a formula to determine each term.

In the usual notation, a general sequence is written as

$$u_1, u_2, u_3, \dots, u_n, \dots,$$

where each term u_1, u_2, u_3, \dots is a number. The subscript gives the position of the term in the sequence.

The TI-84/CE uses the notation $u(n)$ for the n th term of the sequence, rather than u_n , so the general sequence is written

$$u(1), u(2), u(3), \dots, u(n), \dots$$

This was presumably done for ease of display, but it reinforces an important fact about sequences: a sequence is really just a function with domain the positive integers or some subset of them.

There are two ways to give a formula for each term.

- **Recursively:** write the n th term in terms of the previous term or terms. Here we also need to know a value for the first term (or the first few terms) in the sequence.

Examples

1. $u_2 = u_1 + 2, u_3 = u_2 + 2, \dots$ or in general, $u_n = u_{n-1} + 2$.

With $u_1 = 1$, this recursive formula gives the first sequence above, an arithmetic progression (AP).

2. $u_n = u_{n-1}/2$.

With $u_1 = 1$, this gives the second sequence above, a geometric progression (GP).

3. $u_n = u_{n-1} + u_{n-2}$.

With $u_1 = 1$ and $u_2 = 1$, this gives the third sequence above, the famous Fibonacci numbers.

- **Explicitly:** specify the n th term as a function of n , where n takes integer values.

Examples

1. $u_n = 2n - 1, n = 1, 2, 3, \dots$, again giving the first sequence above.

2. $u_n = 0.5^{n-1}, n = 1, 2, 3, \dots$, again giving the second sequence above.

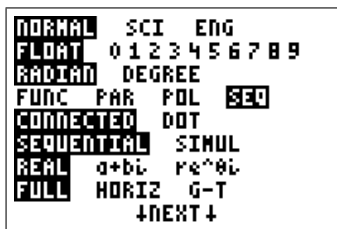
The Fibonacci sequence can also be defined explicitly — see Section 17.5.

There are basically two ways on the TI-84/CE to generate and display terms of a sequence — *using the sequence grapher* and *using LIST commands*. Which method you use will depend on how you teach the topic. Here we'll look at both, using the above examples to illustrate the methods. There are questions to practise on in Section 17.4.

²although we can have sequences of random numbers.

17.2.1 Using the sequence grapher

Sequences can be defined either recursively or explicitly, displayed and graphed by the built-in sequence grapher. To select this, press `mode` and, with the cursor and `enter`, select SEQ as shown below.



TI-84Plus



TI-84CE

The graphing mode SEQ is just one of four possible ways to define a function on the TI-84/CE. Because a sequence is just a function as far as the calculator is concerned, all the calculator graphing keys (top row) are relevant.

Now press the function-definition key `y=` to see the three sequences u , v , w available.

On a TI-84, the n th term $u(n)$ can be written as any combination of n , $u(n-1)$, $u(n-2)$, $v(n-1)$, $v(n-2)$, $w(n-1)$ and $w(n-2)$. There is more flexibility on a CE.

Arithmetic progression

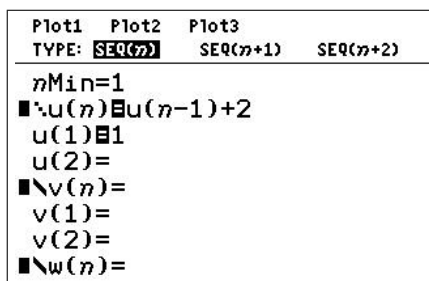
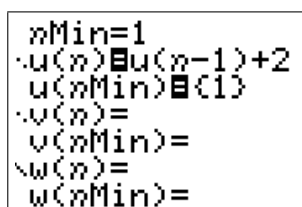
In an arithmetic progression, there is a constant difference between successive terms. In calculator notation, the recursive definition is

$$u(n) = u(n-1) + d,$$

where d is a constant called the common difference.

Example: $u(n) = u(n-1) + 2$, with $u(1) = 1$.

Press `y=` and set this sequence up on your calculator as shown below. The independent variable here, n , is produced by the `X,T,θ,n` key in SEQ mode. The sequence name u is `2nd` `7`. The initial u value on a TI-84, $u(nMin)$, must be in curly brackets.



While we are on this screen, check that you have dots to the left of the definition of $u(n)$, indicating that only the sequence points will be plotted (no joining lines) when we graph the sequence. If not, move the cursor here and press `enter`. On a TI-84, press `enter` until three dots come up. On a CE, use the arrow keys to select bold dots; you can select a different colour here too.

Displaying the sequence

Here we use the table feature of the calculator.

First a table ‘window’: press `tblset` (`(2nd window)`) and set `TblStart=1` and `ΔTbl=1`.

n	$u(n)$
1	1
2	3
3	5
4	7
5	9
6	11
7	13
8	15
9	17
10	19
11	21

TABLE SETUP	
TblStart=1	
ΔTbl=1	
Indpnt:	Auto Ask
Depend:	Auto Ask

Press `table` (`(2nd graph)`). Scroll down in either column to see the terms of the sequence. If you want to scroll up past the top of the screen, you have to be in the n column.

Another way of displaying terms of a sequence u is to type on the Home screen a command of the form `u(start, end [, increment])`. For example, typing `u(1,9,2)` and pressing `enter` will display `u(1), u(3), u(5), u(7)` and `u(9)`.

`increment` is optional, a value of 1 assumed if it is not entered.

To display the value of just one term, just type `u(n value)`, for example `u(6)`, and press `enter`.

Graphing the sequence

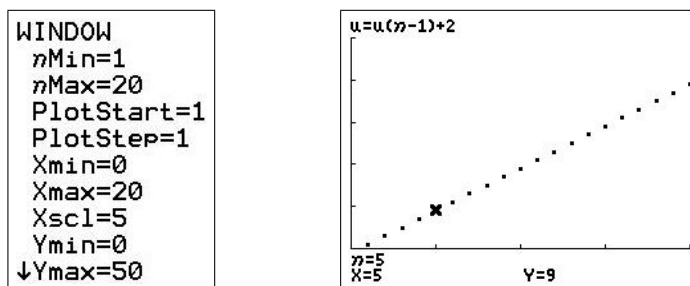
There are several ways to graph sequences, as shown when you press `format` (`(2nd zoom)`): there’s an extra line here (the top one) in SEQmode.

<code>Time</code>	<code>Web</code>	<code>uv</code>	<code>vw</code>	<code>uw</code>
<code>RectGC</code>	<code>PolarGC</code>			
<code>CoordOn</code>	<code>CoordOff</code>			
<code>GridOff</code>	<code>GridOn</code>			
<code>GridColor:</code>	<code>MEDGRAY</code>			
<code>Axes:</code>	<code>BLACK</code>			
<code>LabelOff</code>	<code>LabelOn</code>			
<code>ExprOn</code>	<code>ExprOff</code>			
<code>BorderColor:</code>	<code>1</code>			
<code>Background:</code>	<code>Off</code>			

- *Time* plots $u(n)$ against n , which is what we want here. Select it if necessary with the cursor and `enter`.
- *Web* plots a cobweb plot, $u(n)$ against $u(n-1)$. We’ll look at this in Section 17.5.2.
- If you have more than one sequence, you can plot one sequence against another: uv , vw , uw . These are called *phase plots*.

Next set a graph window: press `window` and you will see a few more parameters than usual. In our *Time* plot, n is plotted along the X axis, so the n range should be contained within the X range.

Set up your window as shown below (not shown on the screen is $Yscl = 10$) and press `trace`. Use the arrow keys to move along the points of the sequence.



Shortcut: Specify $nMax$, then press `zoom` `0` (*ZoomFit*); this sets appropriate X and Y scales. You may want to change these a bit in `window` after the graph has been plotted. Other zoom options, such as *Zoom In* and *Zoom Box*, work too.

Defining the AP explicitly

The n th term for a general arithmetic progression in calculator notation is

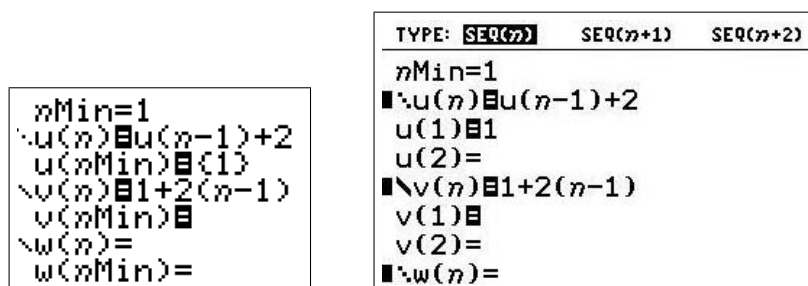
$$u(n) = a + (n-1)d,$$

where a is the first term ($n=1$) and d is the common difference.

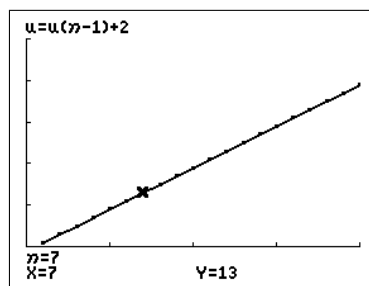
For our example, we have $a=1$ and $d=2$, so that

$$u(n) = 1 + 2(n-1).$$

We can compare the two definitions of the sequence by putting the n th term in sequence v , as shown below. Note that $v(nMin)/v(1)$ is not required when a sequence is defined explicitly **and should be cleared**. Plot v as a solid line so you see which sequence is being plotted.



Now press `graph` and compare u and v . `trace` might be useful. You can also use `table` to compare values of u and v .



Using the APGP/APGPCE program

This program calculates the N th term of an AP, $T(N) = A + (N-1)D$, and the sum of N terms. Useful if you need to do more than one or two of these.

Geometric progression

In a geometric progression, each term is a constant multiple of the previous term. In calculator notation, the recursive definition is

$$u(n) = ru(n-1),$$

where r is a constant called the common ratio or common multiplier.

Exercise: Display a table and graph the geometric sequence $u(n) = 0.5u(n-1)$, with $u(1) = 1$. For the graph, use *ZoomFit* after specifying $nMax$ in .

Solutions to the exercises are in Section 17.6.

You may want to go back to after the graph is plotted to adjust Xscl and Yscl so there aren't too many tick marks on the axes.

Press or to return to the graph after changing the window.

The n th term of a geometric progression is given *explicitly* by

$$u(n) = ar^{n-1},$$

where a is the first term and r is the common multiplier.

For the sequence in the exercise above, the n th term is given by

$$u(n) = 0.5^{n-1}.$$

Exercise: Put this explicit definition in sequence v and compare with the recursive definition.

Using the APGP/APGPCE program

This program calculates the N th term of an GP, $T(N) = AR^{(N-1)}$, and the sum of N terms. Useful if you need to do more than one or two of these.

Fibonacci sequence

The Fibonacci sequence is defined, in calculator terms, by

$$u(n) = u(n-1) + u(n-2) \quad \text{with} \quad u(1) = u(2) = 1.$$

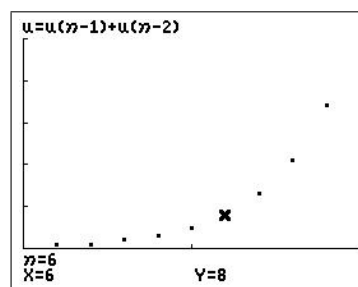
Press and enter this formula (below left).

On a TI-84, the two starting values $u(1)$ and $u(2)$ are stored in $u(nMin)$ as a list: $\{1, 1\}$.

TYPE:	<input type="text" value="SEQ(n)"/>	<input type="text" value="SEQ(n+1)"/>	<input type="text" value="SEQ(n+2)"/>
$nMin=$	<input type="text" value="1"/>		
$\blacktriangleright u(n) =$	<input type="text" value="u(n-1)+u(n-2)"/>		
$u(1) =$	<input type="text" value="1"/>		
$u(2) =$	<input type="text" value="1"/>		
$\blacktriangleright v(n) =$			
$v(1) =$			
$v(2) =$			
$\blacktriangleright w(n) =$			

n	$u(n)$
1	1
2	1
3	2
4	3
5	5
6	8
7	13
8	21
9	34
10	55
11	89

$n=1$



The graph here is of the first 10 points of the sequence ($nMax = 10$) plotted with $Xmin = 0$, $Xmax = 10$, $Xscl = 5$, $Ymin = 0$, $Ymax = 50$ and $Yscl = 10$: i.e. a window $[0, 10, 5] \times [0, 50, 10]$.

17.2.2 Using LIST commands

The *seq* command in the `[list]` (`[2nd]` `[stat]`) OPS menu generates a sequence (list) specified by an explicit general term. The syntax is

`seq` (general term, variable, start, end [, step]),

where variable can be any letter and *step* is assumed 1 if not given.³ The total number of terms calculated must be less than 1000.⁴

With STAT WIZARDS on, you get prompts for each of the entries in the *seq* command. CtlgHelp (scroll down to *seq* in the `[list]` OPS menu and press `[+]`) tells you what to input in the *seq* command and in what order.

Arithmetic progression

In the following sections, there is an implicit `[enter]` after each command.

To generate an AP: `seq(1+2(N-1),N,1,10)`

```
seq(1+2(N-1),N,1,10)
{1 3 5 7 9 11 13 15 17 19}
```

If the sequence goes off the right-hand side of the screen, you can scroll across using the right-arrow key.

A nice way to use the general expression for an AP is

`1 → A : 2 → D : seq(A+(N-1)D,N,1,10)` : is `[alpha]` `[.]`

```
1→A:2→D:seq(A+(N-1)D,N,1,10)
{1 3 5 7 9 11 13 15 17 19}
```

Press `[entry]` (`[2nd]` `[enter]`)⁵ or use the up-arrow key and `[enter]` to recall the command, change the values of A and D, and press `[enter]` to re-execute the command.

Storing to a list

For analysis and graphing, it is often convenient to store a sequence in a named list, which is then stored in memory. The TI-84/CE has six built-in lists: L₁–L₆; on the keyboard `[2nd]` `[1]` – `[2nd]` `[6]`. Lists can also be given any name of up to five characters.

To store a sequence to list L₁,

`seq(2N-1,N,1,10) → L1,`

where `→` is the `[sto]` key. If you have already generated the sequence, just press `→` L₁.

Now press `[stat]` `[1]` (Edit...) to enter the list editor. You can scroll up and down, graph, edit and carry out other operations on any list here.

³The variable in a *seq* command doesn't have to take integer values, e.g. `seq(sin(X),X,0,1,0.05)` generates a sequence of values of the sine function.

⁴A list on the TI-84/CE can have a maximum of 999 entries. The same restriction applies to the *seq* command even if you are not storing the sequence in a list, because the TI-84/CE keeps the sequence in an internal list *Ans*.

⁵Pressing `[entry]` once recalls the previous command, twice the command before that, and so on. There is limited memory for this, so you can't go back indefinitely.

Geometric progression

Try the following commands (`enter` after each).

`seq(0.5^(N-1), N, 1, 10)`

`math 1` a useful command for sequences containing fractions

`1 → A : 0.5 → R : seq(AR^(N-1), N, 1, 10)`

Exercise: Use `entry` (`2nd enter`) to produce terms 11 to 20 in the sequence.

Solutions to the exercises are in Section 17.6.

A neat trick for sequences

What we'd really like is an endless display of the sequence terms. This is provided in `table` by evaluating the continuous function corresponding to our sequence at integer values of its argument. This is a good way to start sequences — see the Class Activity below.

Select FUNC(TION) in `mode` to return to the normal type of graph.

In `y=`: $Y_1 = 0.5^{(X-1)}$.

In `tblset`: $\text{TblStart} = 1 \quad \Delta\text{Tbl} = 1$.

Press `table`.

Scroll down in the Y_1 column (the sequence terms) to see more digits for each entry at the bottom of the screen.

Provided TblStart and ΔTbl are integers, we get sequence values in the table. This is an alternative to using the sequence grapher to generate a table, and is usually faster.

17.2.3 Using the SEQUENCE/SEQNCECE program

Given an infinite sequence $\{A_N\}$, what is the limit of the sequence, i.e. what is $\lim_{N \rightarrow \infty} A_N$?

The program evaluates A_N for $N = 1, 2, 4, 8, 16, 32, \dots$ as an aid to *estimating* the limit.

Use: Store the N th term A_N in Y_1 .

Run the program. Press `enter` to see successive values. Be wary of round-off error for large N . Stop the program at any time by pressing `on` Quit.

We could do the calculation using Sequence mode on the calculator, but this is much slower.

Example: $A_N = (1 + 1/N)^N \rightarrow Y_1 = (1 + N^{-1})^N$

$$A_1 = 2$$

$$A_2 = 2.25$$

$$A_4 = 2.4414065$$

$$A_8 = 2.565784514$$

$$A_{16} = 2.637928497$$

⋮

$$\lim_{N \rightarrow \infty} A_N = \quad ?$$

You may see the effect of roundoff error if you let N become large enough.

Class Activity

Give students one calculator between two. Have them press `tblset` (`2nd` `window`) and set both TblStart and Δ Tbl to 1.

On the viewscreen calculator, with the OHP turned off, set $Y_1 = 2X + 1$ and press `table`. Turn the OHP on and ask the students: *What's the rule?* If you don't have a viewscreen (or a TI-84/CE emulator on a computer), just write the numbers up on the board.

When they have worked out the rule as a class, press `y=` to show them (or just tell them) how the rule is entered. Have them enter the rule and generate the table. Write on the board that the rules to follow are all of the form $_ X + _$.

Now, again with the OHP off, enter a different rule and show them the table. Ask them to make the table on their calculators the same.

Give them various rules to find, moving eventually to negative numbers for the coefficients. Ask them to summarise their findings regarding the two numbers in the rules.

17.3 Series

A series is a sequence with $+$ signs between successive terms. The sum of a series is the actual value when we carry out the additions. To calculate the sum of a series, we need to find the terms in the corresponding sequence, then add them up.

If a series has a finite number of terms, we just add them up to give the sum. All of what we do below can be applied to finite series. However, the more interesting series are infinite — we can't calculate the sum for these by carrying out the additions because there is an infinite number of them.⁶ However, we can work out the sum of a finite number of terms, called a *partial sum* — the n th partial sum is the sum of the first n terms of the series. The behaviour of the partial sums as n gets bigger tells us something about the convergence of the series — whether the sum may be a finite number or infinite. The sum of an infinite series is defined as the limit of its partial sums as $n \rightarrow \infty$.

Both the sequence grapher and the `list` commands can be used to sum a series, but, unless we are quite clever (see Section 17.3.3 below), both methods require an explicit n th term. Because of this, the `list` commands are probably simpler, and the 'neat trick' above, suitably modified (see Section 17.3.2), allows us to use `table`. However, the sequence grapher is easier to use if we want a graph of the partial sums.

The SERIES/SERIESCE program (Section 17.3.3) combines the speed of the `list` commands with the flexibility of the sequence grapher. It is available at *www.XXX*.

⁶There are algebraic methods for summing some infinite series, one of the triumphs of Calculus.

17.3.1 Using the sequence grapher

The n th partial sum S_n of a series is the sum of the first n terms:

$$S_n = u_1 + u_2 + u_3 + \cdots + u_n = \sum_{i=1}^n u_i.$$

Recursively, S_n is the sum of the previous n terms, S_{n-1} , plus the n th term:

$$S_n = S_{n-1} + u_n \quad S_1 = u_1.$$

We'd like to generate the sequence of partial sums

$$S_1, S_2, S_3, S_4, S_5, \dots$$

We'll use the sequence v in the sequence grapher for the partial sums: $v(n)$ will be the n th partial sum of the sequence we put in $u(n)$.

Re-select SEQ in mode to return to the sequence grapher if necessary.

Arithmetic progression

For our AP, with n th term $u(n) = 1+2(n-1)$, we have for the n th partial sum

$$v(n) = v(n-1) + u(n), \quad \text{with } v(1) = 1,$$

as shown below centre.

Note: On a TI-84, the sequence grapher only lets us use the $(n-1)$ th and $(n-2)$ th terms in the definition of an n th term, so we have to use the following for v (left-hand figure below):

$$v(n) = v(n-1) + 1+2(n-1), \quad \text{with } v(1) = 1.$$

nMin=1
u(n)≡1+2(n-1)
u(nMin)≡(1)
v(n)≡v(n-1)+1+2(n-1)
v(nMin)≡(1)

TYPE: SEQ(n) SEQ(n+1)
nMin=1
u(n)≡1+2(n-1)
u(1)≡1
u(2)=
v(n)≡v(n-1)+u(n)
v(1)≡1
v(2)=

n	u(n)	v(n)
1	1	1
2	3	4
3	5	9
4	7	16
5	9	25
6	11	36
7	13	49
8	15	64
9	17	81
10	19	100
11	21	121

The table of partial sums can now be displayed as usual (above right); the terms of the sequence are shown in the $u(n)$ column and the corresponding partial sums in the $v(n)$ column.

Geometric progression

Exercise: Set up the table of partial sums for our GP with $u(n) = 0.5^{n-1}$. To what value does the (infinite) series appear to converge. Confirm your answer algebraically (using the usual formula for the infinite sum of a GP).

Solutions to the exercises are in Section 17.6.

17.3.2 Using LIST commands

The *sum* command in the `list` MATH menu sums a sequence (list). The syntax is

$$\text{sum}(list),$$

where *list* can also be a *seq* command.

The *cumSum* command in the `list` OPS menu generates a sequence of partial sums. The syntax is

$$\text{cumSum}(list).$$

Arithmetic progression

$\text{sum}(\text{seq}(1+2(N-1), N, 1, 20))$ finds the sum of the first 20 terms in our AP.

$\text{cumSum}(\text{seq}(1+2(N-1), N, 1, 20))$ generates the first 20 partial sums of our AP: the *i*th entry is the sum of the first *i* terms.

```
sum(seq(1+2(N-1),N,1,20)
.....400
cumSum(seq(1+2(N-1),N,1,10
))
{1 4 9 16 25 36 49 64 81 ...
```

The sum of an infinite series is defined as the limit as $n \rightarrow \infty$ of the *n*th partial sum. Scrolling across a list like the one above can give an idea of what that limit might be. Here, of course, there is no limit — the *n*th partial sum $\rightarrow \infty$ as $n \rightarrow \infty$.

If you prefer scrolling up and down, rather than side to side, store the partial sums to a list (`sto` `L1` `enter`) and use `stat` Edit.

Geometric progression

$$\text{sum}(\text{seq}(0.5^{(N-1)}, N, 1, 20))$$

$$\text{cumSum}(\text{seq}(0.5^{(N-1)}, N, 1, 20))$$

```
sum(seq(0.5^(N-1),N,1,20))
.....1.999998093
cumSum(seq(05^(N-1),N,1,20
))
{1 6 31 156 781 3906 1953...
```

Exercise: What's the sum of the GP $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$?

You might want to use `entry` to change the end value for N in the first command above to be (almost) sure.

Solutions to the exercises are in Section 17.6.

A neat trick for partial sums

Again we'd really like an endless sequence of partial sums. Try this.

Select FUNC(TION) in `mode`.

In `y=`: $Y_1 = \text{sum}(\text{seq}(0.5^{(N-1)}, N, 1, X))$.

In `tblset`: $\text{TblStart} = 1 \quad \Delta\text{Tbl} = 1$.

Press `table`. Scroll down in the Y_1 column (the partial sums) to see more digits at the bottom. *What's your best estimate for the sum?*

Why does this work? In generating the table, the calculator takes a given value of X , an integer in this case generated by the values of TblStart and ΔTbl , and puts it in the formula for Y_1 , giving the partial sum of X terms. It then displays the result in the Y_1 column of the table. TblStart and ΔTbl must be integers.

17.3.3 Using the SERIES/SERIESCE program

We can get around the problem of not being able to sum recursively defined sequences, such as the Fibonacci sequence, by a combination of methods. If we define the sequence as usual in $u(n)$, then use the 'neat-trick' idea by setting $Y_2 = \text{sum}(\text{seq}(u(n), n, 1, X))$, we end up with the Y_2 column in the table giving the partial sums of the sequence/series. If we also set $Y_1 = u(X)$, we have the sequence values in the Y_1 column.

The SERIES/SERIESCE program sums series defined either recursively (using the above method) or explicitly. Run the program,⁷ select how the n th term is specified, enter the appropriate definition and:

- generate a table of sequence values and corresponding partial sums. Once a table has been generated, you can change TblStart to look at the partial sums starting at different values of N , and increase ΔTbl to move through the partial sums more quickly, for example to examine the convergence or otherwise of the series.
- graph the partial sums. Sometimes a graph gives a better idea of where the partial sums are heading than a table.
- use FAST SUM to calculate a particular partial sum faster.

The SERIES/SERIESCE program is a useful tool for investigating infinite series.

Exercise: Use the SERIES/SERIESCE program to generate a table and a plot of the first 20 partial sums of:

1. the GP with n th term $u(n) = 0.5^{n-1}$;
2. the Fibonacci sequence.

Solutions to the exercises are in Section 17.6.

⁷Press `prgm` and select the SERIES/SERIESCE program by pressing the number against the program name or by scrolling down and pressing `enter`. Press `enter` again to run the program.

17.4 Questions

Solutions to the questions are in Section 17.6.

1. Generate a sequence of the cubes of first 10 positive integers using a LIST command. Store the sequence in a list. Use the *sum* command to evaluate $1^3 + 2^3 + \cdots + 10^3$.
2. Find the sum of the first 10 terms of the series $1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots$.
3.
 - (a) Use a single command to find the sum of the first 100 positive integers. Store the answer in memory S for use in (c).
 - (b) Edit your command in (a) to find the sum of the *cubes* of the first 100 positive integers.
 - (c) Which is bigger: the sum of the cubes of the first 100 positive integers or the cube of the sum of the first 100 positive integers?
4. The half-life of a certain radioactive substance is 1 week. This means that, of the amount present at a particular time, only half will be left a week later. Suppose 1000 grams of the substance exists today, the beginning of Week 1.
 - (a) Write down the amount left at the beginning of Week 2, Week 3, \dots , Week 10.
 - (b) Determine an infinite geometric sequence (recursive or explicit) that is a model of the amount of the substance at the beginning of Week n , where $n = 1, 2, 3, \dots$. What is the common ratio of this sequence?
 - (c) When will there be only 0.005 grams remaining?
 - (d) How much of the substance was there a week ago (beginning of Week 0)?
 - (e) When will the substance be reduced to nothing *according to this model*?
5. The height of a particular fast-growing plant increases at the rate of 2.5% per month. Assume the plant is 30 cm high today and that it dies after 12 months.
 - (a) Determine a finite geometric sequence that is a model of the height of the plant after n months. Write out all the terms of the sequence. What is the common ratio?
 - (b) How long would the plant have to live to double in height?
6. Sue had \$1250 in a savings account 3 years ago. How much will be in her account 2 years from now, assuming no deposits or withdrawals are made and the account earns 6.5% interest compounded annually?
7. Frank has \$12,876 in a savings account today. He made no deposits or withdrawals during the last 6 years. What was the value of his account 6 years ago? Assume that the account earned 5.75% interest compounded monthly.

PTO

8. Generating sequences recursively is equivalent to another mathematical process called *iteration*, in which we do the same operation over and over. Try some of the following sequences/iterations.

Generate some terms in each sequence. What happens to $u(n)$ in each sequence as n becomes large? Be careful with brackets.

(a) $u(n) = (u(n-1))^2$. Try $u(1) < -1$; $u(1) = \pm 1$; $-1 < u(1) < 1$; $u(1) > 1$.

(b) $u(n) = (u(n-1))^2 - 1$ with $-2 < u(1) < 2$.

What happens if we use set $u(1) = \frac{1}{2}(1 + \sqrt{5}) = 1.618\dots$?

$u(1) = \frac{1}{2}(1 - \sqrt{5}) = 0.618\dots$?

(c) $u(n) = \sqrt{u(n-1)}$.

(d) $u(n) = \cos(u(n-1))$.

(e) $u(n) = \tan(u(n-1))$.

9. What value does the sequence $\left(1 + \frac{1}{n}\right)^n$ approach as n gets larger and larger?

Hint: Use a sequence command with a step of at least 1000. Or apply the neat-trick method for sequences: set $Y_1 = (1 + 1/X)^X$ and use a suitably large ΔT bl, say 1000.

10. What is $\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$? factorial ! is in the math PRB menu.

The *cumSum(seq* combination works well here, as we don't need too many terms to see the convergence of the partial sums. Store 20 or so terms in a list to make it easier to see the partial sums or use the neat trick for series. Note that the sum starts at $n=0$.

What is $\sum_{n=0}^{\infty} \frac{2^n}{n!}$? $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, where x is any number?

The following questions are from Chapter 9 of *Intermediate Algebra: Functions and Graphs* by K. Yoshiwara and B. Yoshiwara, Thomson Brooks/Cole, 2004.

11. Generate a table of values for each of the following recursive sequences. What happens to the terms as n gets larger? Do you recognise the number?

Hint: The answers for the first two sequences are square roots of particular numbers. *Can you make and test a conjecture here? Write down a sequence whose limit is 2 ; $\sqrt{5}$.*

The answers for the last two sequences should be obvious. *Can you make and test a conjecture here? Write down a sequence whose limit is 4 ; 5 .*

(a) $a_1 = 1$ $a_n = \frac{1}{1 + a_{n-1}} + 1$ (b) $b_1 = 1$ $b_n = \frac{2}{1 + b_{n-1}} + 1$

(c) $s_1 = 1$ $s_n = \frac{1}{2} \left(s_{n-1} + \frac{4}{s_{n-1}} \right)$ (d) $t_1 = 1$ $t_n = \frac{1}{2} \left(t_{n-1} + \frac{9}{t_{n-1}} \right)$

PTO

- 12.** A rubber ball is dropped from a height of 8 metres and returns to three-quarters of its previous height on each bounce.
- (a) How high does the ball bounce after hitting the floor for the 3rd time? for the 10th time? for the n th time?
 - (b) How far has the ball travelled vertically when it hits the floor for the 4th time? for the 20th time? for the b th time?
- 13.** According to legend, a man who had pleased the Persian king asked for the following reward. The man was to receive a single grain of wheat for the first square of the chessboard, 2 grains for the second square, 4 grains for the third square, and so on, doubling the amount for each square up to the 64th square. How many grains would he receive in all. (Fortunately the king had a good sense of humour.)
- 14.** Find the sum of all integral multiples of 6 between
- (a) 10 and 100.
 - (b) between 1 and 10,000.

PTO

17.5 Further investigations

17.5.1 Fibonacci numbers

Fibonacci (fib-on-archie), real name Leonardo Pisano (Leonardo of Pisa), was born in about 1170. He too thought about populations, but much earlier than Malthus.⁸ One of his problems concerning a rabbit population led to the famous Fibonacci numbers or Fibonacci sequence

$$1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \quad \dots$$

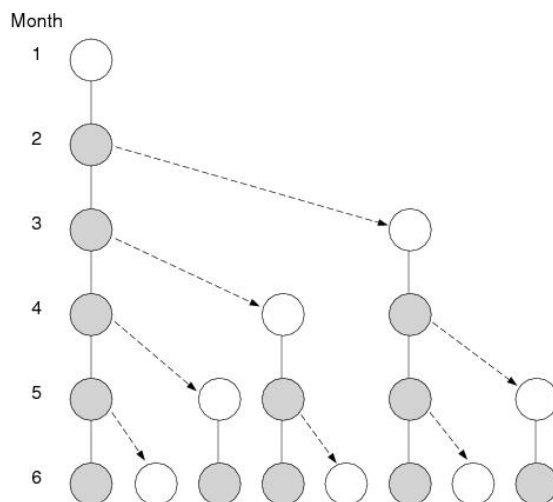
The Rule: Add the previous two numbers to get the next number (after starting with 1 1) .

Exercise: Write down the first twenty Fibonacci numbers.

Solutions to the exercises in this section are in Section 17.6.

Here is Fibonacci's rabbit problem. See if you can understand why the Fibonacci numbers give the number of *pairs* of rabbits each month and answer the question. The diagram might help.

Exercise: A pair of new-born rabbits is put in a pen. How many pairs of rabbits are there after a year if, every month, each adult pair produces a new pair? The rabbits become adult one month after birth.



In the diagram, an open circle represents a *pair* of immature rabbits (too young to breed) and a shaded circle a *pair* of mature breeding rabbits. The arrows lead to offspring. Adding the number of circles for each month gives the number of pairs of rabbits — the Fibonacci numbers.

Note that the growth here is not exponential (we are not multiplying by a constant to obtain the next number), but the number of rabbits still increases rapidly.

PTO

⁸see *Population Modelling 1* in Volume 1 of this book

The Fibonacci sequence and the Golden Ratio

Here is one illustration of the many interesting properties of the Fibonacci sequence.

1. Generate a table of values of the sequence (be very careful with brackets)

$$u_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n.$$

Do you recognise this sequence?

2. One of the many interesting things about the Fibonacci sequence becomes apparent when we look at ratios of successive terms. Using the sequence grapher, set

$$u(n) = u(n-1) + u(n-2) \quad u(1) = 1, \quad u(2) = 1^9$$

to generate the Fibonacci sequence, and

$$v(n) = u(n-1)/u(n-2)$$

to calculate the ratios of successive terms.

Put TblStart = 3, ΔTbl = 1 and generate the table of values.

Scroll down and look at the $v(n)$ column containing a sequence of the ratios of successive terms of $u(n)$. Does this sequence appear to be approaching a particular value? What value? Find this value accurate to six decimal places.

Now find a decimal value for the Golden Ratio or Golden Section, $\frac{1+\sqrt{5}}{2}$.

What conclusion do you reach?

The Golden Number/Ratio/Section

The Golden Ratio (also called the Divine Proportion) is denoted by the Greek letter ϕ or τ . If a length is divided in the Golden Ratio $1:\phi$, the ratio of the longer part to the whole, $\phi:1+\phi$, is also the Golden Ratio. When used to construct a rectangle, this ratio was thought to make the rectangle pleasing to the eye. Hence the Golden Ratio occurs everywhere, and has supposedly been used to design buildings from the Parthenon to the United Nations building in New York, as well as by artists and musicians. It is closely connected with the Fibonacci series, and has a value of $(1+\sqrt{5})/2=1.61803\dots$

Fibonacci numbers are evident in particular (equi-angular or logarithmic) spirals which appear frequently throughout the natural world — in things as small as the double helix and other microscopic twisting structures, to the galaxies that move in equi-angular spirals. These spirals also account for gastropods growing while maintaining their shape. In order for all leaves on a stem to catch sunlight, they are arranged in equiangular spirals that incorporate the Fibonacci numbers. Other examples of this natural phenomenon include pine-cone seeds, flower petals, sunflower seeds, the horns of mountain goats, elephant tusks, lions' claws, scales in pineapples, and so on. Find out more at www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fib.html.

⁹On a TI-84, set $u(nMin) = \{1, 1\}$.

17.5.2 The logistic sequence

The logistic sequence or logistic map has become famous because it is one of the simplest sequences that exhibits chaotic behaviour. It also turns up in a number of areas such as population modelling.¹⁰

The logistic sequence is defined by (be very careful with brackets)

$$u(n) = Au(n-1)(1-u(n-1)),$$

where A is a constant.

The following figures show the set up for *time plots*, $u(n)$ vs n , on the left and for *cobweb plots*, $u(n)$ vs $u(n-1)$, on the right. The top line of the `format` screen is where you choose which plot you want.

- Note that $n\text{Min}=0$ here and we have taken $u(0)=0.5$.¹¹ The value for $u(0)$ can be changed in subsequent plots.
- Store a value for A ($0 < A < 4$) in memory A and press `trace`.
- For a cobweb plot, keep pressing the right arrow to generate the plot. The curves $y = Ax(1-x)$ and $y = x$ are also plotted; the cobweb lines move between these two curves. `2nd` `quit` returns you to the Home screen.
- The TI-84/CE program¹² CHAOS/CHAOSCE makes the process of graphing the logistic sequence a bit easier, allowing you to concentrate on the effects of changing A and $u(0)$.

```

TYPE: SEQ(n)   SEQ(n+1)   SEQ(n+2)
nMin=0
■\u(n)■Au(n-1)(1-u(n-1)
u(0)■0.5

```

```

Time Web uv vw uw
RectGC PolarGC
CoordOn CoordOff
GridOff GridDot GridLine
GridColor: MEDGRAY
Axes: BLACK
LabelOff LabelOn
ExprOn ExprOff
BorderColor: 1
Background: Off

```

format

```

Time Web uv vw uw
RectGC PolarGC
CoordOn CoordOff
GridOff GridDot GridLine
GridColor: MEDGRAY
Axes: BLACK
LabelOff LabelOn
ExprOn ExprOff
BorderColor: 1
Background: Off

```

PTO

¹⁰For examples, see *Population Modelling 2* in Volume 2 of this book.

¹¹ $u(n\text{Min})=\{0.5\}$ on an 84

¹²available at *www.XXX*

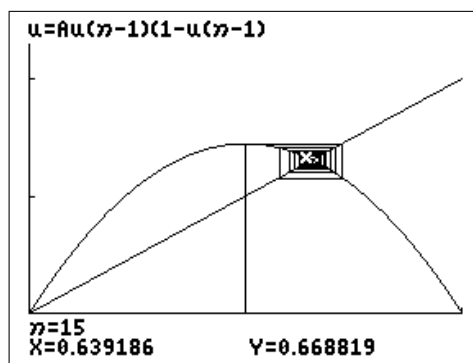
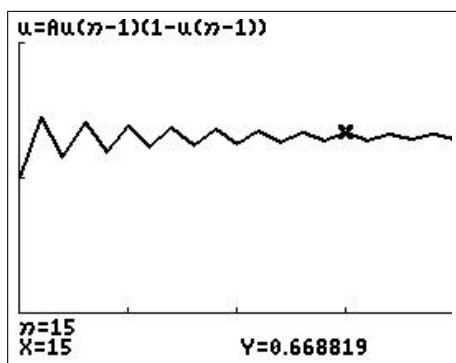
Time

Web

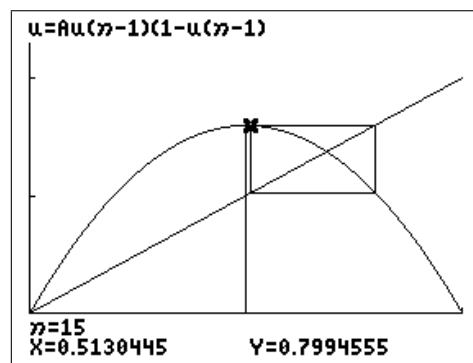
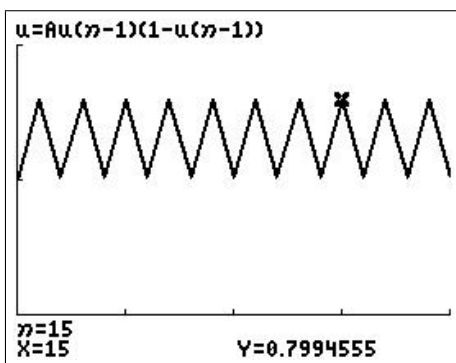
```
WINDOW
nMin=0
nMax=20
PlotStart=1
PlotStep=1
Xmin=0
Xmax=20
Xscl=5
Ymin=0
↓Ymax=1
```

```
WINDOW
nMin=0
nMax=20
PlotStart=1
PlotStep=1
Xmin=0
Xmax=1
Xscl=0.5
Ymin=0
↓Ymax=1.15
```

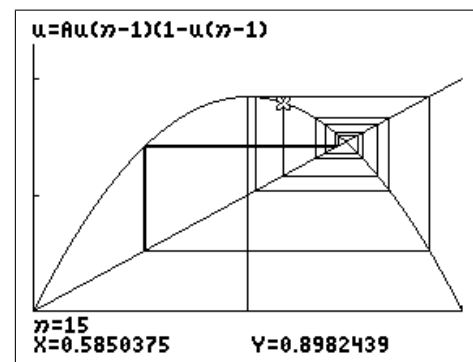
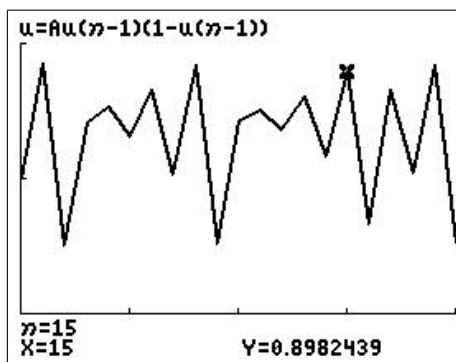
window



$A = 2.9$



$A = 3.2$



$A = 3.7$

Exercise: Confirm graphically (by choosing appropriate values for A) that the logistic sequence:

- converges to 0 for $0 < A < 1$;
- converges to $1 - 1/A$ for $1 < A < 3$;
- oscillates between 2 values, then 4 values, 8 values and so on, as A is increased above 3;
- becomes chaotic for values of A greater than about 3.568;
- diverges rapidly to $\pm\infty$ for values of A greater than 4 or values of a_1 less than 0 or greater than 1

Try both Time and Web plots.

Solutions are in Section 17.6.

PTO

17.5.3 The harmonic series

The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

has many interesting properties.¹³

1. Use the neat-trick method to generate the partial sums, starting with the first.

Then press `tblset` and change *Indpnt* to *Ask* using the cursor and `enter`. Press `table`. Now you can input the number of terms (X) in the partial sum, up to X = 999, and the calculator will calculate the corresponding partial sum (Y).

Does the series appear to be converging?

2. The *seq* command can only generate up to 999 terms. To sum more terms than this, we need a program. Use FAST SUM in the SERIES/SERIESCE program (Section 17.3.3) to find the sum of the harmonic series to 2000 and 4000 terms. Record the number of terms and the partial sum in a table, continuing on from your calculator table in Q1. Continue the table for as long as you have patience. Doing something else while the program runs is a good idea. The CE is actually quite quick.

Does the series appear to be converging?

3. Graph the data using the SERIES/SERIESCE program.

What can you conclude about the harmonic series?

4. There is an approximation to the partial sums of the harmonic series,

$$\sum_{n=1}^m \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots + \frac{1}{m} \approx \frac{1}{2m} + \ln(m) + \gamma,$$

where $\gamma = 0.5772\dots$ is the Euler or Euler-Mascheroni constant.

Check this approximation against the values you found in Q1 and Q2. To compare with your values in 1, set $Y_2 = (2X)^{-1} + \ln(X) + 0.5772$ and use the table.

What do you conclude?

5. Find the next few decimal places in the Euler-Mascheroni constant.
6. Find out what you can about the Euler-Mascheroni constant.

Solutions are in Section 17.6.

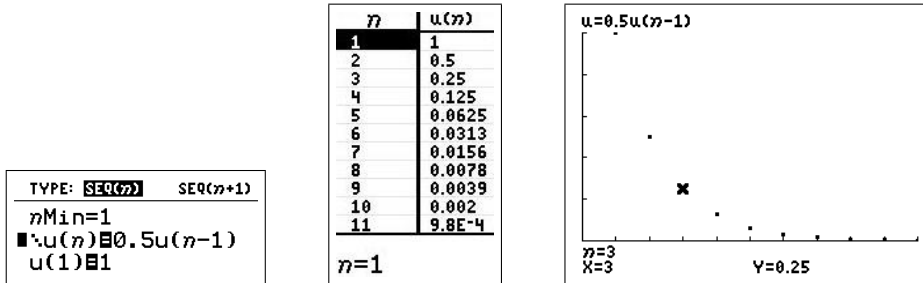
¹³The series does not converge — the more more terms you take, the larger the sum, despite the fact that the terms tend to 0 as $n \rightarrow \infty$.

17.6 Solutions to exercises, questions and activities

Exercises

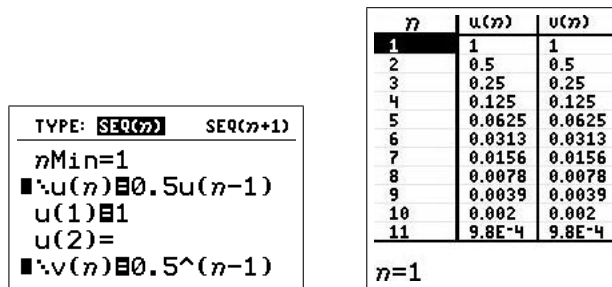
Exercise page 7

Display a table and graph the geometric sequence $u(n) = 0.5 u(n - 1)$, with $u(1) = 1$.



Exercise page 7

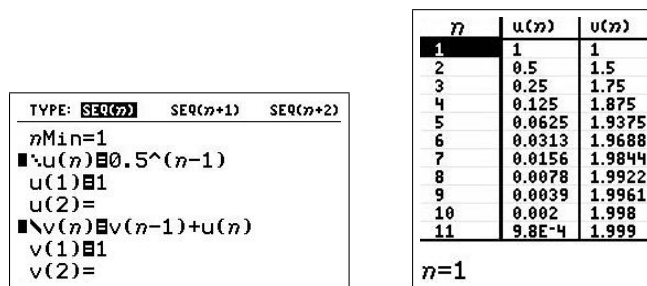
Put this explicit definition in sequence v and compare with the recursive definition.



The two sequences are identical.

Exercise page 11

Set up the table of partial sums for our GP with $u(n) = 0.5^{n-1}$.



To what value does the (infinite) series appear to converge: 2.

Confirm your answer algebraically (using the usual formula for a GP).

Here $a=1$ and $r=0.5$, so that

$$S_{\infty} = \frac{a}{1-r} = \frac{1}{1-0.5} = 2.$$

Exercise page 12

What's the sum of the GP $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$?

You might want to use to change the end value for N to be (almost) sure.

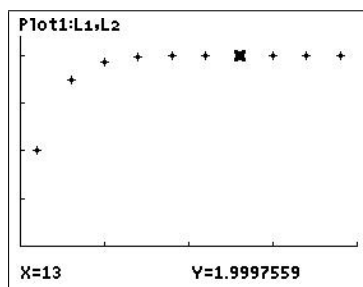
```
sum(seq(0.5^(N-1),N,1,20))
.....1.999998093
sum(seq(0.5^(N-1),N,1,50))
.....2
sum(seq(0.5^(N-1),N,1,100))
.....2
)
```

From the figure, we are pretty sure the sum of the GP is 2.

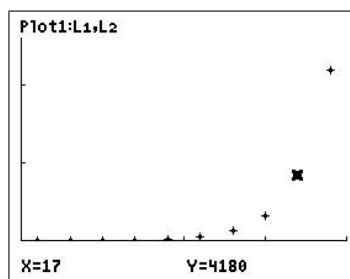
Exercise page 13

Use the SERIES/SERIESCE program to generate a table and a plot of the first twenty partial sums of:

- the GP with n th term $u(n) = 0.5^{n-1}$



- the Fibonacci sequence: $u(n) = u(n-1) + u(n-2)$; $u(1) = u(2) = 1$.



Questions page 14

1. The command is $\text{seq}(X^3, X, 1, 10)$ $\boxed{\text{sto}}$ L1. You can use any letter instead of X: X is easiest to use because it requires only one key press. If you just execute the *seq* command, the combination $\boxed{\text{sto}}$ L1 on the next line produces the output Ans \rightarrow L1 and stores the sequence. To calculate the sum, use $\text{sum}(L1)$ to produce the answer 3025.

```
seq(X^3,X,1,10,1)→L1
{1 8 27 64 125 216 343 512}
sum(L1)
3025
```

2. The command is $\text{sum}(\text{seq}(1/X^3, X, 1, 10))$, giving 1.197531986.

```
sum(seq(1/X^3,X,1,10))
1.197531986
```

3. (a) The sum of the first 100 positive integers is $\text{sum}(\text{seq}(X, X, 1, 100)) = 5050$.
- (b) The sum of the *cubes* of the first 100 positive integers is $\text{sum}(\text{seq}(X^3, X, 1, 100)) = 25,502,500 = 2.55 \times 10^7$ to 3 significant digits.
- (c) The cube of the sum of the first 100 positive integers is $5050^3 \approx 1.29 \times 10^{11}$, larger than the sum of the cubes of the first 100 positive integers.

```
sum(seq(X,X,1,100))
5050
Ans^3
1.28787625E11
sum(seq(X^3,X,1,100))
25502500
```

4. (a) The amount of radioactive substance in grams at the beginning of successive weeks, starting at Week 1, is (to two decimal places)

1000, 500, 250, 125, 62.5, 31.25, 15.63, 7.81, 3.91, 1.95.

- (a) Explicitly, the amount at the beginning of Week n is $u_n = 1000(0.5)^{n-1}$. Recursively, $u(n) = 0.5u(n-1)$, with $u(1) = 1000$. The common ratio here is 0.5.
- (b) Using a calculator table with $Y_1 = 1000(0.5)^{X-1}$ or a table generated from the recursive definition using the sequence grapher, there is 0.005 g remaining sometime in Week 18, i.e. between $n=18$ and $n=19$.

X	Y ₁
9	3.9063
10	1.9531
11	0.9766
12	0.4883
13	0.2441
14	0.1221
15	0.061
16	0.0305
17	0.0153
18	0.0076
19	0.0038

n	u(n)
9	3.9063
10	1.9531
11	0.9766
12	0.4883
13	0.2441
14	0.1221
15	0.061
16	0.0305
17	0.0153
18	0.0076
19	0.0038

- (c) A week ago (beginning of Week 0), there was twice as much as there is now (beginning of Week 1), that is 2000 g.
- (d) According to this model, there will always be some of the substance left, although the amount becomes small very rapidly. You can't reduce any number to 0 by dividing it by 2 or raising it to a power.
5. (a) The height n months from now is $u_n = 30 \left(1 + \frac{2.5}{100}\right)^n = 30(1.025)^n$, $n = 1, 2, \dots, 12$. The table below shows months (top row) and corresponding heights, rounded to one decimal place.

0	1	2	3	4	5	6	7	8	9	10	11	12
30	30.8	31.5	32.3	33.1	33.9	34.8	35.7	36.6	37.5	38.4	39.4	40.4

The common ratio is 1.025.

- (b) Using the calculator table (remember to use X instead of n if you don't use the sequence grapher), the plant would double in height in the 29th month.
6. Let S_n be the amount of money in Sue's account at the start of year n , with $n = 1$ corresponding to 3 years ago. Then,

$$S_n = 1250 \left(1 + \frac{6.5}{100}\right)^{n-1} = 1250(1.065)^{n-1}.$$

In 2 years' time, $n = 1 + 5 = 6$, so the amount of money in her account will be

$$S_5 = 1250(1.065)^5 = \$1712.61 \quad \text{to the nearest cent.}$$

You could also use the calculator table to reach this answer.

7. The amount in Frank's account is given by

$$S_n = 12876 \left(1 + \frac{5.75}{12 \times 100}\right)^{12(n-1)} = 12876 \left(1 + \frac{0.0575}{12}\right)^{12(n-1)},$$

where $n = 1$ corresponds to now.

Six years ago, $n = -5$, and the amount in Frank's account was

$$S_{-5} = 12876 \left(1 + \frac{0.0575}{12}\right)^{-72} = \$9126.56 \quad \text{to the nearest cent.}$$

You could also use the calculator table (scroll up) to reach this answer.

8. (a) Set $u(n) = (u(n-1))^2$ in the sequence grapher, set $u(n\text{Min})/u(1)$ to an appropriate starting value and use the table to see what happens as n becomes large, the limit of the sequence.

Initial value	Limit of sequence
$u(1) < 1$	∞
$u(1) = -1$	1
$-1 < u(1) < 1$	0
$u(1) = 1$	1
$u(1) > 1$	∞

(b) Set $u(n) = (u(n-1))^2 - 1$.

Initial value	Limit of sequence
$u(1) < -1.618\dots$	∞
$u(1) = -1.618\dots$	$-1.618\dots$ (constant)
$-1.618\dots < u(1) < -0.618\dots$	oscillates between 0 and -1
$u(1) = -0.618\dots$	$-0.618\dots$ (constant)
$-0.618\dots < u(1) < 0.618\dots$	oscillates between 0 and -1
$u(1) = 0.618\dots$	$-0.618\dots$ (constant)
$0.618\dots < u(1) < 1.618\dots$	oscillates between 0 and -1
$u(1) = 1.618\dots$	$1.618\dots$ (constant)
$u(1) > 1.618\dots$	∞

(c) $u(n) = \sqrt{u(n-1)}$ gives 0 if $u(1) = 0$, and tends to 1 otherwise.

(d) $u(n) = \cos(u(n-1))$ tends to $0.7391\dots$ in radian mode and $0.99985\dots$ in degree mode.

(e) $u(n) = \tan(u(n-1))$ looks several times as though it is going to settle down to a limit, but never does.

9. The sequence $\left(1 + \frac{1}{n}\right)^n$ approaches the value $e = 2.71828\dots$ as n goes to ∞ .

10. $\sum_{n=0}^{\infty} \frac{1}{n!} = e$; $\sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2$; $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$, the Taylor-MacLaurin series for e^x .

11. (a) approaches $\sqrt{2}$; (b) approaches $\sqrt{3}$; (c) approaches 2; (d) approaches 3.

12. (a) The following table gives the rebound height after bounce n .

n	1	2	3	4	...
Height (m)	6	$\frac{9}{2}$	$\frac{27}{8}$	$\frac{81}{32}$...

The rebound height after the 3rd bounce is therefore $27/8$ m.

The rebound height after bounce n is $6(0.75)^{n-1}$ m. Therefore, after the 10th bounce, the rebound height is $6(0.75)^9 \approx 0.45$ m.

Alternatively, put $Y_1 = 6(0.75)^{X-1}$ and use the table.

X	Y ₁
1	6
2	4.5
3	3.375
4	2.5313
5	1.8984
6	1.4238
7	1.0679
8	0.8009
9	0.6007
10	0.4505
11	0.3379

(b) The total distance travelled is $8 + 2 \times$ each rebound height, i.e.

$$8 + 2 \left(6 + \frac{9}{2} + \frac{27}{8} + \frac{81}{32} + \dots \right).$$

When the ball hits the floor for the 4th time, it has travelled

$$8 + 2 \left(6 + \frac{9}{2} + \frac{27}{8} + \frac{81}{32} \right) = 35.75 \text{ m.}$$

Therefore, the distance travelled when it hits the floor for the b th time ($b \geq 2$) is

$$8 + 2 \sum_{n=1}^{b-1} 6(0.75)^{n-1} = 8 + 12 \sum_{n=1}^{b-1} 0.75^{n-1}.$$

Therefore, when the ball hits the floor for the 20th time, it has travelled

$$8 + 12 \sum_{n=1}^{19} 0.75^{n-1} \approx 55.8 \text{ m.}$$

Here, use the command $8 + 12 \text{ sum } (\text{seq}(0.75 \wedge (B-1), B, 1, 19))$ to do the calculation.

13. Number of grains = $1 + 2 + 4 + 8 + 16 + \dots = \sum_{n=1}^{64} 2^{n-1} \approx 1.89 \times 10^{19}$.

Use $\text{sum}(\text{seq}(2 \wedge (N-1), N, 1, 64))$ to calculate this.

To put this number into context, 10^{18} grains would be about the same volume as the Great Wall of China and the five Great Pyramids combined.¹⁴

14. We need to evaluate $\sum_{n=a}^b 6n$, where

(a) $6a$ is the smallest multiple of 6 greater than 10 and $6b$ is the largest multiple of 6 less than 100. Clearly, $a=2$ and $b=16$, so that the required sum is

$$\sum_{n=2}^{16} 6n = 810,$$

where we used the command $\text{sum}(\text{seq}(6N, N, 2, 16))$.

PTO

¹⁴From *Large Numbers* by Victor Scharaschkin, Australian Senior Mathematics Journal 4 (2), 111–125 (1990).

- (b) $6a$ is the smallest multiple of 6 greater than 1 and $6b$ is the largest multiple of 6 less than 10,000. Clearly, $a = 1$ and b is the integer part of $10,000/6$, that is 1666. The required sum is

$$\sum_{n=1}^{1666} 6n = 8,331,666.$$

We can't use the single command `sum(seq(6N, N, 1, 1666))` here because there are more than 999 terms in the series. Either break the sum up into two parts, `sum(seq(6N, N, 1, 999))` and `sum(seq(6N, N, 1000, 1666))`, and add the two answers, or use the `SERIES/SERIESCE` program.

PTO

Activities

Rabbits and Fibonacci Numbers

The first twenty Fibonacci numbers are

1 1 2 3 5 8 13 21 34 55 89 144 233 377 610 987 1597 2584 4181 6765.

How many pairs will there be in one year?

1. There is only 1 pair of immature rabbits during the first month.
2. At the beginning of the 2nd month they mate, but there is still only 1 pair during the month.
3. At the beginning of the 3rd month the female produces a new pair, so now there are 2 pairs of rabbits in the pen.
4. At the beginning of the 4th month, the original female produces a second pair, making 3 pairs in all in the pen.
5. At the beginning of the 5th month, the original female has produced yet another new pair, while the female born 2 months ago produces her first pair, making 5 pairs in all. and so on.

The number of pairs of rabbits after one year is the 12th Fibonacci number, 144 pairs.

Why do the Fibonacci numbers appear as the number of rabbits in the pen each month?

If we let $f(n)$ be the number of pairs of rabbits in the pen at the start of the n th month, we will show that $f(1)=1$, $f(2)=1$ and $f(n)=f(n-1)+f(n-2)$, which is exactly the definition of the Fibonacci numbers.

We start in month 1 with one newly born pair, so $f(1)=1$.

There is also only 1 pair during month 2, since although the adults mate at the start of month 2, babies are not born until the start of month 3: $f(2)=1$.

Now look at the n th month.

All the rabbits from the previous month ($f(n-1)$ pairs of them) survive. Any rabbit (pair) that was alive 2 months ago is now able to produce a new pair; we assume they produce 1 and only 1 new *pair* per month. Thus, the number of newly born pairs is the same as the number of pairs alive 2 months ago, $f(n-2)$. The total number of rabbits this month is the sum of all the rabbits alive last month and those that are newly born this month, that is $f(n)=f(n-1)+f(n-2)$ — the definition of the Fibonacci numbers.

The Fibonacci sequence and the Golden Ratio page 18

1. Generate a table of values of the sequence (be very careful with brackets)

$$u_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n$$

TYPE:	SEQ(<i>n</i>)	SEQ(<i>n</i> +1)	SEQ(<i>n</i> +2)
<i>n</i> Min=	1		
u(<i>n</i>)	((1+√(5))/2)^ <i>n</i> - ((1-√(5))/2)^ <i>n</i> / √(5)		
u(1)	1		
u(2)	1		

<i>n</i>	u(<i>n</i>)
1	1
2	1
3	2
4	3
5	5
6	8
7	13
8	21
9	34
10	55
11	89

n=1

Do you recognise this sequence? It's the Fibonacci sequence.

2. One of the many interesting things about the Fibonacci sequence becomes apparent when we look at ratios of successive terms. Using the sequence grapher, set

$$u(n) = u(n-1) + u(n-2) \quad u(1) = 1, \quad u(2) = 1^{15}$$

to generate the Fibonacci sequence, and

$$v(n) = u(n-1)/u(n-2)$$

to calculate the ratios of successive terms.

Put TblStart = 3, ΔTbl = 1 and generate the table of values.

<i>n</i>	u(<i>n</i>)	v(<i>n</i>)
3	2	1
4	3	2
5	5	1.5
6	8	1.6667
7	13	1.6
8	21	1.625
9	34	1.6154
10	55	1.619
11	89	1.6176
12	144	1.6182
13	233	1.618

n=3

Scroll down and look at the v(*n*) column containing a sequence of the ratios of successive terms of u(*n*).

<i>n</i>	u(<i>n</i>)	v(<i>n</i>)
9	34	1.6154
10	55	1.619
11	89	1.6176
12	144	1.6182
13	233	1.618
14	377	1.6181
15	610	1.618
16	987	1.618
17	1597	1.618
18	2584	1.618
19	4181	1.618

v(18)=1.6180344478217

¹⁵On a TI-84, set u(*n*Min) = {1, 1}.

Does this sequence appear to be approaching a particular value? What value? Find this value accurate to six decimal places.

From the table, the ratio is approaching 1.618034 (to 6 decimal places); all values after $v(17)$ round to this value.

Now find a value for the Golden Ratio or Golden Section, $\frac{1+\sqrt{5}}{2}$.

$\frac{1+\sqrt{5}}{2} = 1.618034$ to 6 decimal places.

What conclusion do you reach?

The ratio of successive terms in the Fibonacci sequence tends to the Golden Ratio as $n \rightarrow \infty$.

The harmonic series page 22

The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

has many interesting properties.

1. Use the neat-trick method to generate the partial sums, starting with the first.

`Y1=Σsum(seq(1/N,N,1,X))`

Then press `tblset` and change *Indpnt* to *Ask* using the cursor and `enter`. Press `table`. Now you can input the number of terms (X) in the partial sum, up to $X = 999$, and the calculator will calculate the corresponding partial sum (Y).

X	Y1		
100	5.1874		
200	5.878		
300	6.2827		
400	6.5699		
500	6.7928		
600	6.975		
700	7.129		
800	7.2625		
900	7.3802		
999	7.4845		

Y1=7.4844708605505

Does the series appear to be converging?

Not obvious one way or the other.

PTO

2. The *seq* command can only generate up to 999 terms. To sum more terms than this, we need a program. Use FAST SUM in the SERIES/SERIESCE program (Section 17.3.3) to find the sum of the harmonic series to 2000 and 4000 terms. Record the number of terms *M* and the partial sum in a table, continuing on from your calculator table in Question 1. Continue the table for as long as you have patience.

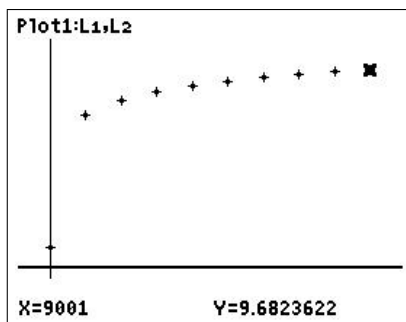
Table values rounded to 3 decimal places. *Approx* refers to Question 4.

M	Sum	Approx
100	5.187	5.187
200	5.878	5.878
300	6.283	6.283
400	6.570	6.570
500	6.793	6.793
600	6.975	6.975
700	7.129	7.129
800	7.263	7.263
900	7.380	7.380
999	7.484	7.484
2000	8.178	8.178
4000	8.871	8.871
6000	9.277	9.277
8000	9.564	9.564
10,000	9.788	9.788
20,000	10.481	10.481
30,000	10.886	10.886
40,000	11.174	11.174

Does the series appear to be converging?

Still not clear.

3. Graph the data using the SERIES/SERIESCE program.



What can you conclude about the harmonic series?

Still not clear.

PTO

4. There is an approximation to the partial sums of the harmonic series,

$$\sum_{n=1}^M \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{M} \approx \frac{1}{2M} + \ln(M) + \gamma,$$

where $\gamma = 0.5772\dots$ is the Euler or Euler-Mascheroni constant.

Check this approximation against the values you found in Questions 1 and 2. To compare with your values in 1, set $Y_2 = (2X)^{-1} + \ln(X) + 0.5772$ and use the table.

See the table in Question 2.

What do you conclude?

The approximation is accurate to 3 decimal places for M up to at least 40,000.

5. Find the next few decimal places in the Euler-Mascheroni constant.

Rewriting the equation above for the approximation to the partial sums of the harmonic series,

$$\gamma = \sum_{n=1}^M \frac{1}{n} - \left(\frac{1}{2M} + \ln(M) \right).$$

Setting $Y_1 = \text{sum}(\text{seq}(1/N, N, 1, X)) - ((2X)^{-1} + \ln(X))$ and using the table gives, from comparing the values for $X = 900$ and $X = 999$, $\gamma = 0.5772156$.

To go further, we need to use the program.

$$M = 10,000 \quad \gamma = 0.5772156641$$

$$M = 20,000 \quad \gamma = 0.5772156647$$

$$M = 30,000 \quad \gamma = 0.5772156648$$

$$M = 40,000 \quad \gamma = 0.5772156648$$

From this process, we obtain $\gamma = 0.5772156648$, probably accurate to this number of decimal places ± 1 in the last place.

Wolfram Mathworld: $\gamma = 0.577215664901532860606512090082402431042\dots$

Evidently, the larger M , the better the approximation to the partial sums of the harmonic series, and hence the more accurate the value we obtain for γ .

As a corollary, $\ln(M) \rightarrow \infty$ as $M \rightarrow \infty$, so the harmonic series must diverge.

18 Probability and Statistics 2

Probability Distributions and Hypothesis Testing

18.1 Introduction

Probability and Statistics 1 in the first volume of this book dealt with descriptive statistics, that is using the calculator to store, process and display statistical data. Here, we deal with discrete and continuous probability distributions, and hypothesis testing. We end with a section on gambling systems as a ‘practical’ example.

The most comprehensivel book I have seen on this topic is *Probability and Statistics with the TI-83 Plus: For A-Level Mathematics* by Peter Jones and Chris Barling.¹⁶ Sadly, this is no longer readily available; if you have a copy, treasure it. They explain everything in great detail, with large numbers of screen shots. The exposition here is somewhat briefer but I’ll follow their general outline, and even borrow some of their examples (labelled JB). Most of the exercises are also from there.

Some of the examples and exercises are from the course notes for a first-year introductory Statistics course by Dr Leesa Sidhu of UNSW Canberra.

What follows is not a textbook on Probability and Statistics (you’ll need one) but how to do the various operations in such a course on a TI-84/CE graphics calculator. This replaces the various tables usually used in such courses, and makes many of the calculations quite simple. As always, students should do all these operations by hand first, so that they understand the process but, ultimately, always having to do the calculations manually stands in the way of doing any meaningful modelling and interesting applied problems.

18.1.1 Australian Curriculum

References are given here to the texts *Nelson Senior Maths Methods 11 for the Australian Curriculum* (NSM11) and *Nelson Senior Maths Methods 12 for the Australian Curriculum* (NSM12) used in the ACT.

The material here is relevant to the topics *Discrete random variables* (Chapter 2 in NSM12), *Binomial distributions* (Chapter 5 in NSM12), *Continuous random samples and the normal distribution* (Chapter 8 in NSM12), *Random samples and proportions* (Chapter 9 in NSM12), and *Confidence intervals* (Chapter 10 in NSM12).

18.1.2 Calculator note

There are four options on the calculator that can help with the calculations here.

MATHPRINT

In CLASSIC mode, commands are typed on one line, with arguments in brackets. In MATHPRINT mode, the calculator tries to display commands in mathematical notation, with small boxes in the relevant positions for the inputs. This really only applies to the powers and roots (blue) commands on the keyboard and to many of the commands in the `math` NUM menu, and so is of limited use in the Probability and Statistics operations here; where it applies, it is illustrated with a screen shot. Set CLASSIC or MATHPRINT in `mode`.

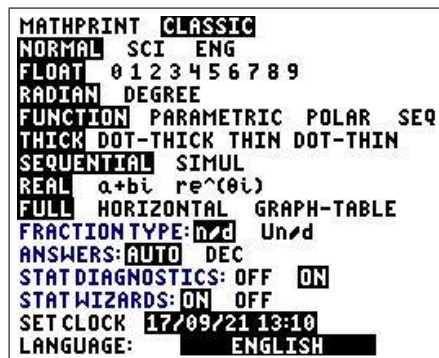
¹⁶Cambridge University Press, 2002, ISBN 0 521 52531 4

Catalog Help

An app on the TI-84Plus,¹⁷ built in to the operating system of a TI-84CE, this tells you what arguments are required for most of the commands on the calculator in CLASSIC mode, allows you to input the arguments and paste the whole command onto the Home screen. Activated by scrolling down to the particular command and pressing $\boxed{+}$.

STAT WIZARDS

An alternative to *Catalog Help*, and much more useful here than MATHPRINT, this provides a menu for inputs to most of the commands and tests in $\boxed{\text{stat}}$ CALC and $\boxed{\text{stat}}$ TESTS, rather than having to type these into a one-line command in the correct order. It also works for some operations in the $\boxed{\text{math}}$ PROB menu. Turn STAT WIZARDS on in $\boxed{\text{mode}}$ and leave on. It is used in most of the examples here.



Prob Sim app

This app¹⁷ simulates a number of different ways of obtaining random data normally done with physical materials (see the figure). You can simulate single or multiple occurrences, with the outcomes summarised in on-screen histograms. *Random Numbers* can be used to simulate Lotto draws. Lots of data can be generated and stored in lists on the calculator for subsequent analysis. You can even change the probability of an event occurring. Highly recommended.



PTO

¹⁷download from *education.ti.com*

18.2 Discrete probability distributions

18.2.1 Counting methods

The relevant commands here are all in the $\boxed{\text{math}}$ PROB menu.

```
MATH NUM CMLX PROB FRAC
1:rand
2:nPr
3:nCr
4:!
5:randInt(
6:randNorm(
7:randBin(
8:randIntNoRep(
```

To evaluate $6!$ $\boxed{6}$ $\boxed{\text{math}}$ PROB $\boxed{4}$

To evaluate ${}^{10}P_2$ $\boxed{10}$ $\boxed{\text{math}}$ PROB $\boxed{2}$ $\boxed{2}$

To evaluate ${}^{10}C_2$ $\boxed{10}$ $\boxed{\text{math}}$ PROB $\boxed{3}$ $\boxed{2}$

```
6! ..... 720
10 nPr 2 ..... 90
10 nCr 2 ..... 45
```

Both the CLASSIC and MATHPRINT versions are shown here. In MATHPRINT, select the commands before inserting any numbers; however, note what happens here if you do this with the factorial command !.

```
6! ..... 720
10P2 ..... 90
10C2 ..... 45
```

18.2.2 Random numbers

A random number between 0 and 1 is generated by the *rand* command in the $\boxed{\text{math}}$ PROB menu; an integer in brackets after the *rand* command generates a list of that many random numbers. You can store this list to *listname* using $\boxed{\text{sto}}$ *listname* either with the *rand* command or after the list of random numbers has been generated.

Random integers are generated by the *randInt* command: to generate three random integers between 1 and 17, use the command *randInt*(1, 17, 3). If you leave off the 3, you get a single random integer.

With STAT WIZARDS on, you get a menu for the input. Enter the respective numbers, then move the cursor to paste the command to the Home screen and press $\boxed{\text{enter}}$ to execute it.

```
randInt
lower:1
upper:17
n:3
Paste
```

```
randInt(1,17,3) {11 10 12}
```

Note the commands in the $\boxed{\text{math}}$ PROB menu (above) generating random numbers that are normally or binomially distributed. The last command in the menu generates random integers with no repeats, useful, for example, if you are simulating a raffle draw.

Exercises Solutions are in Section 18.6

1. Evaluate each of the following by hand, then check with your calculator.

(a) $6!$ (a) 6P_4 (a) 8C_5 .

2. Find: (a) $13!$ (b) ${}^{15}P_9$ (c) ${}^{35}C_{22}$.

3. Generate 100 random integers between 1 and 6, simulating throwing a die. Store them in an appropriately named list.¹⁸ Find: (a) the mean; (b) the standard deviation of the mean; (c) the median. Comment on your answers.

¹⁸see *Probability and Statistics 1* in Volume 1

18.2.3 Expected value and variance

Theory

The *probability distribution function* (pdf) of a discrete random variable X is

$$p(x) = P(X=x),$$

for each possible value of x ; P stands for (*the*) *probability (that)*.¹⁹

The *cumulative distribution function* (cdf) is

$$F(x) = P(X \leq x).$$

The *mean* (μ or μ_X) or *expected value* ($E[X]$) of a discrete random variable X is

$$\mu = \mu_X = E[X] = \sum_{\text{all } x} xp(x).$$

The *variance* of a discrete random variable X with mean μ is

$$V[X] = \sum_{\text{all } x} (x-\mu)^2 p(x) = \sum_{\text{all } x} x^2 p(x) - \mu^2.$$

The *standard deviation* of a random variable is the square root of the variance.

List operations

To calculate the expected value and variance of a discrete distribution, we must first enter the distribution values into lists.

Example: Consider the following distribution of a discrete random variable X .

x	0	1	2	3	4
$p(x) = P(X=x)$	0.2	0.3	0.1	0.3	0.1

Put the two sets of numbers in lists X and PX (see Probability and Statistics 1 in Volume 1 of this book).

X	PX
0	0.2
1	0.3
2	0.1
3	0.3
4	0.1

Then, the cdf $P(X \leq 2)$ is given by $p(0) + p(1) + p(2) = 0.2 + 0.3 + 0.1 = 0.6$.

On the calculator: `sum(LPX, 1, 3)`.

The calculator command *sum* is in the list MATH menu. The 1 and 3 indicate the first and third terms of LPX.

<code>sum(LPX, 1, 3)</code>	0.6
------------------------------	------------

¹⁹Pr in JB

Mean: On the calculator, the product LXL PX is a new list in which each element is the product of the corresponding elements of the individual lists, i.e. a list containing the values $xp(x)$.

If we now sum this list, $\text{sum}(\text{LXL PX})$, we obtain the mean value μ or the expected value $E[X]$ of x , 1.8 in this case (figure below).

Variance: Similarly, the variance $V[X]$ is $\text{sum}(\text{LX}^2 \text{L PX}) - \text{sum}(\text{LXL PX})^2$, 1.76 here.

Standard deviation: The standard deviation, the square root of the variance, is 1.33 here.

sum(LXL PX)	1.8
sum(LX ² L PX) - sum(LXL PX) ²	1.76
√(Ans)	1.326649916

18.2.4 The Binomial Distribution

1. Calculating binomial probabilities

If a discrete random variable X has a binomial distribution, $X \sim \text{bin}(n, p)$, the *probability distribution function* (pdf) is given by

$$P(X=x) = {}^n C_x p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \dots, n.$$

The Binomial Distribution is defined by two parameters, the number of independent trials n and the probability p of success at one trial.

If $X \sim \text{bin}(n, p)$, the calculator command to calculate $P(X=x)$ is $\text{binompdf}(n, p, x)$.

DISTR	DRAW
4↑	invT(
5:	tPdf(
6:	tcdf(
7:	X ² pdf(
8:	X ² cdf(
9:	Fpdf(
0:	Fcdf(
▣	binompdf(
▾	binomcdf(

Example: If $X \sim \text{bin}(20, 0.6)$, calculate $P(X=8)$.

Press `distr` (`2nd` `vars`) and scroll down to *binompdf*.

Press `enter` to generate an input menu.

Put in the three arguments, move the cursor to *Paste*, press `enter` to paste the command to the Home screen and `enter` again to execute it.

binompdf
trials:20
p:0.6
x value:8
Paste

`binompdf(20,0.6,8)`
0.0354974396

The binomial *cumulative distribution function* (cdf), denoted $F(x)$, is $F(x) = P(X \leq x)$.

To calculate $P(X \leq x)$, we use the binomial cdf command $\text{binomcdf}(n, p, x)$ (DISTR menu).

Example: If $X \sim \text{bin}(20, 0.6)$, find $P(X \leq 11)$.

`binomcdf(20,0.6,11)`
0.4044012768

Example: If $X \sim \text{bin}(20, 0.6)$, find $P(X \geq 11)$:
 $P(X \geq 11) = 1 - P(X \leq 10)$.

`1-binomcdf(20,0.6,10)`
0.7553372032

Example: If $X \sim \text{bin}(20, 0.6)$, find $P(2 \leq X \leq 14)$:

$$P(2 \leq X \leq 14) = P(X \leq 14) - P(X \leq 1).$$

```
binomcdf(20,0.6,14)-binomcdf(20,0.6,1)
0.8744006864
```

Exercises

- If $X \sim \text{bin}(8, 0.4)$, find
 - $P(X=4)$
 - $P(X \leq 1)$
 - $P(X \geq 7)$
 - $P(1 \leq X \leq 3)$
 - $P(X < 6) = P(X \leq 5)$
- If $X \sim \text{bin}(25, 0.2)$, find
 - $P(X=4)$
 - $P(X \leq 10)$
 - $P(X \geq 7)$
 - $P(5 \leq X \leq 8)$
 - $P(X > 4) = 1 - P(X \leq 4)$

Solutions are in Section 18.6.

2. Graphing a Binomial Distribution

Example: If $X \sim \text{bin}(20, 0.6)$, graph the distribution of X for $0 \leq X \leq 20$.

To do this, we need to store values of x in the X list and the corresponding values of $P(x = X)$ in the PX list, as we did in Section 18.2.3.

A quick way to fill the X list is to move the cursor to the list name, press `enter` and type in the command `seq(X, X, 0, 20)` (below left); `seq` is in the `list` OPS menu. Press `enter` again to fill the list (below right).

X	PX	L1
-----	-----	-----

`x=seq(X,X,0,20)`

X	PX
0	-----
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

`X(1)=0`

Similarly, move the cursor to the header of the PX list, press `enter` and insert the `binomcdf` command as shown below left. `LX` comes from the `list` NAMES menu. Press `enter` again to fill the list (below right).

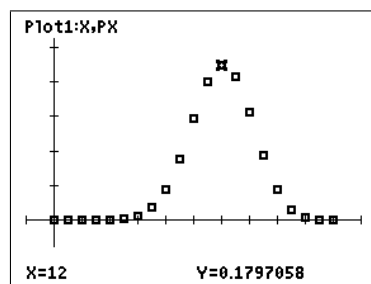
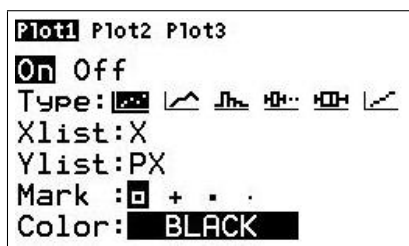
X	PX	L1	L2	L3
0	-----	-----	-----	-----
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				

`PX=binompdf(20,0.6,LX)`

X	PX	L1	L2	L3
0	1.1E-8	-----	-----	-----
1	3.3E-7			
2	4.7E-6			
3	4.2E-5			
4	2.7E-4			
5	0.0013			
6	0.0049			
7	0.0146			
8	0.0355			
9	0.071			
10	0.1171			

`PX(1)=1.0995116277759E-8`

Set up the `stat plot` as shown below left and press `zoom` `9` (ZoomStat) to plot the graph (below right). Adjust Xscl and Yscl if necessary. `trace` has been pressed and the cursor moved to $X = 12$.



Exercises

1. If $X \sim \text{bin}(10, 0.4)$, display the distribution of X graphically.
2. On the one graph, plot the distribution of:²⁰
 - $X_1 \sim \text{bin}(10, 0.1)$ use Plot1 with a square marker;
 - $X_2 \sim \text{bin}(10, 0.5)$ use Plot2 with a + marker;
 - $X_3 \sim \text{bin}(10, 0.9)$ use Plot3 with a dot marker.

Comment on how changing the value of p changes the shape of the distribution.

Solutions are in Section 18.6.

18.2.5 The Geometric Distribution

1. Calculating geometric probabilities

If a discrete random variable X has a Geometric Distribution, $X \sim \text{geom}(p)$, the *probability distribution function* (pdf)

$$P(X=x) = p(1-p)^{x-1} \quad \text{for } x = 1, 2, \dots$$

The Geometric Distribution is defined by a single parameter p .

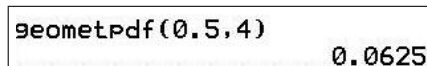
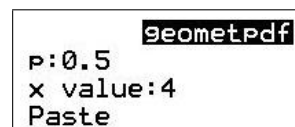
If $X \sim \text{geom}(p)$, the calculator command to calculate $P(X=x)$ is `geometpdf(p, x)`.

Example: If $X \sim \text{geom}(0.5)$, calculate $P(X=4)$.

Press `distr` (`2nd` `vars`) and go to `geometpdf`.

Press `enter` to generate an input menu.^a

Put in the two arguments, move the cursor to `Paste`, press `enter` to paste the command to the Home screen and `enter` again to execute it.



^aSTAT WIZARDS on

²⁰You might try line graphs, set in `stat plot`, rather than scatterplots to show the three distributions better.

To calculate $P(X \leq x)$, we use the *geometric cdf* command `geometcdf(p, x)`.

Example: If $X \sim \text{geom}(0.5)$, calculate $P(X \leq 6)$.

```
geometcdf(0.5,6)
0.984375
```

Example: If $X \sim \text{geom}(0.5)$, calculate $P(X \geq 3)$:

$$P(X \geq 3) = 1 - P(X \leq 2).$$

```
1-geometcdf(0.5,2)
0.25
```

Example: If $X \sim \text{geom}(0.5)$, find $P(2 \leq X \leq 5)$:

$$P(2 \leq X \leq 5) = P(X \leq 5) - P(X \leq 1).$$

```
geometcdf(0.5,5)-geometcdf(0.5,1)
0.46875
```

Exercises

If $X \sim \text{geom}(0.2)$, find

- (a) $P(X = 5)$ (b) $P(X \leq 8)$ (c) $P(X \geq 7)$ (d) $P(3 \leq X \leq 9)$.

Solutions are in Section 18.6.

2. Graphing a Geometric Distribution

Example: If $X \sim \text{geom}(0.2)$, graph the distribution of X for $x = 1, 2, \dots, 10$.

To do this, we need to store values of x in the X list and the corresponding values of $P(x = X)$ in the PX list, as we did in Section 18.2.4.

Here, to fill the X list, execute the command `seq(X, X, 1, 10)`.

Move the cursor to the header of the PX list, press `enter` and insert the *geometpdf* command as shown. Press `enter` again to fill the list.

X	PX	L1	L2
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			

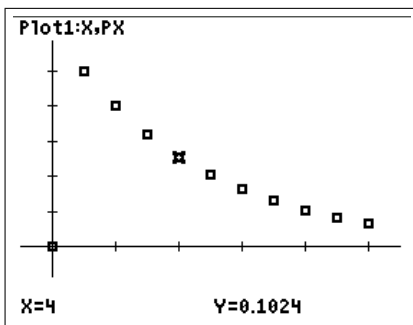
PX = `geometpdf(0.2, LX)`

X	PX
1	0.2
2	0.16
3	0.128
4	0.1024
5	0.0819
6	0.0655
7	0.0524
8	0.0419
9	0.0336
10	0.0268

PX(1) = 0.2

```
Plot1 Plot2 Plot3
On Off
Type: [ ] [ ] [ ] [ ] [ ] [ ]
Xlist: X
Ylist: PX
Mark: [ ] + [ ] [ ]
Color: BLACK
```

Set up the `stat plot` as shown above right and press `zoom` ZoomStat to plot the graph below. `trace` has been pressed and the cursor moved to $X = 4$.



Exercises

1. If $X \sim \text{geom}(0.4)$, display the distribution of X graphically for $1 \leq x \leq 8$.
2. On the one graph, plot the distribution of:
 - $X_1 \sim \text{geom}(0.1)$ use Plot1 with a square marker;
 - $X_2 \sim \text{geom}(0.5)$ use Plot2 with a + marker;
 Comment on how changing the value of p changes the shape of the distribution.

Solutions are in Section 18.6.

18.2.6 The Hypergeometric Distribution**1. Calculating hypergeometric probabilities**

If a discrete random variable X has a Hypergeometric Distribution, $X \sim \text{hypg}(N, k, n)$, the *probability distribution function* (pdf)

$$P(X=x) = \frac{{}^k C_x {}^{N-k} C_{n-x}}{{}^N C_n} \quad \text{with } 0 \leq x \leq n \text{ and } x \leq k \quad (\text{see Section 18.2.1}).$$

The Hypergeometric Distribution is defined by three parameters, the population size N , the number of successes in the population k and the sample size n .

There is no command for the pdf or cdf of the Hypergeometric Distribution but the calculations can be done both numerically and graphically by putting the pdf in Y_1 . M has been used for n .

$$\boxed{\text{Y}_1 \left(\frac{{}^k C_x {}^{N-k} C_{n-x}}{{}^N C_n} \right)}$$

Example: If $X \sim \text{hypg}(50, 10, 8)$, calculate $P(X=2)$.

Store values of N , k and M in the relevant memories, then evaluate $Y_1(2)$.

50→N	50
10→M	10
8→K	8
Y ₁ (2)	0.3217246579

To calculate $P(X \leq x)$, we have to add up the probabilities from 0 up to and including x .

Example: If $X \sim \text{hypg}(50, 10, 8)$, calculate $P(X \leq 1)$.

$P(X \leq 1) = P(X=0) + P(X=1)$. Following on from the previous example, calculate $Y_1(0) + Y_1(1)$.

$$\boxed{Y_1(0) + Y_1(1) = 0.4905024348}$$

Example: If $X \sim \text{hypg}(50, 10, 8)$, calculate $P(X \geq 2)$.

$P(X \geq 2) = 1 - P(X \leq 1) = 1 - 0.4905 = 0.509$ from the previous example.

Exercises

If $X \sim \text{hypg}(100, 15, 5)$, find

- (a) $P(X=3)$ (c) $P(X > 3)$
 (b) $P(X \leq 3)$ (d) $P(1 \leq X \leq 2)$.

Solutions are in Section 18.6.

2. Graphing a Hypergeometric Distribution

Example: If $X \sim \text{hypg}(50, 10, 8)$, graph the distribution of X for $x = 0, 2, \dots, 8$.

To do this, we need to store values of x in the X list and the corresponding values of $P(x=X)$ in the PX list, as we did in Section 18.2.4.

Here, to fill the X list, execute the command `seq(X, X, 0, 8)`.

Store values of N , k (K) and n (M) in the relevant memories. Move the cursor to the header of the PX list, press `enter` and insert the command as shown below left. Press `enter` again to fill the list.

X	PX
0	-----
1	
2	
3	
4	
5	
6	
7	
8	

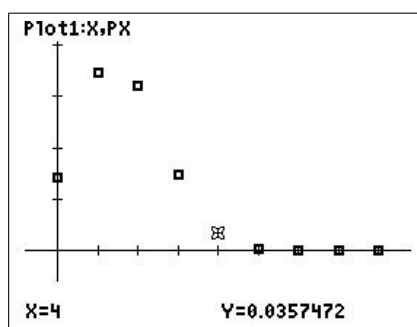
PX = Y1(LX)

X	PX	L1	L2	L3
0	0.1432	-----	-----	-----
1	0.3473			
2	0.3217			
3	0.1471			
4	0.0357			
5	0.0046			
6	3.1E-4			
7	8.9E-6			
8	8.4E-8			

PX(1) = 0.14324407387032

Plot1	Plot2	Plot3
On	Off	
Type:		
Xlist:	X	
Ylist:	PX	
Mark:		
Color:	BLACK	

Set up the stat plot as shown above right and press `zoom` ZoomStat to plot the graph. `trace` has been pressed in the figure below and the cursor moved to $X = 4$.



Exercises

- If $X \sim \text{hypg}(100, 15, 8)$, display the distribution of X graphically for $0 \leq x \leq 8$.
- On the one graph, plot the distribution of:
 - $X \sim \text{hypg}(100, 15, 8)$ use Plot1 with a square marker;
 - $X \sim \text{hypg}(100, 30, 8)$ use Plot2 with a + marker;

Comment on how changing the value of k changes the shape of the distribution.

Solutions are in Section 18.6.

18.2.7 The Poisson Distribution

1. Calculating Poisson probabilities

If a discrete random variable X has a Poisson Distribution, $X \sim \text{pois}(\mu)$, the *probability distribution function* (pdf)

$$P(X=x) = \frac{\mu^x}{x!} e^{-\mu} \quad \text{for } x = 0, 1, 2, \dots$$

The Poisson Distribution is defined by a single parameter μ .

If $X \sim \text{pois}(\mu)$, the calculator command to calculate $P(X=x)$ is `poissonpdf(μ, x)`.

Example: If $X \sim \text{pois}(3)$, calculate $P(X=5)$.

Press `distr` (`2nd` `vars`), scroll down to `poissonpdf` and press `enter` to generate an input menu.

Put in the two arguments, move the cursor to `Paste`, press `enter` to paste the command to the Home screen and `enter` again to execute it.

```
DISTR DRAW
9↑Fpdf(
0:Fcdf(
A:binomPdf(
B:binomcdf(
C:invBinom(
D:poissonPdf(
E:poissoncdf(
F:geometPdf(
G:geometcdf(
```

```
poissonPdf
μ:3
x value:5
Paste
```

```
poissonPdf(3,5)
0.1008188134
```

To calculate $P(X \leq x)$, we use the *Poisson cdf* command `poissoncdf(μ, x)`.

Example: If $X \sim \text{pois}(3)$, calculate $P(X \leq 6)$.

```
poissoncdf(3,6)
0.9664914647
```

Example: If $X \sim \text{pois}(3)$, calculate $P(X \geq 2)$:

$$P(X \geq 2) = 1 - P(X \leq 1).$$

```
1-poissoncdf(3,1)
0.8008517265
```

Example: If $X \sim \text{pois}(3)$, calculate $P(3 \leq X \leq 9)$:

$$P(3 \leq X \leq 9) = P(X \leq 9) - P(X \leq 2).$$

```
poissoncdf(3,9)-poissoncdf(3,2)
0.5757074307
```

Exercises

If $X \sim \text{pois}(0.5)$, find

- (a) $P(X=0)$ (c) $P(X \geq 1)$
 (b) $P(X \leq 3)$ (d) $P(1 \leq X \leq 4)$.

Solutions are in Section 18.6.

2. Graphing a Poisson Distribution

Example: If $X \sim \text{pois}(4)$, graph the distribution of X for $x = 0, 1, \dots, 10$.

To do this, we need to store values of x in the X list and the corresponding values of $P(x = X)$ in the PX list (Section 18.2.4).

Here, to fill the X list, execute the command `seq(X, X, 0, 10)`.

Move the cursor to the header of the PX list, press `enter` and insert the `poissonpdf` command as shown. Press `enter` again to fill the list.

X	PX	L1	L2
0			
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			

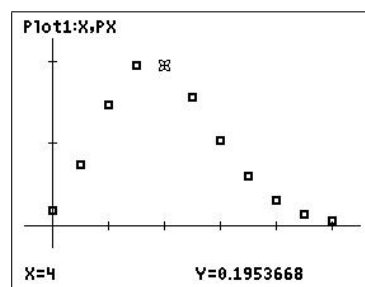
PX = poissonpdf(4, LX)

X	PX	L1	L2	L3
0	0.0183			
1	0.0733			
2	0.1465			
3	0.1954			
4	0.1954			
5	0.1563			
6	0.1042			
7	0.0595			
8	0.0298			
9	0.0132			
10	0.0053			

PX(1) = 0.018315638888734

Set up the `stat plot` as shown and press `zoom` `9` (ZoomStat) to plot the graph. `trace` has been pressed in the figure and the cursor moved to $X = 4$.

Plot1	Plot2	Plot3
On	Off	
Type:		
Xlist:	X	
Ylist:	PX	
Mark:	□ + .	
Color:	BLACK	



Exercises

1. If $X \sim \text{pois}(1.5)$, display the distribution of X graphically for $0 \leq x \leq 6$.
2. On the one graph, for $0 \leq x \leq 10$, plot the distribution of:
 - $X_1 \sim \text{pois}(2)$ use Plot1 with a square marker;
 - $X_2 \sim \text{pois}(5)$ use Plot2 with a + marker;
 - $X_2 \sim \text{pois}(9)$ use Plot2 with a dot marker.

Comment on how changing the value of λ changes the shape of the distribution.

Solutions are in Section 18.6.

18.2.8 Discrete-distribution spinner

A spinner,²¹ here a program SPDISPDS/SPDSCDCE,²² allows you to vary the parameters on screen of each of the four types of discrete probability distributions (pdf and cdf) here, by pressing a key, and see the distribution replotted immediately. The program contains the necessary information (in HELP) on how to vary the parameters.

²¹From a feature in an earlier version of Excel used by Ed Staples and Margie Smith in SpinnaMaths to explore the effects of varying parameters in a variety of applications; sadly, SpinnaMaths is no longer available because of subsequent 'upgrades' to Excel.

²²available at www.XXX

18.3 Continuous probability distributions

For continuous random variables, we do not specify the probability for every value in the range. In fact, $P(X = a) = 0$ for any value of a but we can work out the probability that X lies in a range of values. For example, the probability of finding a person whose height is exactly 2 metres is zero, but the probability of finding someone with a height between 2 and 2.01 metres might be 0.05.

For a continuous random variable X , we define a *probability density function* (pdf) $f(x)$. The continuous analogue of a summation is an integral, so the *cumulative distribution function* (cdf), $F(x) = P(X \leq x)$ is naturally defined as

$$F(x) = \int_{-\infty}^x f(u) du.$$

The probability that X lies in an interval is then the area under the pdf curve over that interval.

18.3.1 The Normal Distribution

A continuous random variable X with a Normal Distribution has pdf

$$f(z) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{z-\mu}{\sigma}\right)^2\right) \quad -\infty < z < \infty.$$

The Normal Distribution is defined by two parameters, the mean μ and variance σ^2 ; we write $X \sim N(\mu, \sigma^2)$.

1. Calculations with the Normal Distribution

The basic (cdf) calculation is

$$P(a \leq X \leq b) = \int_a^b f(z) dz.$$

This is achieved by the calculator command (with appropriate numbers inserted)

$$\text{normalcdf}(a, b, \mu, \sigma).$$

The *normalcdf* command is found in the distr menu.

If μ and σ are omitted, the standard values $\mu = 0$ and $\sigma = 1$ are assumed. *It's best to always put them in.* You can also use commands in the distr DRAW menu to show geometrically the calculation you have done. Both are shown below.

If the lower limit a is $-\infty$, set $a = -\text{E}99 = -1 \times 10^{99}$, the smallest number on the calculator.²³ Similarly, if the upper limit b is ∞ , set $b = \text{E}99 = 1 \times 10^{99}$, the largest number on the calculator.

Always think geometrically here: What area under the normal-distribution curve (pdf) am I calculating? The graphical representation below should help. You need to do a certain number of examples by hand first to understand this.

²³The symbol E is EE (on the comma key).

Example: Find $P(Z \leq 0.7) = P(-\infty < Z \leq 0.7)$, where $Z \sim N(0, 1)$, the Standard Normal Distribution (STAT WIZARDS on).

```
DISTR DRAW
1:normalPdf(
2:normalcdf(
3:invNorm(
4:invT(
5:tPdf(
6:tcdf(
7:χ²Pdf(
8:χ²cdf(
9↓Fpdf(
```

```
normalcdf
lower:-1E99
upper:0.7
μ:0
σ:1
Paste
```

```
normalcdf(-E99,0.7,0,1)
0.7580364218
```

Example: Find $P(168 \leq X \leq 196)$, where $X \sim N(180, 8^2)$.

Normally you would rewrite this problem in terms of the standard Normal Distribution, then use tables to find the answer. However, because we can specify the mean and standard deviation in the relevant calculator command, we can do this calculation in one go. Learn how to do both.

```
normalcdf(168,196,180,8)
0.9104427093
```

Example: Find $P(X \geq 13.5)$, where $X \sim N(10, 2^2)$.

Two ways to do this.

(a) $P(X \geq 13.5) = P(13.5 \leq X < \infty)$ (top line below).

```
normalcdf(13.5,E99,10,2)
0.0400591135
1-normalcdf(-E99,13.5,10,2)
)
0.0400591135
```

(b) $P(X \geq 13.5) = 1 - P(X \leq 13.5)$ (second line above).

Example: *An inverse-normal problem*

Find c such that $P(Z \leq c) = 0.975$, where $Z \sim N(0, 1)$

Here we could generate a table of normal-distribution cdf values using `table` on the calculator and search for the appropriate value (figures below). This requires some experimentation with the values in `tblset` (`2nd` `window`).

```
Y1:normalcdf(-E99,X,0,1)
```

```
TABLE SETUP
TblStart=1.9
ΔTbl=0.01
Indent: Auto Ask
Depend: Auto Ask
```

X	Y1
1.9	0.9713
1.91	0.9719
1.92	0.9726
1.93	0.9732
1.94	0.9738
1.95	0.9744
1.96	0.975
1.97	0.9756
1.98	0.9761
1.99	0.9767
2	0.9772

X=1.96

However, there is an easier way. The command $\text{invNorm}(M, \mu, \sigma)$ (in the `distr` menu) finds c such that $P(Z \leq c) = M$, the inverse of the normal-distribution function.

There are actually three options: the one above, find c such that $P(Z \leq c) = M$ (LEFT); find c such that $P(-c \leq Z \leq c) = M$ (CENTER); and find c such that $P(c \leq Z) = M$ (RIGHT).

Selecting invNorm from the `distr` menu gives an input menu (below left). Entering area $M = 0.975$, selecting LEFT, CENTER and RIGHT successively and executing the commands gives the screen below right.

```

invNorm
area:0.975
μ:0
σ:1
Tail: LEFT CENTER RIGHT
Paste

```

```

invNorm(0.975,0,1,LEFT)
1.959963986
.....
invNorm(0.975,0,1,CENTER)
{-2.241402725 2.241402725}
.....
invNorm(0.975,0,1,RIGHT)
-1.959963986

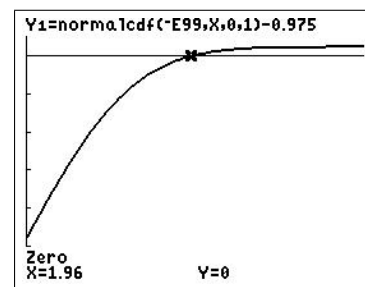
```

Another way to find c such that $P(Z \leq c) = M$ is to solve the equation (in calculator terms)

$$\text{normalcdf}(-E99, X, 0, 1) - 0.975 = 0$$

for X . This can be done using the Solver (last item in the `math` menu) but this requires a reasonable guess, especially if there is more than one solution (there isn't here).

Much better is to solve the equation by graphing the left-hand side and using zero in the `calc` menu, as in the figure.



Exercises

- If $Z \sim N(0, 1)$, find
 - $P(-1.5 \leq Z \leq 2)$
 - $P(Z \leq -0.8)$
 - $P(Z \geq 1.6)$
 - $P(-1 \leq Z \leq 1)$.
- If $X \sim N(100, 20)$, find
 - $P(88 \leq X \leq 112)$
 - $P(100 \leq X \leq 105)$
 - $P(X \leq 107)$
 - $P(X \geq 97)$.
- If $Z \sim N(0, 1)$, find the value of z to 2 decimal places if
 - $P(Z \leq z) = 0.8413$
 - $P(Z \leq z) = 0.95$
 - $P(Z \geq z) = 0.9772$
 - $P(|Z| \leq 0.95)$.

You'll need to think geometrically here.
- If $X \sim N(10, 4)$, find the value of x to 1 decimal place if
 - $P(X \leq x) = 0.05$
 - $P(X \geq x) = 0.90$
 - $P(X \geq x) = 0.025$

Here too.

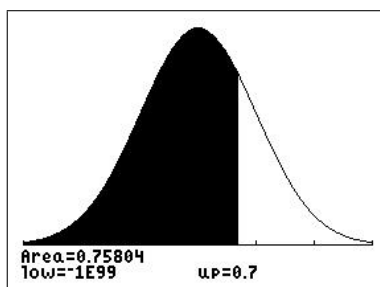
Solutions are in Section 18.6.

2. Graphical representation

Example: Graph $P(Z \leq 0.7) = P(-\infty < Z \leq 0.7)$, where $Z \sim N(0, 1)$.

WINDOW Xmin=-3 Xmax=3 Xscl=1 Ymin=0 Ymax=0.4 Yscl=0.1	ShadeNorm lower: -1E99 upper: 0.7 μ: 0 σ: 1 Color: BLACK Draw	ShadeNorm(-E99, 0.7, 0, 1, BLACK)
--	---	-----------------------------------

ShadeNorm is in the distr DRAW menu. *BLACK* (CE only) is in the vars COLOR menu. If you omit it, the shading comes out blue, the default.



Exercises

If $X \sim N(60, 25)$, display each of the following probabilities as an area under the normal pdf curve: (a) $P(54 \leq X \leq 58)$; (b) $P(X \leq 70)$; (c) $P(X \geq 62.5)$

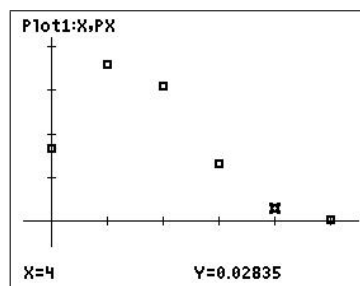
Solutions are in Section 18.6.

3. Using the Normal Distribution to approximate a Binomial Distribution

When n is large, the Binomial Distribution $\text{bin}(n, p)$ can be approximated by the Normal Distribution $N(np, np(1-p))$. This can be seen by plotting the two distributions on the one graph.

Example: Compare $\text{bin}(5, 0.3)$ with $N(1.5, 1.05)$.

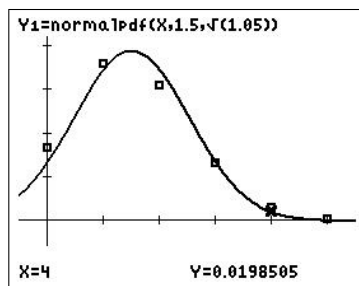
Construct a scatterplot of $\text{bin}(5, 0.3)$ (Section 18.2.4).



Put $N(1.5, 1.05)$ into Y_1 as shown in the figure. Note that $\sigma = \sqrt{1.05}$.

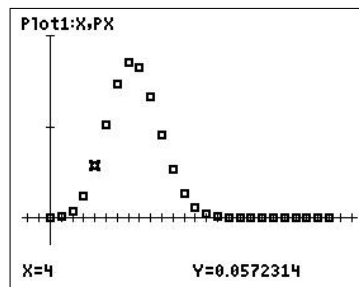
\blacksquare $Y_1 = \text{normalpdf}(X, 1.5, \sqrt{1.05})$
--

Press `graph` to display both graphs. A reasonable but not good approximation with $n=5$.

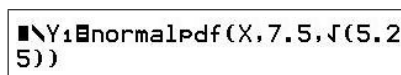


Example: Compare $\text{bin}(25, 0.3)$ with $N(7.5, 5.25)$.

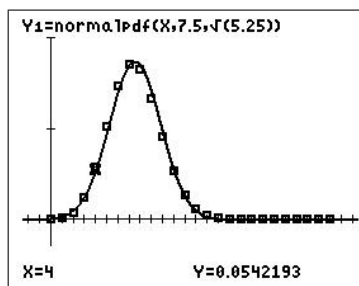
Construct a scatterplot of $\text{bin}(25, 0.3)$ (Section 18.2.4).



Put $N(7.5, 5.25)$ into Y_1 as shown in the figure. Note that $\sigma = \sqrt{5.25}$.



Press `graph` to display both graphs. A much better approximation with $n=25$.



Exercise

Graphically compare the normal approximation to the binomial for $p = 0.5$ when $n = 5$ and $n = 20$.

Solutions are in Section 18.6.

18.3.2 The Exponential Distribution

A continuous random variable X with an Exponential Distribution has pdf

$$p(x) = \lambda e^{-\lambda x} \quad 0 < x < \infty.$$

The Exponential Distribution is defined by one parameter, λ . We write $X \sim \text{Exp}(\lambda)$.

The cdf $P(a \leq X \leq b)$ is given by

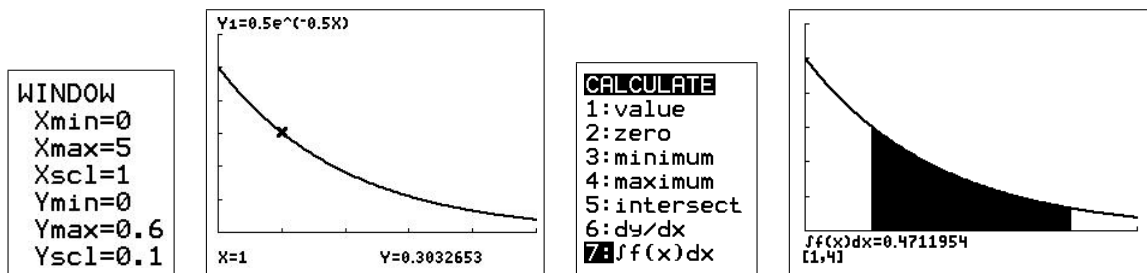
$$P(a \leq X \leq b) = \int_a^b \lambda e^{-\lambda x} dx = e^{-\lambda a} - e^{-\lambda b}.$$

Example: If $X \sim \text{Exp}(0.5)$, find $P(1 \leq X \leq 4)$.

$$P(1 \leq X \leq 4) = e^{-0.5 \times 1} - e^{-0.5 \times 4} = e^{-0.5} - e^{-2} = 0.471 \text{ to 3 significant digits.}$$

To obtain a visual representation of this calculation, use $\int f(x)dx$ in the calc menu.

First set a suitable window and graph Y_1 . Then, select the command and specify the integration limits. This shades the area between the function $Y=0$ and Y_1 for X between 1 and 4; it also gives you the integral.



Exercises

If $X \sim \text{Exp}(0.2)$, find to 4 decimal places

- (a) $P(3 \leq X \leq 5)$ (b) $P(X \leq 5)$ (c) $P(X \geq 2)$

Solutions are in Section 18.6.

Continuous-distribution spinner

A spinner,²⁴ here a program SPCTSPDS/SPCTSDCE,²⁵ allows you to vary the parameters on screen of each of the two types of continuous probability distributions (both their pdf and cdf) here (more may be added), by pressing a key, and see the distribution replotted immediately. The program contains the necessary information (in HELP) on how to vary the parameters.

²⁴From a feature in an earlier version of Excel used by Ed Staples and Margie Smith in SpinnaMaths to explore the effects of varying parameters in a variety of applications; sadly, SpinnaMaths is no longer available because of subsequent 'upgrades' of Excel.

²⁵available at www.XXX

18.4 Statistical inference

18.4.1 Hypothesis testing and confidence intervals for means

1. Single-sample mean drawn from a normal population: σ known

Testing the significance of the difference of a sample mean from a hypothesised population mean: Z test

Example: (JB) A sample of 25 workers in a factory spent, on average, 2.2 minutes to assemble the co-processor of a personal computer. Assuming that the assembly times are normally distributed with a standard deviation $\sigma = 0.4$ seconds, can we conclude that the average assembly time exceeds 2 minutes? Test at the 5% level of significance.

Steps

$$H_0: \mu = 2.$$

$$H_1: \mu > 2.$$

Significance level: 5%.

$$\text{Test statistic: } Z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}.$$

Given: $\mu = 2$; $\sigma = 0.4$; $\bar{x} = 2.2$; and $n = 25$.

Then: $z = 2.5$ the observed value of Z , from the calculator (see below); and

$$p = P(Z \geq 2.5) = 0.0062 \quad \text{from the calculator.}$$

As $p < 0.05$, reject H_0 and conclude that, on average, assembly times exceed 2 minutes.

Calculator operations

Go to the `[stat]` TESTS menu and select Z-Test (below left). Press `[enter]`.

```

EDIT CALC TESTS
1:Z-Test...
2:T-Test...
3:2-SampZTest...
4:2-SampTTest...
5:1-PropZTest...
6:2-PropZTest...
7:ZInterval...
8:TInterval...
9:2-SampZInt...
  
```

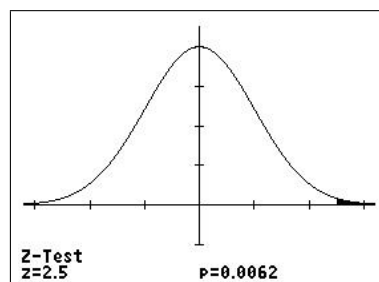
```

Z-Test
Inpt:Data Stats
μ₀:2
σ:0.4
x̄:2.2
n:25
μ:≠μ₀ <μ₀ >μ₀
Color:BLACK
Calculate Draw
  
```

Complete the menu items as shown (above right): *Stats* because we are given the sample mean rather than the raw data; $>\mu_0$ because $H_1: \mu > 2$. Put the cursor on *Calculate* and press `[enter]` to obtain the screen below left. If you select *Draw* instead of *Calculate*, the result appears graphically (below right): shading at the right-hand end of the curve.

```

Z-Test
μ>2
z=2.5
p=0.0062096799
x̄=2.2
n=25
  
```



Estimate the population mean from the sample mean

Use a 95% confidence interval to form an interval estimate for the population mean μ .

Steps

Given: $\sigma=0.4$; $\bar{x}=2.2$; $n=25$; and C(onfidence)-Level 0.95.

From the calculator (see below), we are 95% confident that the interval $2.04 < \mu < 2.36$ captures the population mean.

Calculator operations

Go to the `[stat]` TESTS menu and select ZInterval (below left). Press `[enter]`.

```

EDIT CALC TESTS
1:Z-Test...
2:T-Test...
3:2-SampZTest...
4:2-SampTTest...
5:1-PropZTest...
6:2-PropZTest...
7:ZInterval...
8:TInterval...
9↓2-SampZInt...
  
```

```

ZInterval
Inpt:Data Stats
σ:0.4
x̄:2.2
n:25
C-Level:0.95
Calculate
  
```

```

ZInterval
(2.0432,2.3568)
x̄=2.2
n=25
  
```

Complete the menu items as shown (above centre) if necessary. Put the cursor on *Calculate* and press `[enter]` to obtain the screen above right giving the confidence interval.

2. Single-sample mean drawn from a normal population: σ unknown

Testing the significance of the difference of a sample mean from a hypothesised population mean: *T* test

Example: (JB) A study was conducted to see whether the ascorbic-acid (Vitamin C) content (mg/100g) of a certain brand of frozen vegetables was less than the recommended level of 18.6 mg/100g. Eight packets of frozen vegetables were randomly selected from the production line and were found to have the following ascorbic-acid levels:

14.3 19.1 16.3 19.2 15.8 16.2 18.7 14.7.

Assuming that assembly times are normally distributed, can we conclude that the average ascorbic-acid level of this brand of vegetables is less than 18.6 mg/100g?

Steps

$H_0: \mu = 18.6$.

$H_1: \mu < 18.6$.

Significance level: 5%.

Test statistic: $T = \frac{\bar{x} - \mu}{s/\sqrt{n}}$, with $n - 1$ degrees of freedom.

Then, from the calculator (see below), $\bar{x} = 16.79$; $s = 1.961$; and $n = 8$, giving

$t = -2.614$, the observed value of T with $df = n - 1 = 7$, and

$p = P(T \leq -2.614) = 0.017$.

As $p < 0.05$, reject H_0 and conclude that the mean ascorbic-acid level of this brand of vegetables is less than 18.6 mg/100g.

Calculator operations

Enter the data into a list called ACID (below left).

ACID
14.3
19.1
16.3
19.2
15.8
16.2
18.7
14.7

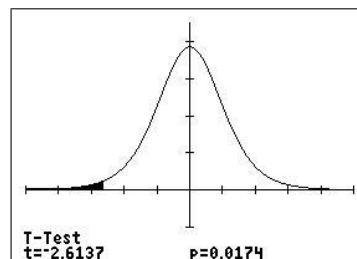
EDIT CALC TESTS
1:Z-Test...
2:T-Test...
3:2-SampZTest...
4:2-SampTTest...
5:1-PropZTest...
6:2-PropZTest...
7:ZInterval...
8:TInterval...
9:2-SampZInt...

Go to the **stat** TESTS menu and select T-Test (above right). Press **enter**.

Complete the menu items (below left): *Data* because we are given the raw data; $<\mu_0$ because $H_1: \mu < 2$. Put the cursor on *Calculate* and press **enter** to obtain the screen below center. If you select *Draw* instead of *Calculate*, the result appears graphically (below right).

T-Test
Inpt: Data Stats
μ_0 : 18.6
List: ACID
Freq: 1
μ : $\neq \mu_0$ < μ_0 > μ_0
Color: BLACK
Calculate Draw

T-Test
$\mu < 18.6$
$t = -2.613749312$
$p = 0.0173617129$
$\bar{x} = 16.7875$
$Sx = 1.961367963$
$n = 8$



Estimate the population mean from the sample mean

Example: Use a 95% confidence interval to form an interval estimate for the mean ascorbic-acid level μ of the brand of vegetables.

Steps

Given: C(onfidence)-Level 0.95.

From the calculator (see below), we are 95% confident that the interval $15.1 < \mu < 18.4$ captures the population mean.

Calculator operations

Go to the `stat` TESTS menu and select TInterval (below left). Press `enter`.

```
EDIT CALC TESTS
1:Z-Test...
2:T-Test...
3:2-SampZTest...
4:2-SampTTest...
5:1-PropZTest...
6:2-PropZTest...
7:ZInterval...
8:TInterval...
9↓2-SampZInt...
```

```
TInterval
Inpt:Data Stats
List:ACID
Freq:1
C-Level:0.95
Calculate
```

```
TInterval
(15.148,18.427)
x̄=16.7875
Sx=1.961367963
n=8
```

Complete the menu items as shown (above centre) if necessary. Put the cursor on *Calculate* and press `enter` to obtain the screen above right giving the confidence interval.

18.4.2 Hypothesis testing and confidence intervals for population proportion

Single-sample proportion

1. Significance test using the binomial distribution (exact test)

Example: (JB) An electronic-component manufacturer claims that fewer than 10% of components have manufacturing faults. A random sample of 15 components contains 4 faulty ones. Does this challenge the manufacturer's claim? Test at the 5% level of significance.

Steps

$H_0: \pi = 0.10$.

$H_1: \pi > 0.10$.

Significance level: 5%.

Let X be the number of faulty components.

Then, under the null hypothesis, $X \sim \text{bin}(15, 0.10)$.

Then, from the calculator (see below), $p = P(X \geq 4) = 1 - P(X \leq 3) = 0.056$.

As $p > 0.05$, do not reject H_0 . There is not sufficient evidence that the percentage of faulty components exceeds 10%.

Calculator operation:

```
1-binomcdf(15,0.1,3)
0.0555556298
```

2. Significance test using the normal approximation to the binomial distribution:

Z test with no continuity correction

For n large, $n\pi > 10$ and $n(1-\pi) > 10$, the binomial distribution $\text{bin}(n, \pi)$ can be approximated by a normal distribution with $\mu = n\pi$ and $\sigma^2 = n\pi(1-\pi)$ (Section 18.3.1).

Example: (JB) An electronic-component manufacturer claims that fewer than 10% of components have manufacturing faults. A random sample of 120 components contains 23 faulty ones. Does this challenge the manufacturer's claim? Test at the 5% level of significance.

Steps

$H_0: \pi = 0.10$.

$H_1: \pi > 0.10$.

Significance level: 5%.

Given: $x = 23$; $n = 120$.

Test statistic: $Z = \frac{x - n\pi}{s\sqrt{n\pi(1-\pi)}}$.

Then, from the calculator (see below), the observed value of Z , $z = 3.347$, and

$p = P(Z > 3.347) = 0.0004$ to 4 decimal places.

As $p < 0.05$, reject H_0 , and conclude that the percentage of faulty components exceeds 10%.

Calculator operations

Go to the `[stat]` TESTS menu and select 1-PropZTest (below left). Press `[enter]`. Fill in the resulting screen (below centre), put the cursor on *Calculate* and press `[enter]` to obtain the screen below right.

```

EDIT CALC TESTS
1:Z-Test...
2:T-Test...
3:2-SampZTest...
4:2-SampTTest...
5:1-PropZTest...
6:2-PropZTest...
7:ZInterval...
8:TInterval...
9:2-SampZInt...
  
```

```

1-PropZTest
P0:0.1
x:23
n:120
PROP:≠P0 <P0 >P0
Color: BLACK
Calculate Draw
  
```

```

1-PropZTest
PROP>0.1
z=3.347193407
p=4.082264262E-4
p̂=0.1916666667
n=120
  
```

3. *Significance test using the normal approximation to the binomial distribution:*

Z test with continuity correction

For n large, $n\pi > 10$ and $n(1-\pi) > 10$, the binomial distribution can be approximated by a normal distribution with $\mu = n\pi$ and $\sigma^2 = n\pi(1-\pi)$ (Section 18.3.1). The approximation can be improved by using the continuity correction.

Example: (JB) An electronic-component manufacturer claims that fewer than 10% of components have manufacturing faults. A random sample of 120 components contains 23 faulty ones. Does this challenge the manufacturer's claim? Test at the 5% level of significance.

Steps

$H_0: \pi = 0.10$.

$H_1: \pi > 0.10$.

Significance level: 5%.

Given: $x = 23$; $n = 120$.

Test statistic: $Z = \frac{x - n\pi}{\sqrt{n\pi(1-\pi)}}$.

Given $x = 22.5$ (taking into account the continuity correction), $\pi = 0.10$ and $n = 120$, we find from the calculator (see below) that the observed value of Z ,

$$z = \frac{22.5 - 120 \times 0.1}{\sqrt{120 \times 0.1(1-0.1)}} = 3.195, \quad \text{and}$$

$p = P(Z > 3.195) = 1 - P(Z \leq 3.195) = 0.0007$ to 4 decimal places.

As $p < 0.05$, reject H_0 , and conclude that the percentage of faulty components exceeds 10%.

Calculator operations

```
(22.5-120*0.1)/√(120*0.1(1
-0.1))
3.195048252
1-normalcdf(-E99,3.1950,0,
1)
6.992185905E-4
```


4. Estimating the population proportion from a sample proportion: Normal approximation

Example: (JB) An electronic-component manufacturer claims that fewer than 10% of components have manufacturing faults. A random sample of 120 components contains 23 faulty ones. Use a 95% confidence interval to form an interval estimate for the true proportion of components that have manufacturing faults.

Steps

Given: C(onfidence)-Level 0.95.

From the calculator (see below), we are 95% confident that the interval $0.12 < \pi < 0.26$ captures the true population proportion.

Calculator operations

Go to the `stat` TESTS menu and select 1-PropZInt (below left). Press `enter`.

```

EDIT CALC TESTS
4↑2-SampTTest...
5:1-PropZTest...
6:2-PropZTest...
7:ZInterval...
8:TInterval...
9:2-SampZInt...
0:2-SampTInt...
1-PropZInt...
B↓2-PropZInt...

```

```

1-PropZInt
x:23
n:120
C-Level:0.95
Calculate

```

```

1-PropZInt
(0.12124,0.26209)
p̂=0.1916666667
n=120

```

Complete the menu items as shown (above centre) if necessary. Put the cursor on *Calculate* and press `enter` to obtain the screen above right giving the confidence interval.

18.5 Probability and gambling schemes

These are programs I wrote for a first-year university course on Probability taught against a background of the probabilities involved in various modes of gambling.²⁶

18.5.1 Throwing dice

Throwing dice is at the heart of many games and methods of gambling. The DICE/DICECE program simulates rolling any number of six-sided dice.

```

ROLLING SIX-SIDED DICE

PRESS [ENTER] FOR ANOTHER
ROLL, [ON] [1] TO STOP

HOW MANY DICE? 8

```

```

HOW MANY DICE? 8
{6 1 2 4 5 4 1 6}
{1 2 1 5 4 5 1 2}
{5 2 5 3 6 1 4 3}
{5 2 6 6 6 1 1 1}
{1 5 6 4 2 5 4 2}
{1 2 2 5 5 2 5 1}
{3 3 2 5 2 6 4 4}

```

You can also do this in the ProbSim app, and store the accumulated data for later analysis.

18.5.2 Random walks

In the WALK/WALKCE program, two players, A and B, start with a specified bank each. You then specify the probability that A wins at each go. If A wins a go, they take \$1 from B's bank and add it to theirs; if A loses, B receives \$1 from A.

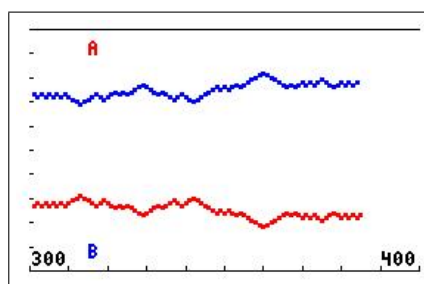
The program then runs until either A or B has no money left. With graphics, the amount that each player has at each go is plotted so you can see how the game is progressing (and cheer on your favourite).

```

RANDOM WALK

STARTING BANKS
A: 50
B: 50
P(A WINS): 0.5

```



For a faster simulation, run the program without the graphics.

²⁶Students were not required to demonstrate their understanding of the course by making money at their chosen form of gambling. In fact, it soon became obvious during the course that gambling is not a way to make money; it's an often costly entertainment.

18.5.3 Basic betting

In the WINBET/WINBETCE program, you specify the amount won for a \$1 bet, the probability of winning and the number of trials. The program then simulates the trials, tells you how much you won (+) or lost (-), and shows the mean win per trial \bar{x} and the variance σ_x^2 .

```

      WIN PROGRAM
      AMOUNT WON FOR A
      1 DOLLAR BET: 2
      PROBABILITY OF
      WINNING: 0.51
      NO OF TRIALS (≤999): 100
      TOTAL WON/LOST: 80
      I=100
                                     [ENTER]
    
```

```

      AMOUNT WON FOR A
      1 DOLLAR BET: 2
      PROBABILITY OF
      WINNING: 0.51
      NO OF TRIALS (≤999): 100
      TOTAL WON/LOST: 80
       $\bar{x}$ : 0.8
       $(\sigma_x)^2$ : 2.16
    
```

18.5.4 Gambling systems

The GAMBSYS/GBSYSCE program simulates five different gambling systems.

```

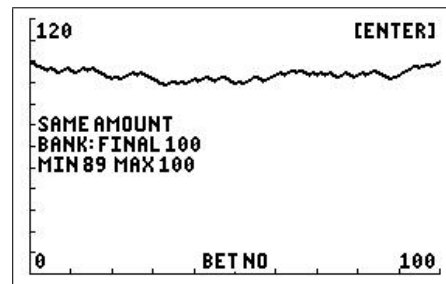
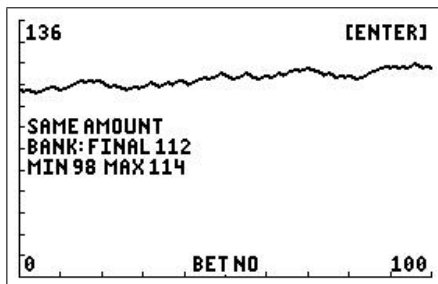
      SYSTEM BETTING
      FIRST BET 1 DOLLAR
      STARTING BANK: 100
      PROBABILITY OF
      A WIN: 0.4929
      NUMBER OF BETS: 100
    
```

```

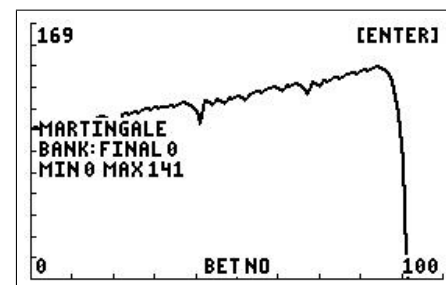
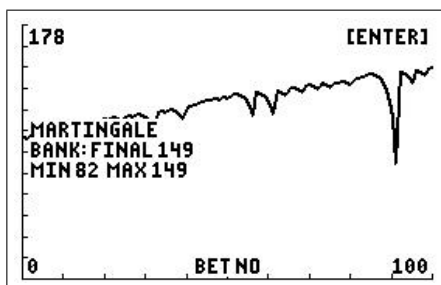
      SYSTEM
      1: BET SAME AMOUNT
      2: MARTINGALE
      3: REVERSE MARTINGALE
      4: LABOUCHERE
      5: REVERSE LABOUCHERE
      6: CHANGE PARAMETERS
      7: QUIT
    
```

Two runs of each system are shown below.

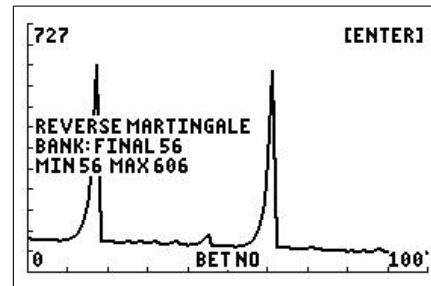
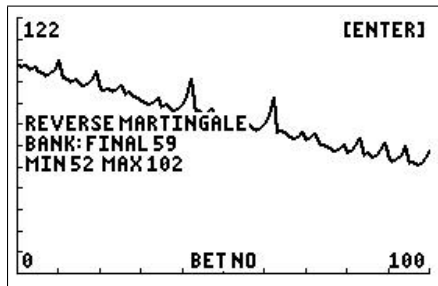
(a) bet the same amount each time.



(b) **Martingale:** double your bet after a win; keep the same bet after a loss.



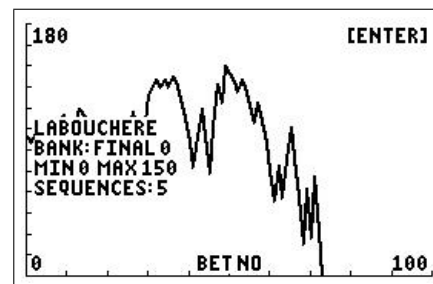
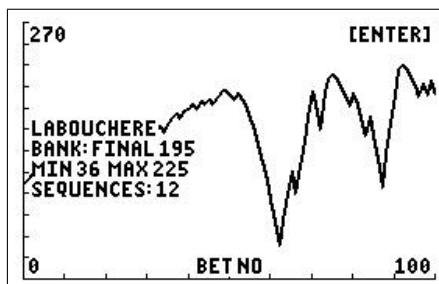
- (c) **Reverse Martingale:** double your bet after a loss; keep the same bet after a win.



- (d) **Labouchère** (sequence 1,2,3,4)

The user of this strategy decides before playing how much money they want to win each bet. He or she then writes down a list or sequence of positive numbers that sum to this amount. With each bet, the player stakes an amount equal to the sum of the first and last numbers on the list (if only one number remains, that number is the amount of the stake).

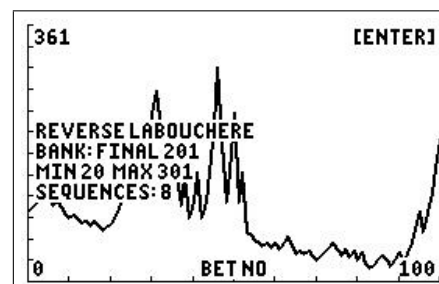
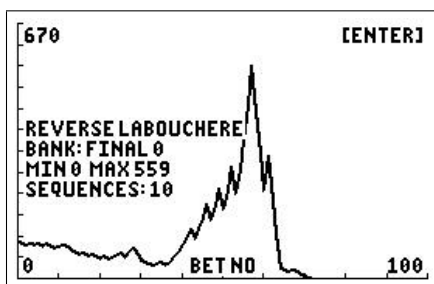
If the bet is successful, the two amounts are removed from the list. If the bet is unsuccessful, the amount lost is appended to the end of the list. This process continues until either the list is completely crossed out, at which point the desired amount of money has been won, or until the player runs out of money to wager.



- (e) **Reverse Labouchère** (sequence 1,2,3,4)

In this version after a win, instead of deleting numbers from the list, the player adds the previous bet amount to the end of the list. He or she continues building up the list until they hit the table maximum. After a loss, the player deletes the outside numbers and continues working on the shorter list. The player starts their list again if they run out of numbers to bet.

The Reverse Labouchère system is often used because where the Labouchère list represents how much the player wants to win each bet, a reverse Labouchère line represents the most that the player is able to lose.



The Martingale and Labouchère systems are dangerous in that they can produce a long string of small wins which don't prepare you for the massive loss which is around the corner.

18.5.5 Tossing coins

Many people misunderstand the Law of Averages.

A sales book said something along the lines of “If 1 in 2 people are interested in buying your product then, if you get a knockback at one house, **the law of averages** says you **will** get a sale at the very next house.” It makes you wonder whether this guy had ever tried it himself.

It’s like saying that if red came up on the roulette wheel last time, black **must** come up next. If that were the case, you could correctly predict the next umpteen trillion spins of the wheel and retire tomorrow.

The Law of Averages (or the Law of Large Numbers) actually states that, if the probability of an event is p , then after n trials, the number of times this event occurs, x say, is such that

$$\frac{x}{n} \rightarrow p \text{ as } n \rightarrow \infty.$$

The Law of Averages does **not** say that one event or a series of events causes another event to occur next in order to ‘even things out’. In fact the opposite is true; the Law of Averages depends for its validity on each event being independent — that is, unaffected by earlier events.

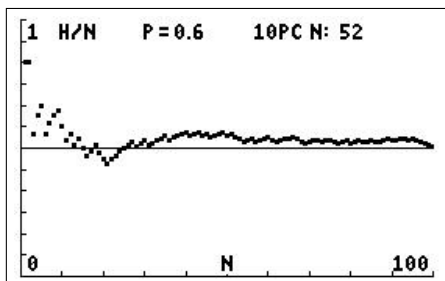
The COINTOSS/CNTOSSCE program simulates tossing a coin (fair or biased). Enter the probability of *heads* and the number of trials (up to 999). The program plots the ratio H/N versus N , where H is the total number of *heads* in N tosses. The program also displays the value of N beyond which H/N remains within 10% of the probability of *heads*.

Values of H/N are stored in list $L1$; you should clear this when you have finished.

```

COINTOSS PROGRAM
PROBABILITY P OF
HEADS: 0.6
NO OF TRIALS
MAX 999: 100

```



The program tells us that the ratio remained within 10% of 0.6 after 52 trials.

18.6 Solutions to exercises

Exercises page 37

- Evaluate each of the following by hand, then check with your calculator.
 - $6! = 720$
 - ${}^6P_4 = 360$
 - ${}^8C_5 = 56$.
- $13! = 6,227,020,800$
 - ${}^{15}P_9 = 1,816,214,400$
 - ${}^{35}C_{22} = 1,476,337,800$.
- Generate 100 random integers between 1 and 6, simulating throwing a die. Store them in an appropriately named list.

```
randInt(1,6,100)→L1RAND
{6 4 1 4 3 1 5 3 5 2 3 4 ...
```

```
1-Var Stats
List:RAND
FreqList:
Calculate
```

```
1-Var Stats
x̄=3.38
Σx=338
Σx²=1422
Sx=1.680428036
σx=1.672004785
n=100
minX=1
↓Q1=2
```

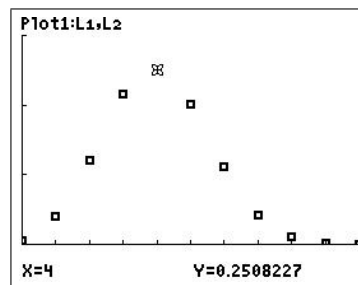
From 1-Var Stats, (a) the mean is 3.38; (b) the standard deviation is 1.67; and (c) the median is 4. Answers will vary here.

Exercises page 170

- If $X \sim \text{bin}(8, 0.4)$, to 4 decimal places,
 - $P(X=4) = 0.2322$
 - $P(X \leq 1) = 0.1064$
 - $P(X \geq 7) = 0.0085$
 - $P(1 \leq X \leq 3) = 0.5773$
 - $P(X < 6) = P(X \leq 5) = 0.9502$.
- If $X \sim \text{bin}(25, 0.2)$, to 4 decimal places,
 - $P(X=4) = 0.1867$
 - $P(X \leq 10) = 0.9944$
 - $P(X \geq 7) = 0.2200$
 - $P(5 \leq X \leq 8) = 0.5326$
 - $P(X > 4) = 1 - P(X \leq 4) = 0.5793$.

Exercises page 170

- If $X \sim \text{bin}(10, 0.4)$, display the distribution of X graphically.



window $[0, 10, 1] \times [0, 0.3, 0.1]$

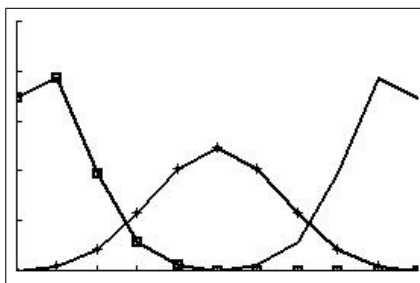
2. On the one graph, plot the distribution of:

$X_1 \sim \text{bin}(10, 0.1)$ use Plot1 with a square marker;

$X_2 \sim \text{bin}(10, 0.5)$ use Plot2 with a + marker;

$X_3 \sim \text{bin}(10, 0.9)$ use Plot3 with a dot marker.

Here, I've done line graphs rather than scatterplots to show the 3 distributions better.



window $[0, 10, 1] \times [0, 0.5, 0.1]$

Comment on how changing the value of p changes the shape of the distribution.

As p increases, the peak of the distribution moves to the right.

Exercises page 42

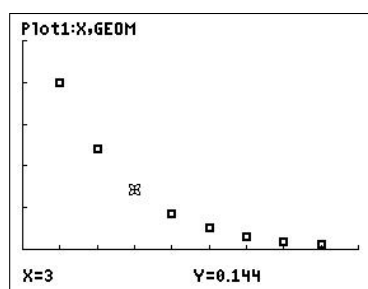
If $X \sim \text{geom}(0.2)$, to 4 decimal places,

(a) $P(X = 5) = 0.0819$ (c) $P(X \geq 7) = 0.2621$

(b) $P(X \leq 8) = 0.8322$ (d) $P(3 \leq X \leq 9) = 0.5058$.

Exercises page 43

1. If $X \sim \text{geom}(0.4)$, display the distribution of X graphically for $1 \leq x \leq 8$.



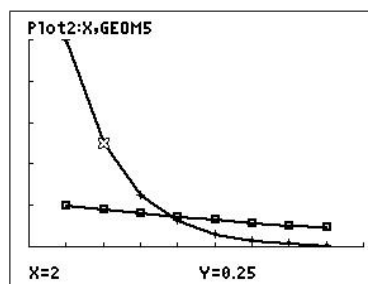
window $[0, 9, 1] \times [0, 0.5, 0.1]$

2. On the one graph, plot the distribution of:

$X_1 \sim \text{geom}(0.1)$ use Plot1 with a square marker;

$X_2 \sim \text{geom}(0.5)$ use Plot2 with a + marker;

Here, I've done line graphs rather than scatterplots to show the two distributions better.



window $[0, 9, 1] \times [0, 0.5, 0.1]$

Comment on how changing the value of p changes the shape of the distribution.

As p increases, the distribution has a larger initial value but decays more rapidly.

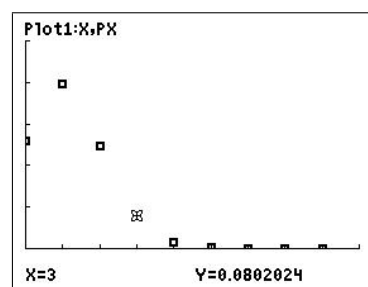
Exercises page 44

If $X \sim \text{hypg}(100, 15, 5)$,

- (a) $P(X=3)=0.0216$ (c) $P(X>3)=0.0016$
 (b) $P(X \leq 3)=0.9984$ (d) $P(1 \leq X \leq 2)=0.5411$.

Exercises page 44

1. If $X \sim \text{hypg}(100, 15, 8)$, display the distribution of X graphically for $0 \leq x \leq 8$.



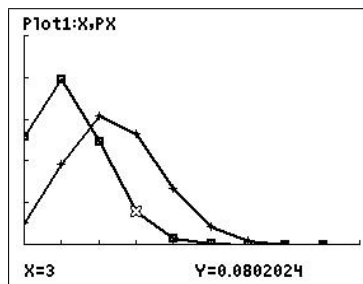
window $[0, 9, 1] \times [0, 0.5, 0.1]$

2. On the one graph, plot the distribution of:

$X \sim \text{hypg}(100, 15, 8)$ use Plot1 with a square marker;

$X \sim \text{hypg}(100, 30, 8)$ use Plot2 with a + marker;

Here, I've done line graphs rather than scatterplots to show the two distributions better.



window $[0, 9, 1] \times [0, 0.5, 0.1]$

Comment on how changing the value of k changes the shape of the distribution.

As k increases, the distribution has a smaller initial value, and a smaller peak, shifted to the right.

Exercises page 132

If $X \sim \text{pois}(0.5)$, to 4 decimal places,

(a) $P(X=0) = 0.6065$

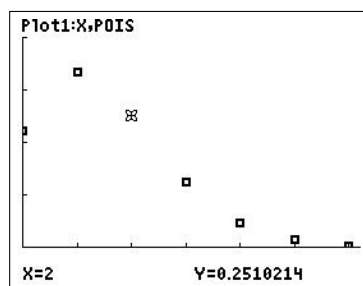
(c) $P(X \geq 1) = 0.3935$

(b) $P(X \leq 3) = 0.9982$

(d) $P(1 \leq X \leq 4) = 0.3933$.

Exercises page 46

1. If $X \sim \text{pois}(1.5)$, display the distribution of X graphically for $0 \leq x \leq 6$.



window $[0, 6.2, 1] \times [0, 0.4, 0.1]$

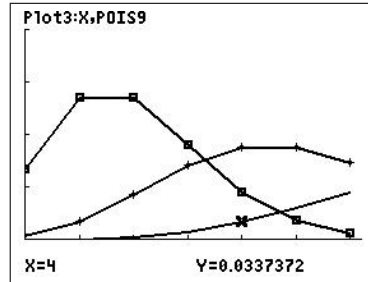
2. On the one graph, for $0 \leq x \leq 10$, plot the distribution of:

$X_1 \sim \text{pois}(2)$ use Plot1 with a square marker;

$X_2 \sim \text{pois}(5)$ use Plot2 with a + marker;

$X_3 \sim \text{pois}(9)$ use Plot2 with a dot marker.

Here, I've done line graphs rather than scatterplots to show the 3 distributions better.



window $[0, 6.2, 1] \times [0, 0.4, 0.1]$

Comment on how changing the value of λ changes the shape of the distribution.

The the peak moves to the right and decreases in amplitude.

Exercises page 49

1. If $Z \sim N(0, 1)$, to 4 decimal places,

(a) $P(-1.5 \leq Z \leq 2) = 0.9104$

(c) $P(Z \geq 1.6) = 0.0548$

(b) $P(Z \leq -0.8) = 0.2119$

(d) $P(-1 \leq Z \leq 1) = 0.6827$.

2. If $X \sim N(100, 20)$, to 4 decimal places,

(a) $P(88 \leq X \leq 112) = 0.9927$

(c) $P(X \leq 107) = 0.9412$

(b) $P(100 \leq X \leq 105) = 0.3682$

(d) $P(X \geq 97) = 0.7488$.

3. If $Z \sim N(0, 1)$, find the value of z to 2 decimal places if

(a) $P(Z \leq z) = 0.8413$: $x = 1.00$

(c) $P(Z \geq z) = 0.9772$: $x = -2.00$

(b) $P(Z \leq z) = 0.95$: $x = 1.64$

(d) $P(|Z| \leq 0.95)$: $x = 1.96$.

4. If $X \sim N(10, 4)$, find the value of x to 1 decimal place if

(a) $P(X \leq x) = 0.05$: $x = 6.7$

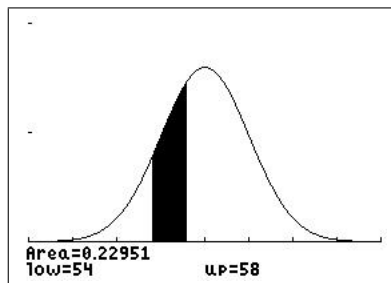
(c) $P(X \geq x) = 0.025$: $x = 13.9$

(b) $P(X \geq x) = 0.90$: $x = 7.4$

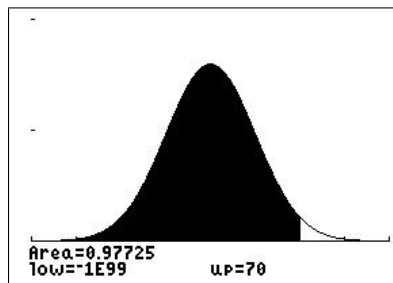
Exercises page 50

If $X \sim N(60, 25)$, display each of the following probabilities as an area under the normal pdf curve.

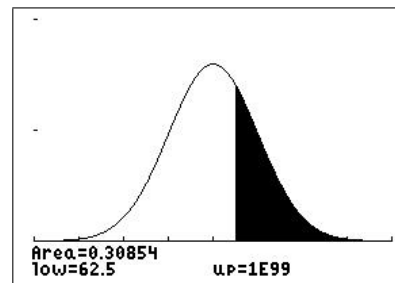
(a) $P(54 \leq X \leq 58)$



(b) $P(X \leq 70)$



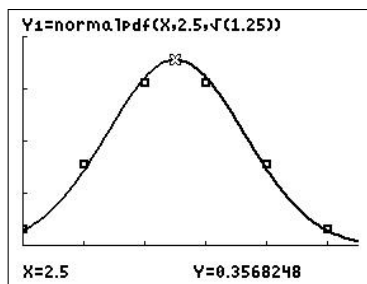
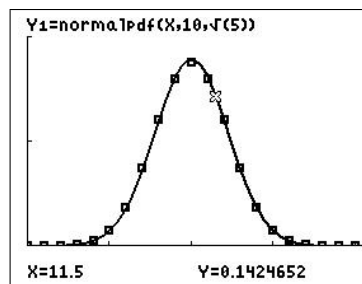
(c) $P(X \geq 62.5)$



window $[40, 80, 5] \times [0, 0.1, 0.05]$

Done using the *ShadeNorm* command.

Graphically compare the normal approximation to the binomial for $p = 0.5$ when $n = 5$ and $n = 20$.

 $n = 5$  $n = 20$ 

window $[0, 5.5, 1] \times [0, 0.4, 0.1]$

window $[0, 20.5, 5] \times [0, 0.2, 0.1]$

Exercises page 52

If $X \sim \text{Exp}(0.2)$, to 4 decimal places,

(a) $P(3 \leq X \leq 5) = 0.1809$

(b) $P(X \leq 5) = 0.6321$

(c) $P(X \geq 2) = 0.6703$

19 Matrix and Vector Operations

The material here is directly relevant to the topics *Vectors in the plane*, *Vectors in three dimensions* and *Matrices* in the Australian Curriculum for Specialist Mathematics.

As with all technology, you should do the first few calculations by hand to make sure you understand the method. Use the technology firstly to check your hand calculations and then when the numbers make hand calculation difficult or prone to error.

19.1 Basic matrix operations

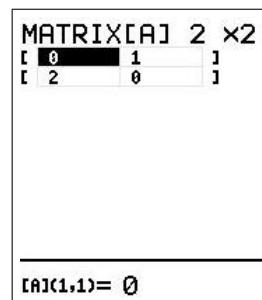
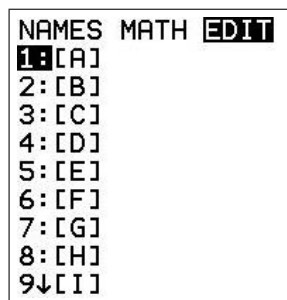
The basic matrix operations on the calculator work just like the number operations with one or two minor exceptions.

Let's start out with matrices $A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$.

19.1.1 Putting matrices in your calculator

Press `matrix`, then `▶▶` (or `◀`) to get to EDIT (figure below left). You are now ready to edit one of the ten matrices [A]–[J]: press `1` to select [A].

Now input the order (rows \times columns) and the elements of [A] — the cursor indicates what is required. Press the desired number and `enter`. The cursor moves on and leads you through the elements row by row. Make sure you press `enter` after the last element (figure below right).



Press `matrix` `◀` again, but press `2` this time to input [B]. Finally input [C].

Press `2nd` `quit` to return to the Home screen.

19.1.2 Displaying matrices

You call up matrices by pressing `matrix` and the appropriate number. For example, pressing `matrix` `2` will display [B] on the screen. Pressing `enter` will show you the elements of (numbers in) [B].



19.1.3 Adding and subtracting matrices

Matrices can only be added or subtracted if they have the same dimensions, that is the same number of rows and the same number of columns. We can add/subtract multiples of [A] and [C], but not with [B].

[A]+[C]	[[-1 3] [5 -4]]
[A]-[C]	[[1 -1] [-1 4]]
2[A]-3[C]	[[3 -4] [-5 12]]

19.1.4 Multiplying matrices

Press `matrix` `1` `matrix` `2` to display [A][B] and press `enter` to display the numerical result

$$\mathbf{AB} = \begin{bmatrix} 4 & 5 & 6 \\ 2 & 4 & 6 \end{bmatrix}.$$

The result is stored in Ans (as is the result of any calculation).

[A][B]	[[4 5 6] [2 4 6]]
[A]Ans	[[2 4 6] [8 10 12]]

If you now want to multiply by \mathbf{A} again, i.e. to find $\mathbf{A}(\mathbf{AB})$, press `matrix` `1` `2nd` `ans` (on the `(-)` key) to display [A]Ans, then `enter` to give the numerical result

$$\mathbf{A}(\mathbf{AB}) = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}.$$

Note: If you want to evaluate the expression $\mathbf{A}(\mathbf{AB})$ using brackets, you need to put in the multiplication sign before the brackets, that is evaluate $[A]*([A][B])$. This is because a bracket following a matrix, $[A](1,2)$ for example, is used to denote the element of a matrix, here the 1,2 element, i.e. the number in the first row and second column.

To evaluate $\mathbf{A}(\mathbf{AB})$, it is much simpler to omit the brackets and just evaluate $[A][A][B]$.

19.1.5 Squaring matrices

We could also work out \mathbf{AAB} as $\mathbf{A}^2\mathbf{B}$.

Press `matrix` `1` `x2` `matrix` `2` to display $[\mathbf{A}]^2[\mathbf{B}]$ and `enter` to give the same result:

$$\mathbf{A}^2\mathbf{B} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}.$$

<code>[\mathbf{A}]^2[\mathbf{B}]</code>	<code>[[2 4 6]</code> <code>[8 10 12]]</code>
<code>[\mathbf{A}]^4</code>	<code>[[4 0]</code> <code>[0 4]]</code>
<code>[\mathbf{A}]^{-1}</code>	<code>[[0 0.5]</code> <code>[1 0]]</code>

Integer powers of a matrix are produced the same way as with numbers (for positive integers). Only square matrices can be raised to a power.

$$\mathbf{A}^4 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}.$$

The only negative integer ‘power’ that works is -1 (using the `x-1` key): however, $[\mathbf{A}]^{-1}$ produces the **inverse** of (square) matrix \mathbf{A} .

The reciprocal of a matrix is not defined, nor is division of one matrix by another.

19.1.6 More-involved expressions

Extensions work just as you would expect. For example, to work out $\mathbf{AB} + 3\mathbf{B}$, press

`matrix` `1` `matrix` `2` `+ 3` `matrix` `2` to display ‘ $[\mathbf{A}][\mathbf{B}] + 3[\mathbf{B}]$ ’, and `enter` to give

$$\mathbf{AB} + 3\mathbf{B} = \begin{bmatrix} 7 & 11 & 15 \\ 14 & 19 & 24 \end{bmatrix}.$$

<code>[\mathbf{A}][\mathbf{B}]+3[\mathbf{B}]</code>	<code>[[7 11 15]</code> <code>[14 19 24]]</code>
<code>Ans→[\mathbf{C}]</code>	<code>[[7 11 15]</code> <code>[14 19 24]]</code>

Note: Again, if you want to evaluate an expression involving brackets, such as $\mathbf{A}(\mathbf{B}+\mathbf{C})$, you need to put in the multiplication sign before the brackets, that is evaluate $\mathbf{A}*(\mathbf{B}+\mathbf{C})$.

19.1.7 Storing matrices

If you wanted to keep the previous answer for later use, you might store it in matrix $[\mathbf{C}]$ by pressing `STO▶` `matrix` `3` `enter` (figure above).

Notice that the calculator automatically gives $[\mathbf{C}]$ the correct order.

19.1.8 Illegal operations

The calculator will not let you do invalid operations. For example, if you try to calculate $\mathbf{A} + \mathbf{B}$ by entering `[matrix] [1] + [matrix] [2]`, you will see `[A]+[B]` on the screen, but pressing `[enter]` produces the message `ERR: DIM MISMATCH`. *Why?*

19.1.9 Other matrix operations

A number of operations are contained in the `[matrix]` math menu.

- *determinant*: $\det([\mathbf{A}]) = -2$.
- *transpose*: $[\mathbf{B}]^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.
- *dimension*: $\dim([\mathbf{B}]) = \{2 \ 3\}$: 2 rows \times 3 columns.

<code>det([A])</code>	<code>-2</code>
<code>[B]^T</code>	<code>[[1 4]</code> <code>[2 5]</code> <code>[3 6]]</code>
<code>dim([B])</code>	<code>{2 3}</code>

<code>Fill(1,[A])</code>	<code>Done</code>
<code>[A]</code>	<code>[[1 1]</code> <code>[1 1]]</code>
<code>identity(2)</code>	<code>[[1 0]</code> <code>[0 1]]</code>

- *'filling' a matrix*: `Fill(1,[A])` produces a matrix of 1s: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.
- *identity matrix*: `identity(n)` produces the $n \times n$ identity matrix.
- *row-echelon form*: `ref([A])` produces the row-echelon form of $[\mathbf{A}]$ (Gaussian elimination — see Section 19.2).
- *reduced-row-echelon form*: `rref([A])` produces the reduced row-echelon form of $[\mathbf{A}]$ (Gauss-Jordan elimination — see Section 19.2).

See the TI-84 Guidebook for details on the other `[matrix]` MATH menu items.

Some applications of matrices in population modelling can be found in *Population Modelling 3: Matrix Models*, Chapter 20.

19.2 Systematic Gauss Elimination

19.2.1 The method

The endpoint of Gauss elimination or Gauss reduction is a matrix in **row-echelon** form, characterised by:

- (a) the first non-zero element in each row is a 1 (called a **pivot**);
- (b) all elements in the column below a pivot are 0.

To reduce a matrix to row-echelon form, we use **elementary row operations**.²⁷ The three elementary row operations are:

- A. exchange two rows: $R_i \rightleftharpoons R_j$
- B. multiply a row by a constant: $R_i \rightarrow cR_i, c \neq 0$
- C. add a constant multiple of one row to another: $R_i \rightarrow R_i + cR_j, c \neq 0$.

Procedure

1. If the first column of the matrix is all zeros, ‘cross out’ this column to leave a smaller matrix.
2. If the element in the first row and first column (top left element) of the matrix is 0, exchange Row 1 with another row (Operation A).
(Sometimes it is convenient to do this even if the top left element is non-zero, so as to avoid fractions in Step 3.)
3. Multiply the top row by a constant to make the first element 1 (Operation B).
4. Use Operation C to obtain 0s **below** the 1 (pivot) by adding multiples of **Row 1** to each successive row ($R_i \rightarrow R_i + cR_1, i = 2, 3, \dots$).
5. ‘Cross out’ the first row and first column to leave a smaller matrix.
6. Go back and start at Step 1 on this smaller matrix.

PTO

²⁷Each row operation corresponds to the multiplication of the matrix by a corresponding elementary matrix.

Notes

- The only choice in this systematic version of Gauss elimination is which rows to interchange in Step 2. All the other operations are prescribed.
- The solutions to the simultaneous equations described by the row-echelon matrix are the same as those of the original matrix, hence the usefulness of the method. Write down the equations corresponding to the row-echelon matrix and use back substitution to find the solutions.
- The matrix changes wherever we perform an elementary row operation, so that we cannot use equal signs between the steps. Use a \sim instead. The matrices are said to be (row-) equivalent.
- We can proceed in a similar manner to obtain 0s above the pivots too, the **reduced row-echelon** matrix. This is called *Gauss-Jordan elimination*. A unique solution can be read directly from the final matrix, without the need for back substitution. By hand, however, Gauss elimination and back substitution is usually quicker than Gauss-Jordan elimination.

19.2.2 Using the calculator

The TI-84/CE has two built-in commands *ref* and *rref* in the $\boxed{\text{matrix}}$ MATH menu.

ref(matrix), where *matrix* is one of the ten matrices used by the calculator, produces the row-echelon form of *matrix* (Gauss elimination).

rref(matrix) produces the reduced row-echelon form of *matrix* (Gauss-Jordan elimination).

Using \blacktriangleright Frac in the $\boxed{\text{math}}$ menu converts the elements to fractions where possible: for example *ref([A])* \blacktriangleright Frac displays the row-echelon form of [A] with elements as fractions.

The GAUSS/GAUSSCE program²⁸

The programs just automate these procedures.

Store the matrix in [A] using $\boxed{\text{matrix}}$ EDIT. Run GAUSS/GAUSSCE. At the pauses, the matrix displayed is in

- row-echelon form — decimal version.
- row-echelon form — fraction version.
- reduced row-echelon form — decimal version.
- reduced row-echelon form — fraction version.

Press $\boxed{\text{enter}}$ to move through the different forms and $\boxed{\text{enter}}$ after the final form to complete the program. If you want to stop at any intermediate stage, press $\boxed{\text{on}}$ Quit.

After running the program, the original matrix remains in [A], with the row-echelon form stored in [D] and the reduced row-echelon form in [E]. Use $\boxed{\text{matrix}}$ to recall either [D] or [E] for further calculations.

²⁸available at *www.XXX*

Examples: Gauss elimination

$$\begin{bmatrix} 10 & 4 & 1 & 1 \\ 6 & 2 & 1 & 4 \\ 1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0.4 & 0.1 & 0.1 \\ 0 & 1 & -1 & -8.5 \\ 0 & 0 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 2/5 & 1/10 & 1/10 \\ 0 & 1 & -1 & -17/2 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

[A]	[[10 4 1 1] [6 2 1 4] [1 0 0 0]]
-----	--

ref([A])	[[1 0.4 0.1 0.1] [0 1 -1 -8.5] [0 0 1 7]]
Ans>Frac	[[1 2/5 1/10 1/10] [0 1 -1 -17/2] [0 0 1 7]]

Using the *ref* command

$$\begin{bmatrix} 0 & 4 & 1 & 1 \\ 6 & 2 & 1 & 4 \\ 1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0.3 & 0.16 & 0.6 \\ 0 & 1 & 0.25 & 0.25 \\ 0 & 0 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 1/3 & 1/6 & 2/3 \\ 0 & 1 & 1/4 & 1/4 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 4 & 1 & 1 \\ 0 & 2 & 1 & 4 \\ 0 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0.25 & 0.25 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1/4 & 1/4 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example: Gauss-Jordan elimination

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 5 & 1 & 3 & 2 \\ 4 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

[A]	[[1 2 3 5] [5 1 3 2] [4 1 1 1] [2 4 3 1]]
-----	--

REDUCED ROW-ECHELON FORM	[[1 0 0 0] [0 1 0 0] [0 0 1 0] [0 0 0 1]]
--------------------------	--

Using the GAUSSCE program

19.2.3 Exercises

These exercises assume you have covered the theory of solving simultaneous linear equations using Gauss elimination and the inverse-matrix method. Solutions are on page 87.

1. Enter the matrix below carefully into [A] using matrix EDIT.

$$\begin{bmatrix} 0 & 1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}$$

This matrix is the augmented matrix $[P|Q]$ for the following system of equations, in matrix form $PX=Q$:

$$\begin{aligned} x_2 + x_3 - 2x_4 &= -3 \\ x_1 + 2x_2 - x_3 &= 2 \\ 2x_1 + 4x_2 + x_3 - 3x_4 &= -2 \\ x_1 - 4x_2 - 7x_3 - x_4 &= -19 \end{aligned}$$

Solve this system with Method:

- A** using the calculator to find the row-echelon form (Gaussian elimination), then back substitution (you have to do this);
- B** using the calculator to find the reduced row-echelon form (Gauss-Jordan elimination), from which you can read off the answer;
- C** using $X=P^{-1}Q$. Do you understand why this works?

A little manipulation on the calculator allows us to evaluate $P^{-1}Q$. Change [A] to the matrix P by changing the column dimension to 4 using matrix EDIT: this chops off the last column of [A]. Store the column matrix Q in [B], then evaluate $[A]^{-1}[B]$.

PTO

2. Solve the following systems of equations using each of the three methods using the calculator to do the Gauss or Gauss-Jordan elimination.

Check your answers by substituting them back into the equations.

Can you use Method C to solve these? What happens in (b) and (c) below? Can you explain? What is $\det(\mathbf{P})$? What does this tell you about the inverse matrix?

$$\begin{array}{lcl} \text{(a)} & 2x - 5y + 5z = 17 & \text{(b)} \quad x_1 + x_2 - 5x_3 = 3 \\ & x - 2y + 3z = 9 & \quad x_1 - 2x_3 = 1 \\ & -x + 3y = -4 & \quad 2x_1 - x_2 - x_3 = 0 \end{array}$$

$$\begin{array}{lcl} \text{(c)} & x_1 - x_2 + 2x_3 = 4 & \\ & x_1 + x_3 = 6 & \\ & 2x_1 - 3x_2 + 5x_3 = 4 & \\ & 3x_1 + 2x_2 - x_3 = 1 & \end{array}$$

Answers

(a) $x=1, y=-1, z=2$

(b) $x_1=1+2t, x_2=2+3t, x_3=t$; t any non-zero number

(c) no solution.

19.3 Eigenvalues and eigenvectors

19.3.1 Theory

Given an $n \times n$ matrix \mathbf{A} , the eigenvalues of \mathbf{A} are the constants λ and the eigenvectors of \mathbf{A} the corresponding (non-zero) n -dimensional vectors \mathbf{X} ($n \times 1$ matrices) satisfying the equation

$$\mathbf{A}\mathbf{X} = \lambda\mathbf{X}.$$

The eigenvalues λ are the zeros or roots of the n th-degree characteristic polynomial

$$p(x) = \det(\mathbf{A} - x\mathbf{I}),$$

where \mathbf{I} is the $n \times n$ identity matrix.

A nice way to find the eigenvalues on a TI-84/CE is graphically: set

$$Y_1 = \det([\mathbf{A}] - X \text{identity}(n)),$$

the characteristic polynomial, where you have to put in a value for n , the dimension of $[\mathbf{A}]$, and graph in the usual way with an appropriate window. Use *zero* (in the `calc` menu) to find the zeros of the polynomial: these are the eigenvalues λ_i , $i = 1, 2, \dots$

To find the eigenvectors, we have to solve the homogeneous matrix equation

$$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{X} = \mathbf{0}$$

for each eigenvalue λ_i found above. On the calculator, we do this by Gauss-Jordan elimination, either using the command *rref* on the matrix $\mathbf{A} - \lambda_i \mathbf{I}$ or using the GAUSS program with $\mathbf{A} - \lambda_i \mathbf{I}$ stored in matrix $[\mathbf{A}]$.

19.3.2 Example

Find the eigenvalues and corresponding eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}.$$

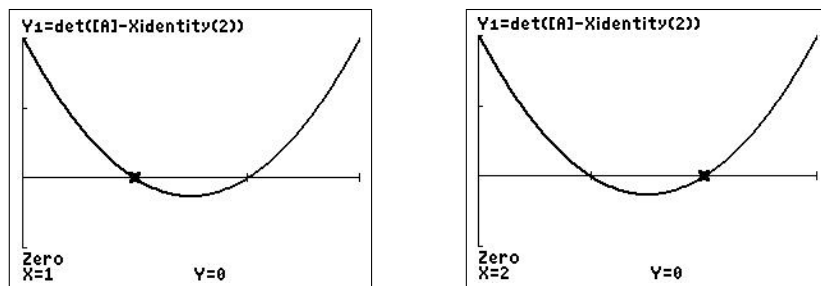
This is a simple problem that can all be done easily by hand. We use it to illustrate the calculator method, which can be used for more complicated problems.

Eigenvalues

Put \mathbf{A} into matrix $[\mathbf{A}]$ in your calculator and set $Y_1 = \det([\mathbf{A}] - X \text{identity}(2))$.

$\blacksquare Y_1 = \det([\mathbf{A}] - X \text{identity}(2))$

Start with $X_{\min} = -5$ and $X_{\max} = 5$. Press `graph` to plot the characteristic polynomial, here a quadratic (because $[\mathbf{A}]$ is 2×2). It is clear that the zeros lie in the range $0 < x < 3$. Change the window to magnify this region.



window $[0, 3, 1] \times [-1, 2, 1]$

Using *zero* in the `calc` menu gives the eigenvalues as $\lambda_1 = 1$ and $\lambda_2 = 2$ (figures above).

In this case, it is easy to show by hand that the characteristic polynomial is $p(\lambda) = \lambda^2 - 3\lambda + 2$, with zeros 1 and 2.

Eigenvectors

$\lambda_1 = 1$: We have to find $\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, such that $\mathbf{A}\mathbf{X} = \mathbf{X}$ or $(\mathbf{A} - \mathbf{I})\mathbf{X} = \mathbf{0}$, i.e.

$$\left(\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (1)$$

To use the GAUSS/GAUSSCE program, put $\mathbf{A} - \mathbf{I}$ in matrix [A].

As we want to use \mathbf{A} , now in [A], for the second eigenvector too, store [A] in [B]. Then store [B] - identity(2) in [A] and run GAUSS/GAUSSCE. Alternatively execute the command `ref([B] - identity(2))`.

We should really use the augmented matrix here, with a third column of 0s from the right-hand side of Eq. (1), but these remain 0 in the Gauss elimination, so we just remember to put them back in at the end.

<pre>[A]→[B] [[3 2] [-1 0]] ----- [B]-identity(2)→[A] [[2 2] [-1 -1]] ----- prgmGAUSS</pre>	<p>REDUCED ROW-ECHELON FORM</p> <pre> [[1 1] [0 0]]</pre>
---	--

The reduced row-echelon form of $\mathbf{A} - \mathbf{I}$, with the third column of 0s now back in, is

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that the bottom row is all zeros. There must be at least one row of zeros (the bottom row) in all eigenvector problems because there is always an infinite number of solutions to the eigenvector equations, Eq. (1) here: eigenvalues are only defined to an arbitrary multiplicative constant.

The corresponding matrix equation is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The bottom row tells us that x_2 is arbitrary, so we set $x_2 = t$, where t is any non-zero number. The top row tells us that $x_1 + x_2 = 0$, so that $x_1 = -x_2 = -t$.

The eigenvector \mathbf{X}_1 corresponding to $\lambda_1 = 1$ is therefore

$$\mathbf{X}_1 = \begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The eigenvectors are arbitrary (non-zero²⁹) multiples of the vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

We usually say that this vector is the eigenvector, with the understanding that all non-zero multiples of it are also eigenvectors.

Check:

$$\mathbf{A}\mathbf{X}_1 = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \mathbf{X}_1.$$

$\lambda_2 = 2$: We have to solve $\mathbf{A}\mathbf{X} = 2\mathbf{X}$ or $(\mathbf{A} - 2\mathbf{I})\mathbf{X} = \mathbf{0}$ for \mathbf{X} . Store [B]–2identity(2) in [A] and run GAUSS. Alternatively execute ref([B]–2identity(2)). Both give the reduced row-echelon form of $\mathbf{A} - 2\mathbf{I}$, with the third column of 0s back in, as

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, $x_2 = t$, where t is any number, and $x_1 + 2x_2 = 0$, so that $x_1 = -2x_2 = -2t$. The eigenvector \mathbf{X}_2 corresponding to $\lambda_2 = 2$ is therefore

$$\mathbf{X}_2 = \begin{bmatrix} -2t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

The eigenvector is an arbitrary (non-zero) multiple of the vector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.
Check that $\mathbf{A}\mathbf{X}_2 = 2\mathbf{X}_2$.

²⁹If $t=0$, we obtain a zero vector, which, by definition, cannot be an eigenvector.

19.3.3 EIGENV5/EIGENV5CE — finds real eigenvalues and eigenvectors

The EIGENV5/EIGENV5CE program³⁰ combines plotting the characteristic polynomial, a root finder to find the eigenvalues and the GAUSS program to find the eigenvectors of the $n \times n$ matrix $[A]$, all semi-automatically. The description below is rather long but the process is relatively quick to carry out once you've had some practice.

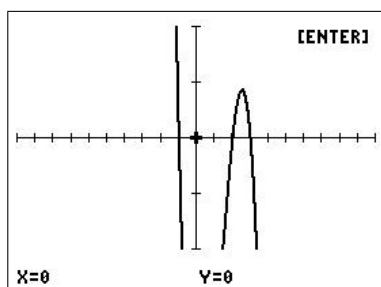
Example

Find the eigenvectors and eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

1. Store \mathbf{A} (here 3×3) in matrix $[A]$ using `matrix` EDIT.
2. Run the EIGENV5/EIGENV5CE program. Choose FIND THE EIGENVALUES. This graphs the characteristic polynomial, whose zeros give the real eigenvalues, using a window $[-10, 10, 1] \times [-5, 5, 1]$.

```
FIND THE
1: EIGENVALUES
2: EIGENVECTORS
3: QUIT
```



```
FIND THE EIGENVALUES
1: AUTOMATICALLY
2: MANUALLY
```

3. After the polynomial has been graphed, you have the option to have its zeros (the real eigenvalues) found automatically or you can do it manually using `zero` in the `calc` menu.
4. If you choose Automatic mode, the calculator first asks if you want to change the X scale (change Xmin/Xmax). The program only finds zeros that are on the screen.

```
CHANGE X SCALE
1: NO
2: YES
```

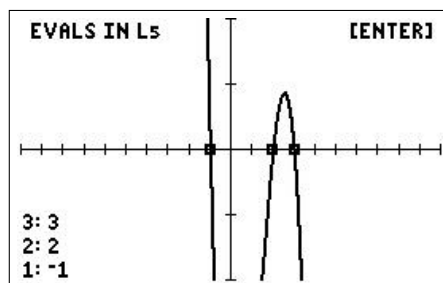
If the zeros are not all on screen (*how many should there be?*) or if they are concentrated in a small part of the X range,³¹ select YES. The calculator will ask you to change Xmin/Xmax, then it will replot the graph. Keep doing this step until all the zeros are on screen and spread out. Then select NO.

³⁰available at www.XXX

³¹The zero finder works along the graph pixel by pixel until it finds a sign change in the function between two adjacent pixels. If the zeros are very concentrated, there may be more than one zero in a pixel. In this case, at least one of these zeros may not be found or the same one may be found twice. It's best to choose Xmin/Xmax so that the zeros are spread out as far as possible ON THE SCREEN.

5. The calculator will find (approximations to) the zeros, place a small square at the relevant point on the graph and list on-screen the values found, rounded to 3 decimal places. The full (but still approximate) values for the zeros/eigenvalues are stored in list L5 (smallest to largest) for later use in finding the eigenvectors.

For the example here, you should obtain the eigenvalues $\lambda_1 = -1$, $\lambda_2 = 2$ and $\lambda_3 = 3$.



6. The program now asks if you want to find the eigenvectors. Answer YES.³²

```

FIND THE EIGENVECTORS?
1: YES
2: NO
3: ADD EIGENVALUE TO L5?

```

7. Choose which eigenvalue you want to use by selecting its index in L5: 1, 2, 3, ...

```

EIGENVECTOR OF MATRIX [A]
EIGENVECTOR -> [B]
USES MATRIX [C]
INDEX OF EIGENVALUE IN L5
I=?1

```

```

EIGENVALUE
L5(1)=-1
EIGENVECTOR [B]
{3 -3 1}

```

8. The program will return the eigenvalue and eigenvector (use the right arrow to scroll if necessary). Alternatively it may return the message

Reduced row-echelon matrix needs interpreting.

This happens if the z component of the eigenvector is zero, if there is more than one row of all zeros or if $\mathbf{A}\mathbf{v} \neq \lambda\mathbf{v}$ (the program checks).

Rather than the program choose between one of several possibilities, it asks you to look at the reduced row-echelon form of the augmented matrix and work out the eigenvector.

9. After it has calculated an eigenvector, the program asks whether you want to find further eigenvectors. The last eigenvector is stored in matrix [B].

In our example, the eigenvectors corresponding to $\lambda_1 = -1$, $\lambda_2 = 2$ and $\lambda_3 = 3$ are

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

³²but see Point 11.

10. You can run EIGENV5/EIGENV5CE at any time to find the eigenvectors (choose FIND THE EIGENVECTORS? at the start), provided the eigenvalues are still in list L5 and the matrix in [A].

Alternatively you can store the eigenvalues in L5 manually (stat Edit), run the program and choose FIND THE EIGENVECTORS from the first menu. The appropriate matrix has to be stored in matrix [A] of course.

11. *Repeated eigenvalues* show up as the graph of the characteristic equation touching, but not crossing, the x axis. The root finder will therefore not pick them up.

If the x value (λ) where the graph touches the x axis is obvious from the graph, you can insert it into L5 via the program. At Point 6 above, there is an additional prompt allowing you to do this. You can then continue to run the program as normal. The program may find an eigenvector, if there is only one, or else it will display the reduced row-echelon form of the augmented matrix to allow you to work out the eigenvector(s).

If the x value is not obvious from the graph, you can use *maximum* or *minimum* in the calc menu to find it. You will need to quit out of the program to do this. Then press graph to replot the graph. Store the value in L5, after the eigenvalues the calculator has already stored.

Then re-run the EIGENV5/EIGENV5CE program, choose FIND THE EIGENVECTORS and put in the index of the eigenvalue you have just entered. If an eigenvalue is repeated, it may have more than one associated eigenvector.

19.3.4 Exercises

1. Find, using the calculator (but not EIGENV5/EIGENV5CE), the eigenvectors and eigenvalues of the matrix

$$\mathbf{A} = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}.$$

Check with EIGENV5/EIGENV5CE and verify that $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ for each eigenvalue.

2. Show, using the calculator (but not EIGENV5/EIGENV5CE), that the eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

are -1 , 1 and 2 , with corresponding eigenvectors

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

3. Find, using the calculator (but not EIGENV5/EIGENV5CE), the eigenvectors and eigenvalues of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

Check with EIGENV5/EIGENV5CE and verify that $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ for each eigenvalue.

Solutions are on page 87.

19.4 Vector operations

19.4.1 Introduction

All the operations here, except one, are based on the dot or scalar product. This can be evaluated with the vectors stored as (column) matrices or as lists. If \mathbf{a} and \mathbf{b} are two vectors of the same dimension n , they can be represented on the calculator by the $n \times 1$ column matrices $[A]$ and $[B]$, say, or the components stored in lists, say, L_1 and L_2 . Then the dot or scalar product is given by the matrix multiplication

$$\mathbf{a} \cdot \mathbf{b} = [A]^T[B],$$

where the superscript T denotes the transpose, or by the list operation

$$\mathbf{a} \cdot \mathbf{b} = \text{sum}(L_1 L_2).$$

The list product $L_1 L_2$ produces a list whose components are the products of the corresponding components of the two lists; summing these then gives the scalar product.

Note that the scalar product is defined for vectors of any length, i.e. $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$; this is handled by both methods on the calculator.

The one exception mentioned above is the cross or vector product, which is only defined for three-dimensional vectors.

The list method is a little easier to use, so we continue with it. The operations described below are contained in the VECTOR/VECTORCE program,³³ which prompts for the vectors, then does the calculations.

19.4.2 Operations

Use stat Edit to input the vectors or enclose them in $\{ \}$, with commas, and store them to the relevant list.

Assume vector \mathbf{a} is in list L_1 , vector \mathbf{b} in list L_2 . *sum* is in the list MATH menu.

Scalar multiplication (\mathbb{R}^n)

If c is a constant, the scalar multiple $c\mathbf{a} = cL_1$.

Scalar or dot product (\mathbb{R}^n)

$$\mathbf{a} \cdot \mathbf{b} = \text{sum}(L_1 L_2).$$

Vector or cross product (\mathbb{R}^3 only)

$\mathbf{a} \times \mathbf{b}$ is a vector, given in terms of the entries in the two lists by

$$\mathbf{a} \times \mathbf{b} = (L_1(2)L_2(3) - L_1(3)L_2(2), L_1(3)L_2(1) - L_1(1)L_2(3), L_1(1)L_2(2) - L_1(2)L_2(1)).$$

³³available at *www.XXX*

Norm (\mathbb{R}^n)

$$\text{norm } \mathbf{a} = \sqrt{(\mathbf{a} \cdot \mathbf{a})} = \sqrt{(\text{sum}(L_1 L_1))}$$

Projection (\mathbb{R}^n)

The (vector) projection of \mathbf{b} onto \mathbf{a} is given by

$$\text{proj}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} \mathbf{a} = \text{sum}(L_1 L_2) / \text{sum}(L_1 L_1) L_2.$$

Angle between vectors (\mathbb{R}^n)

The angle θ between vectors \mathbf{a} and \mathbf{b} is given by

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{\sqrt{(\mathbf{a} \cdot \mathbf{a})} \sqrt{(\mathbf{b} \cdot \mathbf{b})}} = \text{sum}(L_1 L_2) / \sqrt{(\text{sum}(L_1 L_1))} \sqrt{(\text{sum}(L_2 L_2))}.$$

19.4.3 Exercises

1. Given vectors $\mathbf{a} = (1, 2, 3)$ and $\mathbf{b} = (-1, 0, 3)$, find

(a) $\mathbf{a} \cdot \mathbf{b}$ *Answer:* 8.

(b) $\mathbf{a} \times \mathbf{b}$ *Answer:* $(6, 6, -2)$.

(c) norm \mathbf{b} *Answer:* $\sqrt{10}$.

(d) $\text{proj}_{\mathbf{a}} \mathbf{b}$ *Answer:* $\left(\frac{4}{7}, \frac{8}{7}, \frac{12}{7}\right)$.

(e) the angle between \mathbf{a} and \mathbf{b} in radians and degrees. *Answer:* 0.828 rad; 47.5°.

2. Given vectors $\mathbf{a} = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}\right)$ and $\mathbf{b} = \left(\frac{2}{3}, \frac{3}{5}, \frac{2}{7}, -1\right)$, find

(a) $\mathbf{a} \cdot \mathbf{b}$ *Answer:* $\frac{47}{105}$.

(b) norm \mathbf{b} *Answer:* 1.373 (to 3DP).

(c) $\text{proj}_{\mathbf{a}} \mathbf{b}$ *Answer:* $(0.531, 0.354, 0.212, 0.152)$ (to 3DP).

(d) the angle between \mathbf{a} and \mathbf{b} in radians and degrees. *Answer:* 1.05 rad; 59.9°.

19.4.4 Solutions

Gauss elimination page 77

1. Enter the matrix below carefully into [A] using `matrix` EDIT.

$$\begin{bmatrix} 0 & 1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}$$

This matrix is the augmented matrix $[P|Q]$ for the following system of equations, in matrix form $PX=Q$:

$$\begin{aligned} x_2 + x_3 - 2x_4 &= -3 \\ x_1 + 2x_2 - x_3 &= 2 \\ 2x_1 + 4x_2 + x_3 - 3x_4 &= -2 \\ x_1 - 4x_2 - 7x_3 - x_4 &= -19 \end{aligned}$$

- A. Solve this system using the calculator to find the row-echelon form (Gaussian elimination), then back substitution (you have to do this);

Either run the GAUSS/GAUSSCE program (left-hand screen below) or the `ref` command (right-hand screen) to give the row-echelon form of the matrix. Using `math` `1` (`►Frac`) converts decimals to fractions where possible.

```
►Frac FORM
[[1 2 1/2 -3/2 -1]
 [0 1 5/4 -1/12 3 ]
 [0 0 1 -1 -2]
 [0 0 0 1 3 ]]
```

```
ref([A])►Frac
[[1 2 1/2 -3/2 -1]
 [0 1 5/4 -1/12 3 ]
 [0 0 1 -1 -2]
 [0 0 0 1 3 ]]
```

The corresponding (equivalent) equations are

$$\begin{aligned} x_1 + 2x_2 + \frac{1}{2}x_3 - \frac{3}{2}x_4 &= -1 \\ x_2 + \frac{5}{4}x_3 - \frac{1}{12}x_4 &= 3 \\ x_3 - x_4 &= -2 \\ x_4 &= 3 \end{aligned}$$

The last equation gives $x_4=3$.

Substituting this into the third equation gives $x_3=-2+x_4=-2+3=1$.

Substituting these into the second equation gives

$$x_2 = 3 - \frac{5}{4}x_3 + \frac{1}{12}x_4 = 3 - \frac{5}{4} + \frac{3}{12} = 2.$$

Substituting these into the first equation gives

$$x_1 = -1 - 2x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4 = -1 - 4 - \frac{1}{2} + \frac{9}{2} = -1.$$

Therefore, $x_1=-1$, $x_2=2$, $x_3=1$, $x_4=3$.

- B.** using the calculator to find the reduced row-echelon form (Gauss-Jordan elimination), from which you can read off the answer;

Either run the GAUSS/GAUSSCE program again or use the *rref* command to give the reduced-row-echelon form of the matrix.

REDUCED ROW-ECHELON FORM				
[[1	0	0	-1]
[0	1	0	2]
[0	0	1	1]
[0	0	0	1 3]]

The corresponding (equivalent) equations are

$$\begin{aligned}x_1 &= -1 \\x_2 &= 2 \\x_3 &= 1 \\x_4 &= 3,\end{aligned}$$

the answer directly.

- C.** using $\mathbf{X}=\mathbf{P}^{-1}\mathbf{Q}$. Do you understand why this works?

Change [A] from the 4×5 augmented matrix to the 4×4 matrix \mathbf{A} by changing the column dimension to 4 using `[matrix] EDIT`: this chops off the last column of [A] (top matrix in the figure below). Store the column matrix \mathbf{b} in [B], then evaluate $[\mathbf{A}]^{-1}[\mathbf{B}]$.

<table style="margin-left: auto; margin-right: auto;"> <tbody> <tr><td style="text-align: center;">[[</td><td style="text-align: center;">0</td><td style="text-align: center;">1</td><td style="text-align: center;">1</td><td style="text-align: center;">-2]</td></tr> <tr><td style="text-align: center;">[</td><td style="text-align: center;">1</td><td style="text-align: center;">2</td><td style="text-align: center;">-1</td><td style="text-align: center;">0]</td></tr> <tr><td style="text-align: center;">[</td><td style="text-align: center;">2</td><td style="text-align: center;">4</td><td style="text-align: center;">1</td><td style="text-align: center;">-3]</td></tr> <tr><td style="text-align: center;">[</td><td style="text-align: center;">1</td><td style="text-align: center;">-4</td><td style="text-align: center;">-7</td><td style="text-align: center;">-1]]</td></tr> </tbody> </table> <hr style="border-top: 1px dotted black;"/> <p>[B]</p> <table style="margin-left: auto; margin-right: auto;"> <tbody> <tr><td style="text-align: center;">[[</td><td style="text-align: center;">-3]</td></tr> <tr><td style="text-align: center;">[</td><td style="text-align: center;">2]</td></tr> <tr><td style="text-align: center;">[</td><td style="text-align: center;">-2]</td></tr> <tr><td style="text-align: center;">[</td><td style="text-align: center;">-19]]</td></tr> </tbody> </table>	[[0	1	1	-2]	[1	2	-1	0]	[2	4	1	-3]	[1	-4	-7	-1]]	[[-3]	[2]	[-2]	[-19]]	<p>[A]⁻¹[B]</p> <table style="margin-left: auto; margin-right: auto;"> <tbody> <tr><td style="text-align: center;">[[</td><td style="text-align: center;">-1]</td></tr> <tr><td style="text-align: center;">[</td><td style="text-align: center;">2]</td></tr> <tr><td style="text-align: center;">[</td><td style="text-align: center;">1]</td></tr> <tr><td style="text-align: center;">[</td><td style="text-align: center;">3]]</td></tr> </tbody> </table>	[[-1]	[2]	[1]	[3]]
[[0	1	1	-2]																																	
[1	2	-1	0]																																	
[2	4	1	-3]																																	
[1	-4	-7	-1]]																																	
[[-3]																																				
[2]																																				
[-2]																																				
[-19]]																																				
[[-1]																																				
[2]																																				
[1]																																				
[3]]																																				

This method works here because the matrix \mathbf{A} is invertible, corresponding to the case in which there is a unique solution.

Again the answer is $x_1=-1$, $x_2=2$, $x_3=1$, $x_4=3$.

2. Solve the following systems of equations using each of the three methods using the calculator to do the Gauss or Gauss-Jordan elimination.

Check your answers by substituting them back into the equations.

$$\begin{aligned} \text{(a)} \quad & 2x - 5y + 5z = 17 \\ & x - 2y + 3z = 9 \\ & -x + 3y = -4 \end{aligned}$$

In matrix form,

$$\begin{bmatrix} 2 & -5 & 5 \\ 1 & -2 & 3 \\ -1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 17 \\ 9 \\ -4 \end{bmatrix},$$

with augmented matrix

$$\begin{bmatrix} 2 & -5 & 5 & 17 \\ 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \end{bmatrix}.$$

The row-echelon form from GAUSS/GAUSSCE (Method A) is

$$\begin{bmatrix} 1 & -2.5 & 2.5 & 8.5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

The corresponding equations are

$$\begin{aligned} x - 2.5y + 2.5z &= 8.5 \\ y + z &= 1 \\ z &= 2 \end{aligned}$$

The last equation gives $z = 2$ directly. Substituting this into the second equation gives $y = -1$, and these into the first equation $x = 1$.

The unique solution from Method A is $x = 1$, $y = -1$ and $z = 2$.

PTO

Using GAUSS/GAUSSCE for Method B, Gauss-Jordan elimination, starting with the augmented matrix in [A]:

MATRIX[A] 3 x 4				
[2	-5	5	17]
[1	-2	3	9]
[-1	3	0	-4]

REDUCED ROW-ECHELON FORM				
[[1	0	0	1]
[0	1	0	-1]	
[0	0	1	2]	

The right-hand column of the reduced row-echelon form gives the solution; each equation now only contains one variable.

Following the hint in Question 1 for putting \mathbf{A} back in matrix [A], the RHS of the original system in [B] and evaluating $[\mathbf{A}]^{-1}[\mathbf{B}]$ gives the Method C result

[A]	[[2	-5	5]
	[1	-2	3]
	[-1	3	0]]
[B]		
	[[17]		
	[9]		
	[-4]]		

[A] ⁻¹ [B]	[[1]
	[-1]
	[2]]

which gives the solution directly.

Substituting the solution $x=1$, $y=-1$ and $z=2$ back into the original equations:

$$\text{LHS} = 2 \times 1 - 5 \times -1 + 5 \times 2 = 17 = \text{RHS.}$$

$$\text{LHS} = 1 - 2 \times -1 + 3 \times 2 = 9 = \text{RHS.}$$

$$\text{LHS} = -1 + 3 \times -1 = -4 = \text{RHS.}$$

The solution is verified.

PTO

$$\begin{aligned}
 \text{(b)} \quad & x_1 + x_2 - 5x_3 = 3 \\
 & x_1 - 2x_3 = 1 \\
 & 2x_1 - x_2 - x_3 = 0
 \end{aligned}$$

In matrix form,

$$\begin{bmatrix} 1 & 1 & -5 \\ 1 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix},$$

with augmented matrix

$$\begin{bmatrix} 1 & 1 & -5 & 3 \\ 1 & 0 & -2 & 1 \\ 2 & -1 & -1 & 0 \end{bmatrix}.$$

GAUSS/GAUSSCE gives the row-echelon form (Method A)

$$\begin{bmatrix} 1 & -0.5 & -0.5 & 0 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The bottom row of all zeros tells us there is an infinite number of solutions.

The corresponding equations are

$$\begin{aligned}
 x_1 + x_2 - 5x_3 &= 3 \\
 x_2 - 3x_3 &= 2
 \end{aligned}$$

Here, x_3 is a free variable: let $x_3 = t$, $t \in \mathbb{R}$.

The second equation gives $x_2 = 2 + 3t$, and the first $x_1 = 1 + 2t$.

The solution is $x_1 = 1 + 2t$, $x_2 = 2 + 3t$ and $x_3 = t$, $t \in \mathbb{R}$

Using GAUSS/GAUSSCE for Method B, Gauss-Jordan elimination,

MATRIX[A] 3 x 4				
[1	1	-5	3
[1	0	-2	1
[2	-1	-1	0

REDUCED ROW-ECHELON FORM				
[1	0	-2	1
[0	1	-3	2
[0	0	0	0

The right-hand column of the reduced row-echelon form does not give the solution directly here. However, back substitution is easier with the reduced row-echelon form than with the row-echelon form.

Putting \mathbf{A} back in matrix $[A]$, the RHS of the original system in $[B]$ and evaluating $[A]^{-1}[B]$ gives an error message: *No inverse exists*. Method C cannot be used, consistent with the fact that there is not a unique solution.

Substituting the solution $x_1 = 1 + 2t$, $x_2 = 2 + 3t$ and $x_3 = t$ back into the original equations:

$$\text{LHS} = 1 + 2t + 2 + 3t - 5t = 3 = \text{RHS.}$$

$$\text{LHS} = 1 + 2t - 2t = 1 = \text{RHS.}$$

$$\text{LHS} = 2(1 + 2t) - (2 + 3t) - t = 0 = \text{RHS.}$$

The solution is verified.

(c)

$$x_1 - x_2 + 2x_3 = 4$$

$$x_1 + x_3 = 6$$

$$2x_1 - 3x_2 + 5x_3 = 4$$

$$3x_1 + 2x_2 - x_3 = 1$$

In matrix form,

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 2 & -3 & 5 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 4 \\ 1 \end{bmatrix},$$

with augmented matrix

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 1 & 0 & 1 & 6 \\ 2 & -3 & 5 & 4 \\ 3 & 2 & -1 & 1 \end{bmatrix}$$

GAUSS/GAUSSCE gives the row-echelon form (Method A)

$$\begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & -\frac{17}{3} & -\frac{10}{3} \\ 0 & 0 & 1 & \frac{67}{6} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The bottom row tells us there is no solution: the corresponding equation is $0x_1 + 0x_2 + 0x_2 = 1$.

Using GAUSSCE for Method B, Gauss-Jordan elimination,

MATRIX[A] 4 × 4					
[1	-1	2	4]
[1	0	1	6]
[2	-3	5	4]
[3	2	-1	1]

REDUCED ROW-ECHELON FORM					
[1	0	0	0]
[0	1	0	0]
[0	0	1	0]
[0	0	0	1]

Again no solution.

Putting \mathbf{A} back in matrix [A], the RHS of the original system in [B] and evaluating $[\mathbf{A}]^{-1}[\mathbf{B}]$ gives an error message: *No inverse exists*. Method C cannot be used, consistent with the fact that there is not a unique solution.

What is $\det(\mathbf{P})$? What does this tell you about the inverse matrix?

In (b) and (c), $\det(\mathbf{P}) = 0$, confirming that an inverse matrix does not exist.

Eigenvalues and eigenvectors page 84

- Find, using the calculator (but not EIGENV5/EIGENV5CE), the eigenvectors and eigenvalues of the matrix

$$\mathbf{A} = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}.$$

Check with EIGENV5/EIGENV5CE and verify that $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ for each eigenvalue.

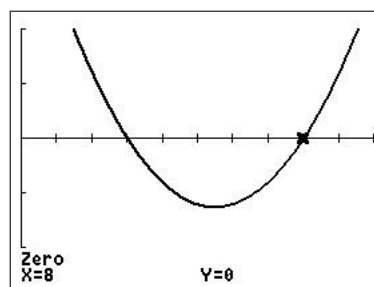
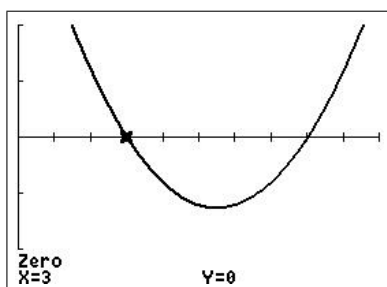
Eigenvalues

Put the 2×2 matrix \mathbf{A} into matrix [A] in your calculator.

Set $Y_1 = \det([\mathbf{A}] - X \text{identity}(2))$.

Y1=det([A]-Xidentity(2))

Start with **zoom** **6**. Press **graph** to plot the characteristic polynomial, a quadratic here. It is clear that the zeros lie in the range $0 < x < 10$.



window $[0, 10, 1] \times [-10, 10, 5]$

Using *zero* in the **calc** menu gives the eigenvalues as $\lambda_1 = 3$ and $\lambda_2 = 8$.

PTO

Eigenvectors **$\lambda_1 = 3$**

Let $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$. Then $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_1 = (\mathbf{A} - 3\mathbf{I})\mathbf{v}_1 = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

To use the GAUSS/GAUSSCE program, we need to put $\mathbf{A} - 3\mathbf{I}$ in matrix [A]. As we want to use \mathbf{A} , now in [A], for the second eigenvector, store [A] in [B]. Then store [B]–identity(2) in [A] and run GAUSS/GAUSSCE.

We should really use the augmented matrix here with a third column of 0s; these remain 0 in the Gauss elimination, so we just remember to put them back in afterwards.

<div style="border: 1px solid black; padding: 5px; display: inline-block;"> $[A]$ </div> <div style="margin-left: 100px; border: 1px solid black; padding: 5px; display: inline-block;"> $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$ </div>	<div style="border: 1px solid black; padding: 5px; display: inline-block;"> REDUCED ROW-ECHELON FORM </div> <div style="margin-left: 100px; border: 1px solid black; padding: 5px; display: inline-block;"> $\begin{bmatrix} 1 & 0.5 & \\ 0 & 0 & \end{bmatrix}$ </div>
--	---

This gives the reduced row-echelon form of $\mathbf{A} - 3\mathbf{I}$, with the third column of 0s back in, as

$$\begin{bmatrix} 1 & 0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that the bottom row is all zeros. There must be at least one row of zeros (the bottom row) in all eigenvector problems.

Therefore, y_1 is a free variable. Set $y_1 = t \in \mathbb{R}$, $t \neq 0$.

The top row gives the equation $x_1 + 0.5y_1 = 0$, so that $x_1 = -0.5y_1 = -0.5t$.

The solution is therefore $y_1 = t$, $x_1 = -0.5t$, where t is any **non-zero** real number.

Therefore, \mathbf{A} has an eigenvalue $\lambda_1 = 3$, with corresponding eigenvector

$$\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -0.5t \\ t \end{bmatrix} = t \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}, \quad t \neq 0 \in \mathbb{R}.$$

Given that t is arbitrary, set $s = -\frac{1}{2}t$, $s \neq 0 \in \mathbb{R}$, so that

$$\mathbf{v}_1 = s \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Check your answer by verifying that $\mathbf{A}\mathbf{v}_1 = 3\mathbf{v}_1$.

$$\begin{aligned} \mathbf{A}\mathbf{v}_1 &= \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ -6 \end{bmatrix} \\ &= 3 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ &= 3\mathbf{v}_1. \end{aligned}$$

$\lambda_2 = 8$

Let $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$. Then $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_1 = (\mathbf{A} - 8\mathbf{I})\mathbf{v}_1 = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Gauss elimination on the augmented matrix:

$$\begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, y_2 is a free variable. Set $y_2 = t \neq 0 \in \mathbb{R}$.

The top row gives the equation $x_2 - 2y_2 = 0$, so that $x_2 = 2y_2 = 2t$.

The solution is therefore $y_2 = t$, $x_2 = 2t$, where t is any **non-zero** real number.

Therefore, \mathbf{A} has an eigenvalue $\lambda_2 = 8$, with corresponding eigenvector

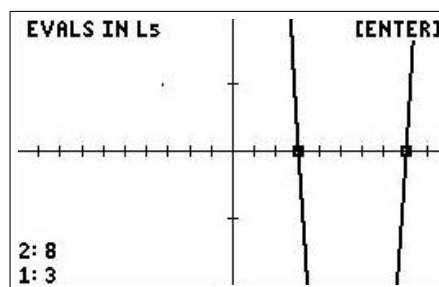
$$\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad t \neq 0 \in \mathbb{R}.$$

Check your answer by verifying that $\mathbf{A}\mathbf{v}_2 = 8\mathbf{v}_2$.

$$\begin{aligned} \mathbf{A}\mathbf{v}_2 &= \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 16 \\ 8 \end{bmatrix} \\ &= 8 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= 8\mathbf{v}_2. \end{aligned}$$

Note that we usually take $t=1$ when writing down the eigenvectors, knowing that any non-zero multiple is also an eigenvector.

Using EIGENV5/EIGENV5CE



EIGENVALUE $L_5(1)=3$ EIGENVECTOR [B] $\{-1/2 \ 1\}$

EIGENVALUE $L_5(2)=8$ EIGENVECTOR [B] $\{2 \ 1\}$
--

2. Show, using the calculator (but not EIGENV5/EIGENV5CE), that the eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

are -1 , 1 and 2 , with corresponding eigenvectors

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

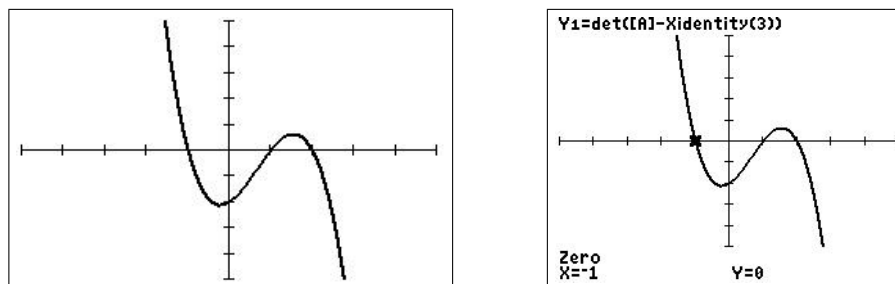
Eigenvalues

Put the 3×3 matrix \mathbf{A} into matrix [A] in your calculator.

Set $Y_1 = \det([\mathbf{A}] - X \text{identity}(3))$.

`Y1=det([A]-Xidentity(3))`

Start with $X_{\min} = -5$ and $X_{\max} = 5$. Press `graph` to plot the characteristic polynomial, here a cubic. It is clear that the zeros lie in the range $-2 < x < 3$.



window $[-5, 5, 1] \times [-5, 5, 1]$

Using `zero` in the `calc` menu gives the eigenvalues as $\lambda_1 = -1$, $\lambda_2 = 1$ and $\lambda_3 = 3$.

Eigenvectors

$\lambda_1 = -1$: We have to find $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, such that $\mathbf{A}\mathbf{v}_1 = -\mathbf{v}_1$ or $(\mathbf{A} + \mathbf{I})\mathbf{v}_1 = \mathbf{0}$,

i.e.

$$\left(\begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

To use the GAUSS/GAUSSCE program, we need to put $\mathbf{A} + \mathbf{I}$ in matrix [A]. Since we want to use \mathbf{A} , now in [A], for the second eigenvector, store [A] in [B]. Then store `[B]+identity(3)` in [A] and run GAUSS/GAUSSCE.

[A] `[[2 1 -2]
[-1 3 1]
[0 1 0]]`

REDUCED ROW-ECHELON FORM `[[1 0 -1]
[0 1 0]
[0 0 0]]`

This gives the reduced row-echelon form of $\mathbf{A}+\mathbf{I}$, with the fourth column of 0s put back in, as

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note again that the bottom row is all zeroes.

The bottom row tells us that x_3 is arbitrary, so we set $x_3 = t$, where t is any number. The second row gives $x_2 = 0$. The top row tells us that $x_1 - x_3 = 0$, so that $x_1 = x_3 = t$.

The eigenvector \mathbf{v}_1 corresponding to $\lambda_1 = -1$ is therefore

$$\mathbf{v}_1 = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

The eigenvectors are arbitrary (non-zero) multiples of the vector $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

$\lambda_2 = 1$: We have to solve $\mathbf{A}\mathbf{v}_2 = \mathbf{v}_2$ or $(\mathbf{A}-\mathbf{I})\mathbf{v}_2 = \mathbf{0}$ for \mathbf{v}_2 .

Store [B] – identity(3) in [A] and run GAUSS/GAUSSCE. The reduced row-echelon form of $\mathbf{A}-\mathbf{I}$, with the fourth column of 0s put back in, is

$$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, $x_3 = t$, where t is any number. The second row gives the equation $x_2 - 2x_3 = 0$, so that $x_2 = 2x_3 = 2t$. The top row gives the equation $x_1 - 3x_3 = 0$, so that $x_1 = 3x_3 = 3t$.

The eigenvector \mathbf{v}_2 corresponding to $\lambda_2 = 1$ is therefore

$$\mathbf{v}_2 = \begin{bmatrix} 3t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

The eigenvectors are arbitrary (non-zero) multiples of the vector $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

$\lambda_3=2$: We have to solve $\mathbf{A}\mathbf{v}_3=2\mathbf{v}_3$ or $(\mathbf{A}-2\mathbf{I})\mathbf{v}_3=\mathbf{0}$ for \mathbf{v}_3 .

Store [B]-2identity(3) in [A] and run GAUSS/GAUSSCE. The row-equivalent reduced row-echelon form of $\mathbf{A}-2\mathbf{I}$, with the fourth column of 0s put back in, is

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, $x_3=t$, where t is any number. The second row gives the equation $x_2-3x_3=0$, so that $x_2=3x_3=3t$. The top row gives the equation $x_1-x_3=0$, so that $x_1=x_3=t$.

The eigenvector \mathbf{v}_3 corresponding to $\lambda_3=2$ is therefore

$$\mathbf{v}_3 = \begin{bmatrix} t \\ 3t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

The eigenvectors are arbitrary (non-zero) multiples of the vector $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$.

3. Find, using the calculator (but not EIGENV5/EIGENV5CE), the eigenvectors and eigenvalues of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

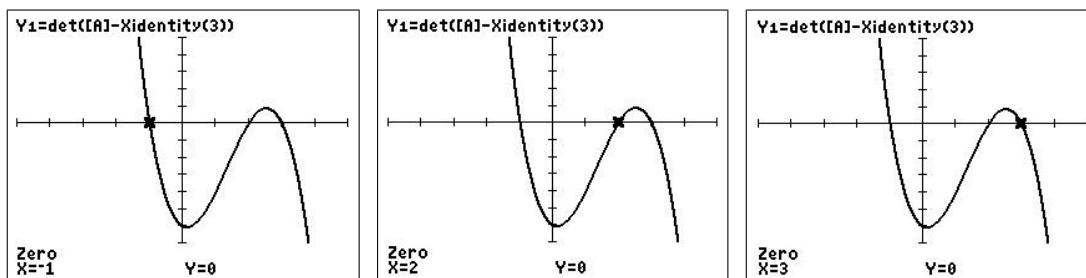
Check with EIGENV5/EIGENV5CE and verify that $\mathbf{A}\mathbf{v}=\lambda\mathbf{v}$ for all the eigenvalues.

Eigenvalues

Put the 3×3 matrix \mathbf{A} into matrix [A] in your calculator.

Set $Y_1 = \det([\mathbf{A}] - X \text{identity}(3))$.

Start with $X_{\min} = -5$ and $X_{\max} = 5$. Press graph to plot the characteristic polynomial, here a cubic.



window $[-5, 5, 1] \times [-7, 5, -1]$

Using *zero* in the calc menu gives the eigenvalues as $\lambda_1 = -1$, $\lambda_2 = 2$ and $\lambda_3 = 3$.

Eigenvectors

$\lambda_1 = -1$: We have to solve $(\mathbf{A} + \mathbf{I})\mathbf{v}_1 = \mathbf{0}$ for \mathbf{v}_1 .

Let $\mathbf{v}_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. We need to find the components x , y and z . Then,

$$(\mathbf{A} + \mathbf{I})\mathbf{v}_1 = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The augmented matrix for this problem is 3×4 , with the 4th column all zeros.

To use the GAUSS/GAUSSCE program, we need to put $\mathbf{A} + \mathbf{I}$ in matrix [A]. Store [A] in [B], then store [B]+identity(3) in [A] and run GAUSS/GAUSSCE.

This gives the reduced row-echelon form of $\mathbf{A} + \mathbf{I}$, with the 4th column of 0s back in, as

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

(note the bottom row of zeros), with corresponding equations

$$\begin{aligned} x + y &= 0 \\ y + 3z &= 0. \end{aligned}$$

Back substitution gives $z = t$, $y = -3t$, $x = 3t$.

Therefore the required eigenvector is $\mathbf{v}_1 = t \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}$, where $t \neq 0 \in \mathbb{R}$.
Check that $\mathbf{A}\mathbf{v}_1 = -\mathbf{v}_1$.

 $\lambda_2 = 2$

Let $\mathbf{v}_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Solve $(\mathbf{A} - 2\mathbf{I})\mathbf{v}_2 = \begin{bmatrix} -1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Gauss elimination

$$\begin{bmatrix} -1 & 2 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

corresponding to equations

$$\begin{aligned} x - 2y &= 0 \\ y &= 0, \end{aligned}$$

with free variable z and solution (back substitution) $z = t$, $y = 0$, $x = 0$, with $t \neq 0 \in \mathbb{R}$.

Therefore, the required eigenvector is $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.
Check that $\mathbf{A}\mathbf{v}_2 = 2\mathbf{v}_2$.

$\lambda_3 = 3$

Let $\mathbf{v}_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Solve $(\mathbf{A} - 3\mathbf{I})\mathbf{v}_3 = \begin{bmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Gauss elimination

$$\begin{bmatrix} -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

corresponding to equations

$$x - y = 0$$

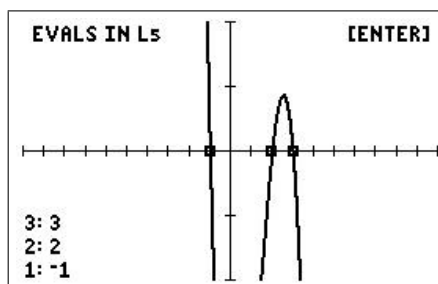
$$y - z = 0,$$

with free variable z and solution (back substitution) $z=t, y=t, x=t$, with $t \neq 0 \in \mathbb{R}$.

Therefore, the required eigenvector is $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Check that $\mathbf{A}\mathbf{v}_3 = 3\mathbf{v}_3$.

Using EIGENV5/EIGENV5CE



EIGENVALUE L5(1)=-1 EIGENVECTOR [B] {3 -3 1}

EIGENVALUE L5(2)=2 EIGENVECTOR [B] {0 0 1}

EIGENVALUE L5(3)=3 EIGENVECTOR [B] {1 1 1}

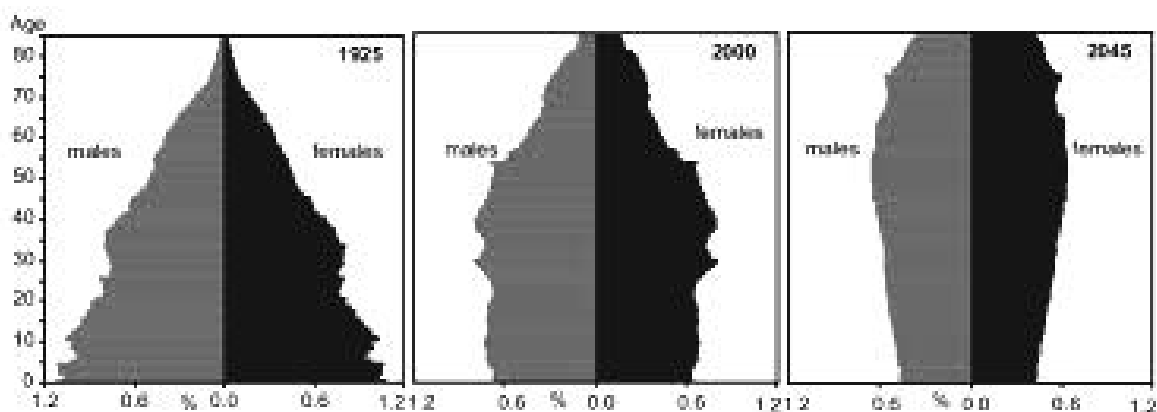
20 Population Modelling 3: Matrix Models

20.1 Population-projection matrices

In 2005, the Australian Government's Productivity Commission released its research report, *Economic Implications of an Aging Australia*³⁴. Planners were concerned that the proportion of older people was increasing, which would eventually put pressure on social services like health care. There would also be fewer people in the working population to pay for the required services and pensions. The distribution of the population with respect to age is changing.

To describe the population, we can divide it into age groups, for example into five-year groups, 0–5 years, 6–10 years, etc. It is also useful to split the population into males and females.

Here are some data taken from the Productivity Report, with the Australian population in one-year age groups and split into males and females.



From pyramid to coffin. Changing age structure of the Australian population, 1925–2045.

Some of the results are measured data (1925 and 2000), while those for 2045 are clearly predictions. The trend is very clear, with a predominance of young people in 1925 (after World War 1) changing to a more even distribution in 2000 up to the age of the ‘baby boomers’ (40–55 years).

We follow this approach in the examples in this section: splitting a population into age classes and seeing how these classes evolve over time.

20.1.1 Leslie matrices

The Leslie matrix is a discrete, age-structured model of population growth that is very popular in population ecology. It was invented by and named after Patrick H. Leslie.

We divide a population into a number of classes — here we shall assume three classes, referring to three age groups, young, adults and seniors, with respective numbers y , a and s . The population is then described by the vector or 3×1 column matrix

$$\mathbf{v} = \begin{bmatrix} y \\ a \\ s \end{bmatrix}.$$

³⁴www.pc.gov.au/inquiries/completed/ageing/report

A 3×3 transition matrix \mathbf{T} tells us how the population evolves. For example, if the population to start with is

$$\mathbf{v}_0 = \begin{bmatrix} y_0 \\ a_0 \\ s_0 \end{bmatrix},$$

after one cycle it is

$$\mathbf{v}_1 = \begin{bmatrix} y_1 \\ a_1 \\ s_1 \end{bmatrix} = \mathbf{T} \begin{bmatrix} y_0 \\ a_0 \\ s_0 \end{bmatrix} = \mathbf{T}\mathbf{v}_0.$$

In problems leading to a Leslie transition matrix:

- in each cycle, members of the other classes produce a certain number of new young in Class 1;
- a certain fraction of each class survives to move into the next class; the rest die;
- all members of the top class die.

This leads to a Leslie matrix \mathbf{T} that is zero everywhere except possibly:

- along the top row after the first element — the birth rates for each class;
- in the elements along the diagonal parallel to and just below the main diagonal — the survival rates for each class.

$$\begin{bmatrix} 0 & * & * & * & * & \cdots & * & * \\ * & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & * & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & * & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & * & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & * & 0 \end{bmatrix}$$

Leslie discovered these matrices in the 1940s when he pioneered this way of exploring how populations can develop. He taught himself matrix algebra while he was in hospital with TB.

Exercise: Leslie matrices and beetles

- (a) During each cycle, each adult beetle produces on average 2.75 young and each senior beetle produces on average 2.5 young; one quarter of the young beetles survive to become adults; and one half of the adult beetles survive to become seniors. In a Leslie-matrix problem, all the seniors die.

Find the Leslie transition matrix \mathbf{T} .

Good strategy: Write out the linear equations for y_1 , a_1 and s_1 in terms of y_0 , a_0 and s_0 , and convert to matrix form.

$$y_1 = ?y_0 + ?a_0 + ?s_0$$

$$a_1 = ?y_0 + ?a_0 + ?s_0$$

$$s_1 = ?y_0 + ?a_0 + ?s_0$$

(b) If we start with 40 young and no adults or seniors, show that after one cycle

$$\mathbf{v}_1 = \begin{bmatrix} y_1 \\ a_1 \\ s_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}.$$

You should have entered \mathbf{T} into [A], the initial \mathbf{v} into [B] and evaluated [A][B].

(c) Multiply repeatedly by \mathbf{T} , and record \mathbf{v} and the total population $P = y + a + s$ after 11, 12 and 13 cycles.³⁵

What happens to the total population? to the ratios of the numbers in the different classes?

Solutions are in Section 20.3.

20.1.2 Populations and oscillations

Workers other than Leslie had independently used matrix algebra in population models. The first was Harro Bernardelli, who published a paper in 1941 in the Journal of the Burma Research Society with the title *Population Waves*. Bernardelli's paper was unusual in focussing not on the eventual stability of the population structure, but on intrinsic oscillations in the population structure. He had observed oscillations in the age structure of the Burmese population between 1901 and 1931.

As an abstract model for such oscillations, he proposed a matrix model for the evolution of the population with

$$\mathbf{T} = \begin{bmatrix} 0 & 0 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix},$$

and showed by numerical calculations that this gave rise to apparently permanent oscillations in the age structure.

Exercise

(a) Record the total population P at each cycle for 12 cycles using Bernardelli's matrix \mathbf{T}

with $\mathbf{v}_0 = \begin{bmatrix} 1 \\ 0.01 \\ 0.01 \end{bmatrix}$, the initial populations in three age groupings in millions.

Do this by hand or using POP/POPCE (Section 20.2).

Plot P versus cycle number, joining up the points with straight lines.

Discuss your findings.

(b) Repeat (a) using $\mathbf{T} = \begin{bmatrix} 0 & 0 & 5 \\ 0.7 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}$.

Describe your results in words. Explain in terms of the entries in \mathbf{T} .

Solutions are in Section 20.3.

³⁵the POP/POPCE program (Section 20.2) helps here

20.1.3 Exercise: Killer whales

Based on a Year-12 project set by Margaret McLaughlan of St Francis Xavier College.

Teachers (and students) may find the programs WHALES/WHALESCE (Section 20.2) useful (but not essential) for experimenting with the population-projection matrix.

Leslie matrices are just a special case of a more general population-projection matrix. In the more general case, animals in a class may remain in that class for more than one cycle. The probability that an animal remains in a particular class for any given cycle is an element on the diagonal of the matrix, immediately above the entry in the Leslie matrix, which gives the probability of moving to the next class in any given cycle.

Therefore we have a matrix whose elements along the top row give the fecundity or birth rate per animal per cycle for each class, whose diagonal elements give the probability of an animal remaining in a particular class in any cycle and whose elements below the diagonal give the probability of an animal moving to the next class in any cycle. The fact that the latter two numbers in any column do not add up to 1 means that some animals in each class die each cycle.

For female killer whales, we have four classes — yearlings (individuals in the first year of life), juveniles (past the first year, but not mature), mature females and post-reproductive females. The mean period in the juvenile stage is 13.4 years and in the mature stage 22.1 years, with an overall lifetime of 80–90 years. Details in Brault and Caswell, *Pod-specific demography of killer whales*, Ecology 74, 1444–1454 (1993).

The population-projection matrix for female killer whales is given below. The time for one cycle is one year.

$$\mathbf{T} = \begin{bmatrix} 0 & 0.0043 & 0.1138 & 0 \\ 0.9775 & 0.9111 & 0 & 0 \\ 0 & 0.0736 & 0.9534 & 0 \\ 0 & 0 & 0.0452 & 0.9804 \end{bmatrix}$$

1. A project you are involved in wants to re-introduce killer whales into an area of ocean from which they have disappeared. The project leader wants to know what is the best combination of juveniles and mature females to re-introduce, assuming an overall total of 50 (plus an appropriate number of adult males). You decide to model three options over 40 cycles (years).
 - (a) 50 female juveniles.
 - (b) 40 female juveniles and 10 mature females.
 - (c) 50 mature females.

What happens to the different population classes of female killer whales over time in each of these options (according to this model)? What is the best strategy for re-establishing the killer-whale population?

Hint: (manual method) If \mathbf{T} is in the 4×4 matrix [A] and the initial population \mathbf{v}_0 in the 4×1 matrix [B], executing the command [A] [B] \rightarrow [B], then repeatedly pressing enter will generate successive population vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, etc.³⁶ The symbol \rightarrow stands for the sto key.

³⁶You could also use the POP/POPCE program (Section 20.2). Run POP/POPCE repeatedly by pressing enter to give the total population directly.

2. Does the trend in the total population depend on which option you choose?
3. The value of 0.1138 in the top row of \mathbf{T} gives the number of live births per mature female per cycle (year). To what value could this birth rate fall before the total population starts to decrease?

This birth rate is clearly important for the overall survival of killer whales.

Hint: See the discussion in Section 20.2 on the eigenvalue λ_p of the matrix \mathbf{T} .

4. How sensitive is the population to the survival rates of yearlings (0.9775), juveniles (0.9111), mature females (0.9534) and post-reproductive females (0.9804)?

You might like to quantify your answers here by determining what percentage decrease in each rate is needed to stop the population growing.

Do your answers make sense?

The WHALES/WHALESCE program (Section 20.2) does these calculations and also plots the populations.

Solution are in Section 20.3.

PTO

20.1.4 Exercise: Age distribution of trees in a forest

The difference-equation version of this problem can be found in *Population Modelling 2* in Volume 2 of this book.

The population of trees in a forest is split into four age groups: b_n is the number of baby trees (0–15 years old) at time-point n ; y_n the number of young trees (16–30 years); m_n middle-aged trees (31–45 years old); and o_n old trees (more than 45 years old).

The time step for our difference equations is 15 years.

In order to simplify the model we make the following assumptions:

- A. a certain percentage of trees in each age group dies in each time interval;
- B. surviving trees age into the next age group each time step; old trees remain old trees (or die);
- C. dead trees are replaced by an equal number of baby trees.

Define $\alpha, \beta, \gamma, \delta$ as the fraction of dead trees in the respective age groups in each time interval. Then, the difference-equation model is

$$b_{n+1} = \alpha b_n + \beta y_n + \gamma m_n + \delta o_n \quad (\text{Assumption C}) \quad (1)$$

$$y_{n+1} = (1-\alpha)b_n \quad (\text{Assumptions A, B}) \quad (2)$$

$$m_{n+1} = (1-\beta)y_n \quad (\text{Assumptions A, B}) \quad (3)$$

$$o_{n+1} = (1-\gamma)m_n + (1-\delta)o_n \quad (\text{Assumptions A, B}). \quad (4)$$

1. If the population of trees in time interval n is $N = b_n + y_n + m_n + o_n$, show that the population stays the same size after one more time step, and so by induction the population of trees is a constant N .
2. Equations (1)–(4) are linear in the dependent variables so that matrices are a way of solving the problem. Define $\mathbf{v}_n = (b_n, y_n, m_n, o_n)^T$ (a column vector) and write down the matrix \mathbf{T} that you would use in the matrix model $\mathbf{v}_{n+1} = \mathbf{T}\mathbf{v}_n$ of these equations.
3. Take $\alpha = 0.2, \beta = 0.5, \gamma = 0.3, \delta = 0.2$ and $N = 1000$. Write down the matrix \mathbf{T} with these values.
4. Four initial conditions are needed in order to fully solve this matrix equation. Assume all baby trees initially.

Run the matrix model in Problems 2 and 3 through 10 cycles (150 years). Alternatively, you could use the program POPMX4/POPMX4CE (Section 20.2) to calculate and plot your results. Do the individual populations appear to be stabilising?

5. Use the EIGENV5/EIGNV5CE program to find the eigenvalues of \mathbf{T} .
6. Find the eigenvector of \mathbf{T} corresponding to $\lambda = 1$ (*explain why we use this value*) and relate this to the behaviour of the model that you found in Problem 4.

Solutions to these problems are in Section 20.3.

20.2 Matrix-modelling programs

POP/POPCE multiplies an $n \times 1$ column vector \mathbf{v} (n age classes) by an $n \times n$ matrix \mathbf{T} (transition matrix), displays the new \mathbf{v} and the sum of the components of \mathbf{v} (total population).

Use: Store \mathbf{T} in matrix [A] and \mathbf{v} in matrix [B]. Run the program for the first step; press `enter` repeatedly for subsequent steps. Press `on` `1` (Quit) to stop the program.

POPMX4/POPMX4CE calculates and plots as a function of cycle number (time) the numbers in *from 2 to 4 age classes* of a population which is modelled using a projection matrix.

WHALES/WHALESCE is a version specifically for the problem on killer whales on page 104. **POPCLEAR/POPCLRCE** deletes all the lists used in WHALES/WHALESCE and resets defaults. Run when you have finished using a program.

POPMXM/POPMXMCE calculates and displays (no plots) as a function of cycle number (time) the numbers *in M age classes* of a population and gives the ratios of the individual populations.

Use: Put your 2×2 , 3×3 , 4×4 or $m \times m$ projection matrix \mathbf{P} in matrix [A] and the initial populations in the different age classes in the 2×1 , 3×1 , 4×1 or $m \times 1$ matrix [B].

Run the relevant program and input the number of cycles for which you wish to calculate the populations. The program will do the calculations.

POPMX4/POPMX4CE pauses in `trace` mode so that you can analyse the graphs. The arrow keys allow you to move along a graph or from graph to graph. Which graph the cursor is on is indicated at the top left of the screen.

Pressing `enter` then gives a plot of the total population versus cycle number. Again you can `trace` this graph. Displayed on this graph is the largest eigenvalue λ_p of the projection matrix. The long-term total population varies as λ_p^n , where n is the cycle number. Therefore, if $\lambda_p > 1$, the population increases; if $\lambda_p = 1$, the population is steady; if $\lambda_p < 1$, the population decreases.

Pressing `enter` again gives you a menu from which you can either quit, keeping the matrices, lists and other settings, or quit, deleting everything. The former allows you to replot data or look at the numbers in `stat` Edit. The latter is a good option when you have finished.

Once POPMX4/POPMX4CE has finished (and you keep all the data), you can replot and `trace` any or all of the graphs of the individual populations versus cycle by selecting the appropriate plot in the `y=` menu. You can change the window either manually in the `window` menu or using `zoom` `9` (ZoomStat). To plot the graph of the fourth population class, you will have to change the Ylist to LCLS4 in one of the plots using `statplot` (`2nd` `y=`).

The data generated in these programs are stored in lists which can be accessed by pressing `stat` Edit. Scrolling across will show the lists not initially on the screen.

20.3 Solutions

Exercise: Leslie matrices and beetles

(a) Writing out the equations for the three beetle age classes,

$$\begin{aligned} y_1 &= 0y_0 + 2.75a_0 + 2.5s_0 \\ a_1 &= 0.25y_0 + 0a_0 + 0s_0 \\ s_1 &= 0y_0 + 0.5a_0 + 0s_0 \end{aligned} \quad \text{or} \quad \begin{bmatrix} y_1 \\ a_1 \\ s_1 \end{bmatrix} = \begin{bmatrix} 0 & 2.75 & 2.5 \\ 0.25 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} y_0 \\ a_0 \\ s_0 \end{bmatrix}.$$

$$\text{Therefore, } \mathbf{T} = \begin{bmatrix} 0 & 2.75 & 2.5 \\ 0.25 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}.$$

$$\text{(b) } \mathbf{v}_1 = \mathbf{T} \begin{bmatrix} 40 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}.$$

(c) **After cycle** **v** **Total pop'n**

1	$\begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}$	10
2	$\begin{bmatrix} 27.5 \\ 0 \\ 5 \end{bmatrix}$	32.5
\vdots	\vdots	\vdots
11	$\begin{bmatrix} 17.320 \\ 4.305 \\ 2.184 \end{bmatrix}$	23.809
12	$\begin{bmatrix} 17.299 \\ 4.330 \\ 2.153 \end{bmatrix}$	23.781
13	$\begin{bmatrix} 17.289 \\ 4.325 \\ 2.165 \end{bmatrix}$	23.779

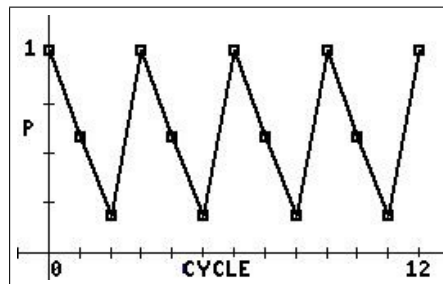
The total population seems to be stabilising at a little under 24, and the ratios of the populations in the 3 classes at about 8:2:1. Divide the first two numbers by the third (smallest) number to see this.

Populations and oscillations

(a) After 1 cycle: $\mathbf{v}_1 = \begin{bmatrix} 0 & 0 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0.01 \\ 0.01 \end{bmatrix} = \begin{bmatrix} 0.08 \\ 0.5 \\ 0.0025 \end{bmatrix}$, so $P=0.5825$.

From successive cycles, we build up a table.

Cycle	0	1	2	3	4	5	6	...
Pop'n	1.02	0.5825	0.185	1.02	0.5825	0.185	1.02	...

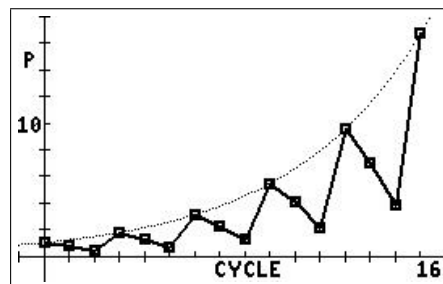


The population is oscillating or going in waves, with no overall growth or decline.

A group of young must first become adults (survival rate 0.5), then seniors (survival rate 0.25) before producing a new group of 8 young; the process then repeats itself. The overall survival rate between young and seniors of $0.5 \times 0.25 = 1/8$ is balanced by a birth rate of 8, so that the overall population is not growing or declining.

- (b) The populations are now 1.02, 0.755, 0.41, 1.785, 1.321, 0.718, 3.124, 2.312, 1.256, 5.467, 4.046, 2.197, 9.566, 7.081, 3.846, 16,741, ...

The population is oscillating, but growing overall. The birth rate of 5 and the survival rate of $0.7 \times 0.5 = 0.35$ gives an overall growth rate of 1.75 (dashed line).

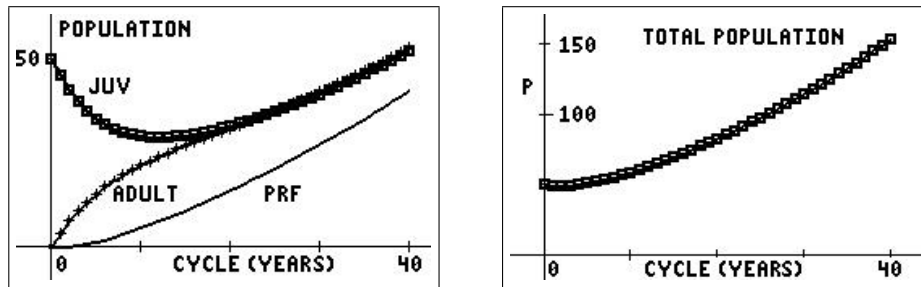


Exercise: Killer whales

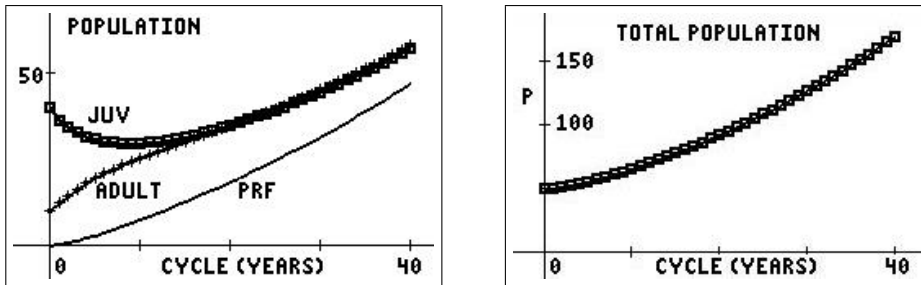
1. A project you are involved in wants to re-introduce killer whales into an area of ocean from which they have disappeared. The project leader wants to know what is the best combination of juveniles and mature females to re-introduce, assuming an overall total of 50 (plus an appropriate number of adult males).

You model three options, with the resulting plots shown below. The left-hand plots show the populations of juvenile females (squares), mature females (crosses) and post-reproductive females (PRF: thin line); the right-hand plots show the total population (including yearlings), both as functions of the number of cycles (1 cycle = 1 year).

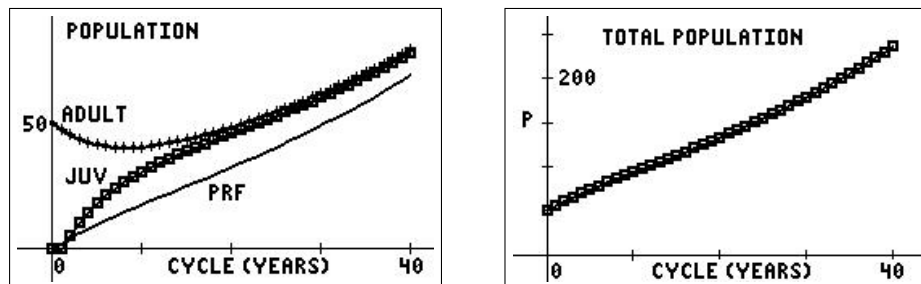
(a) 50 juveniles.



(b) 40 juveniles and 10 mature females.



(c) 50 adult females.



PTO

What happens to the different population classes of female killer whales over time in each of these options (according to this model)? What is the best strategy for re-establishing the killer-whale population?

After about 15 years, in all three options, there are approximately equal numbers of juvenile and mature females, and this remains the case: both populations are increasing. The greater the initial number of mature females, the greater the numbers of both juvenile and mature females at any given cycle. The number of post-reproductive females follows a similar trend in all three options.³⁷

In terms of the best strategy for re-introducing killer whales, it probably comes down to cost. Clearly the best option, according to the model, is to introduce all mature females but it could take quite a long time (and be expensive) to breed up a sufficient number, given it takes at least 14 years to maturity. Releasing all juveniles, a much cheaper option, runs the risk that not enough of the inexperienced juveniles survive long enough to establish a sustainable population. Some combination of juvenile and mature females may be the best option.

2. Does the trend in the total population depend on which option you choose?

In the three options with different initial populations, the trend is the same: the overall female population increases. However, the greater the initial number of mature females, the greater the total population at any given cycle. The fact of an increase in total population (and the eventual ratios of the populations in the four classes) does not depend on the initial populations in each of the classes.

3. The value of 0.1138 in the top row of \mathbf{T} gives the number of live births per mature female per cycle (year). To what value could this birth rate fall before the total population starts to decrease?

This birth rate is clearly important for the overall survival of killer whales.

Experimenting with different values of the birth rate for mature females, either manually or using POPMT4/POPMT4CE, shows that the population becomes steady when the value is about 0.055 ($\lambda_p \approx 1$), i.e. about 48% of the observed value. For birth rates less than 0.055, the total population will decline.

4. How sensitive is the population to the survival rates of yearlings (0.9775), juveniles (0.9111), mature females (0.9534) and post-reproductive females (0.9804)?

The female population is more sensitive to the survival rates than to the birth rates, especially that of the mature females. Reducing the survival rate of 0.9534 for mature females to 0.905 (a reduction of only 5%) is enough to stop the population growing.

For the other classes, the corresponding values are: yearlings 0.9775 down to 0.49 (50%); juveniles 0.9111 down to 0.82 (10%). The growth or otherwise of the population is unaffected by the survival rate of post-reproductive females in this model.

³⁷The ratios of the populations in the different classes are those of the components of the eigenvector of the population-projection matrix corresponding to λ_p .

Exercise: Age distribution of trees in a forest

1. If the population of trees in time interval n is $N = b_n + y_n + m_n + o_n$, show that the population stays the same size after one more time step, and so by induction the population of trees is a constant N .

Adding the four equations gives

$$\begin{aligned} b_{n+1} + y_{n+1} + m_{n+1} + o_{n+1} &= \alpha b_n + \beta y_n + \gamma m_n + \delta o_n + (1-\alpha)b_n \\ &\quad + (1-\beta)y_n + (1-\gamma)m_n + (1-\delta)o_n \\ &= b_n + y_n + m_n + o_n \\ &= N. \end{aligned}$$

Therefore, the population of trees stays the same size after one more time step, and so by induction the population of trees is a constant N .

2. Equations (1)–(4) are linear in the dependent variables so that matrices are a way of solving the problem. Define $\mathbf{v}_n = (b_n, y_n, m_n, o_n)^T$ (a column vector) and write down the matrix \mathbf{T} that you would use in the matrix model $\mathbf{v}_{n+1} = \mathbf{T}\mathbf{v}_n$ of these equations.

Equations (1)–(4) can be written in matrix form as

$$\mathbf{v}_{n+1} = \begin{bmatrix} b_{n+1} \\ y_{n+1} \\ m_{n+1} \\ o_{n+1} \end{bmatrix} = \begin{bmatrix} \alpha & \beta & \gamma & \delta \\ 1-\alpha & 0 & 0 & 0 \\ 0 & 1-\beta & 0 & 0 \\ 0 & 0 & 1-\gamma & 1-\delta \end{bmatrix} \begin{bmatrix} b_n \\ y_n \\ m_n \\ o_n \end{bmatrix},$$

so that

$$\mathbf{T} = \begin{bmatrix} \alpha & \beta & \gamma & \delta \\ 1-\alpha & 0 & 0 & 0 \\ 0 & 1-\beta & 0 & 0 \\ 0 & 0 & 1-\gamma & 1-\delta \end{bmatrix}.$$

3. Take $\alpha = 0.2$, $\beta = 0.5$, $\gamma = 0.3$, $\delta = 0.2$ and $N = 1000$. Write down the matrix \mathbf{T} with these values.

$$\mathbf{T} = \begin{bmatrix} 0.2 & 0.5 & 0.3 & 0.2 \\ 0.8 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.7 & 0.8 \end{bmatrix}.$$

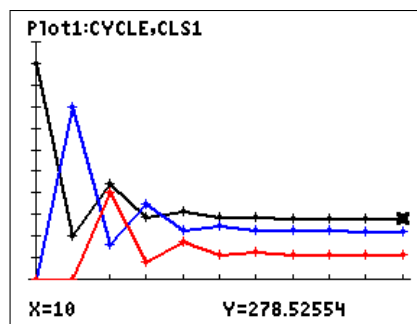
4. Four initial conditions are needed in order to fully solve this matrix equation. Assume all baby trees initially.

Run the matrix model in Problems 2 and 3 through 10 cycles (150 years). Do the individual populations appear to be stabilising?

Run the model $\mathbf{v}_{n+1} = \mathbf{T}\mathbf{v}_n$ through 10 cycles with $\mathbf{v}_0 = \begin{bmatrix} 1000 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, that is all baby

trees initially: with \mathbf{T} in matrix [A] and \mathbf{v}_0 in matrix [B], execute the command [A][B] → [B], then press nine times.

Alternatively, with \mathbf{T} in matrix [A] and \mathbf{v}_0 in matrix [B], run the POPMX4/POPMX4CE program. This produces the following graph of the first three populations versus cycle number. The window is $[-0.2, 10.2, 1] \times [0, 1100, 100]$.



Here, the graph of b_n is in black, y_n in blue and m_n in red.

The number in each population appears to be stabilising: b_n at around 279; y_n at around 223; and m_n around 112. The corresponding value of o_n is 386.

5. Use the EIGENV5/EIGNV5CE program to find the eigenvalues of \mathbf{T} .

The eigenvalues are -0.358 , -0.2 , 0.558 and 1 .

6. Find the eigenvector of \mathbf{T} corresponding to $\lambda = 1$ (*explain why we use this value*) and relate this to the behaviour of the model that you found in Problem 4.

An eigenvector of 1 means the total population does not change, the case here. The corresponding eigenvector from EIGENV5/EIGNV5CE, normalised to a maximum component of 1 , is

$$\begin{bmatrix} 0.7143 \\ 0.5714 \\ 0.2857 \\ 1 \end{bmatrix},$$

giving a total population of 2.5714 .

To relate this to the problem here, we multiply the eigenvector by $1000/2.5714$ to give a total population of 1000 . The new eigenvector, giving the four populations after a large (actually infinite) number of cycles, is (values rounded to integers)

$$\begin{bmatrix} 278 \\ 222 \\ 111 \\ 389 \end{bmatrix},$$

in good agreement with our results after 10 cycles in Problem 4.

21 Fitting Curves to Data 2

21.1 Introduction

Much scientific and other research involves data. Fitting a function to the data is a way of summarising the data; if the fit is good, the fitted curve can be used instead of the data in further calculations, especially useful if Calculus is involved. Sometimes the function chosen is guided by the theory involved, for example motion under gravity. At other times, the choice is empirical: the function which gives the best fit is used.

The material here is especially relevant to the second case. A range of functions is given, together with an example of each in fitting a given dataset. The functions available on the TI-84/CE to fit data are dealt with in *Fitting Curves to Data 1* in the first volume of this book. Here, we look at fitting polynomials to data, then work through the functions available using the CURVEFIT/CRVFITCE program³⁸. In most cases, an indication of the goodness of the fit is given, usually the correlation coefficient r and r^2 or the coefficient of determination R^2 . Finally, we look at time averaging of data.

21.2 Exact fitting of polynomials to data

One very commonly used class of functions used to fit data is polynomials. Examples have already been encountered in *Fitting Curves to Data 1*; the fits there are exact when the number of points fitted is the same as the number of coefficients in the equation to be fitted. If there are more points than coefficients, the curve of best fit (regression) is obtained.

A general polynomial of degree m is of the form

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots + a_mx^m. \quad (1)$$

The data points to be fitted are then used to find the coefficients $a_0, a_1, a_2, \dots, a_m$, using the fact that the x and y values of each point must satisfy the polynomial. A polynomial of degree m has $m+1$ coefficients, and therefore will fit $m+1$ data points exactly. This is what we do here. The method applies to polynomials of any degree.

Note: If critical points ($y'=0$) are given, differentiate Eq. (1) and substitute the coordinates of the point into the derivative (a polynomial). Similarly with points of inflection ($y''=0$).

21.2.1 Manual method

Find a polynomial passing through the 10 points shown in the screen below.

X	Y
0	1
1	2
2	6
3	5
4	6
5	3
6	3
7	8
8	9
9	8

There are 10 data points, so we can fit a unique 9th-degree polynomial

$$y = a_0 + a_1x + a_2x^2 + \dots + a_9x^9, \quad (2)$$

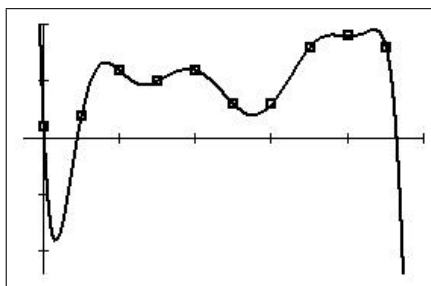
which has the 10 unknowns $a_0, a_1, a_2, \dots, a_9$.

³⁸available at www.XXX

Substituting the x and y values of each data point into Eq. (2) gives 10 linear equations:

$$\begin{aligned} a_0 &= 1 \\ a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 &= 2 \\ a_0 + 2a_1x + 4a_2 + 8a_3 + 16a_4 + 32a_5 + 64a_6 + 128a_7 + 256a_8 + 512a_9 &= 6 \\ a_0 + 3a_1 + 9a_2 + 27a_3 + 81a_4 + 243a_5 + a_63^6 + a_73^7 + a_83^8 + a_93^9 &= 5 \\ a_0 + 4a_1 + 16a_2 + 64a_3 + 256a_4 + a_54^5 + a_64^6 + a_74^7 + a_84^8 + a_94^9 &= 6 \\ a_0 + 5a_1 + 25a_2 + 125a_3 + 625a_4 + a_55^5 + a_65^6 + a_75^7 + a_85^8 + a_95^9 &= 3 \\ a_0 + 6a_1 + 36a_2 + 216a_3 + a_46^4 + a_56^5 + a_66^6 + a_76^7 + a_86^8 + a_96^9 &= 3 \\ a_0 + 7a_1 + 49a_2 + 343a_3 + a_47^4 + a_57^5 + a_67^6 + a_77^7 + a_87^8 + a_97^9 &= 8 \\ a_0 + 8a_1 + 64a_2 + 512a_3 + a_48^4 + a_58^5 + a_68^6 + a_78^7 + a_88^8 + a_98^9 &= 9 \\ a_0 + 9a_1 + 81a_2 + 729a_3 + a_49^4 + a_59^5 + a_69^6 + a_79^7 + a_89^8 + a_99^9 &= 8 \end{aligned}$$

Solving these equations³⁹ gives (to 4 significant digits) $a_0 = 1$, $a_1 = -76.32$, $a_2 = 187.4$, $a_3 = -175.0$, $a_4 = 85.45$, $a_5 = -24.25$, $a_6 = 4.138$, $a_7 = -0.4183$, $a_8 = 0.02307$ and $a_9 = -5.346 \times 10^{-4}$.



window $[-0.5, 10, 2] \times [-12, 10, 5]$

Once we find a formula for the polynomial, we can use it to give y values for x values between the data points — this is called *interpolation*.

We can also use the formula to find y values when x is outside the set of x values for which data are available. This is called *extrapolation*, and can be dangerous, as the curve might in practice behave quite differently beyond the measured data points, as it does here.

21.2.2 Using a program

This process is automated in the polynomial-fit program POLYFIT/POLYFTCE. This fits a polynomial of degree N to $N+1$ points, then plots the polynomial, displays the coefficients and calculates points on the polynomial. The polynomial is stored in Y_7 .

PTO

³⁹by Gauss Elimination or the inverse-matrix method— see *Matrix and Vector Operations*, Chapter 19

Use

Run the program. Select INPUT POINTS and enter the points on the prompts. The points are copied to lists LX and LY.⁴⁰ Press CALCULATE FIT to do the calculations.

```

POLYNOMIAL FIT
1:INPUT POINTS
2:CALCULATE FIT
3:GRAPH POLYNOMIAL
4:CHANGE WINDOW
5:DISPLAY COEFFICIENTS
6:CALCULATE POINT
7:QUIT

```

Select GRAPH to graph the polynomial and the points. CHANGE WINDOW allows you to change the X range for the graph, with the calculator automatically choosing the Y range. DISPLAY COEFFICIENTS displays the coefficients of the polynomial in ascending powers of x . CALCULATE POINT allows you to calculate a point on the polynomial. QUIT does just that: not deleting the data leaves everything as it is; deleting the data removes everything, but you can still plot the polynomial in Y7.⁴¹

Try it out on the 10 points above. You'll find that the polynomial is not defined at $x = 0$ because the calculator cannot calculate 0^0 in Y7. The curve disappears near this point but clearly passes through the point $(0, 1)$ if you expand the window out in both directions.

Exercise: Fit a polynomial to the points $(0, 1)$, $(1, 6)$, $(-1, 0)$, $(2, 63)$, $(-2, -21)$ and $(-3, -182)$.

Answer: $y = x^5 + x^4 + x^3 + x^2 + x + 1$.

PTO

⁴⁰If you need to change a point, Quit out of the program without deleting the data. Press stat Edit, change the point(s), then rerun the program and recalculate if necessary.

⁴¹the coefficient list LCOEF is not deleted, as it is used by Y7

21.3 Using the CURVEFIT/CRVFITCE program

Reference: *Curve Fitting for Programmable Calculators* by William M. Colb, Syntec, 1984.

The CURVEFIT/CRVFITCE program⁴² fits 19 different functions (curves) to the data in lists L1 (x) and L2 (y), and displays the correlation coefficient (coefficient of determination) R^2 and conditional correlation coefficient⁴³ R_c^2 (CE only) for the fit. The screens below show use of the program up to the selection of the function.

The second option in the main menu, FIND BEST FIT, fits all 19 functions to the data, then tells you the function of best fit, the one with the largest value of R^2 (Section 21.3.20).

MULTIPLE CURVE FIT

DATA IN L1,L2

CURVES 9-19 USE LOGS:
NEGATIVE DATA CANNOT BE
USED.

[ENTER]

MAIN MENU

1:SELECT A CURVE
2:FIND BEST FIT
3:QUIT

CURVES #1

1:Y=A+BX
2:Y=1/(A+BX)
3:Y=A+BX+C/X
4:Y=A+B/X
5:Y=X/(AX+B)
6:Y=A+B/X+C/X²
7:Y=A+BX+CX²
8:Y=1/(A(X+B)+C)
9:MORE...

CURVES #2 (ADD 8)

1:Y=A+Bln(X)
2:Y=1/(A+Bln(X))
3:Y=AX^B
4:Y=AX^(BX)
5:Y=AX^(B/X)
6:Y=AB^X*X^C
7:Y=AB^(1/X)X^C
8:Y=Ae^{((B-ln(X))²/C)}
9:MORE...

CURVES #3 (ADD 16)

1:Y=AB^X
2:Y=AB^(1/X)
3:Y=Ae^{((X-B)²/C)}
4:BACK TO START
5:MAIN MENU
6:QUIT

The origin and application of some of these functions are often somewhat esoteric but all can be used to fit data. Which one to use depends on the characteristics of your data (e.g. maxima and/or minima) and the behaviour of the different functions. Clearly, there's no point in fitting a quadratic function to data with an obvious maximum and minimum. Ultimately, you can use the correlation coefficient (coefficient of determination) R^2 to give you some indication, and the best-fit option to pick out the mathematically best fit.

We use the data shown below⁴⁴ for all the fits. The window for all the plots showing the function fitted to the data is $[0, 10, 2] \times [0, 10, 2]$. Also given in many cases is a plot showing the general shape of the function when this is not obvious from the first plot.

L1	L2
1	2
2	6
4	5
4	6
4	3
6	3
6	8
7	9
7	8
9	4

⁴²Both programs and their three subprograms, MCFSUB0, MCFSUB1 and MCFSUB2, are available individually or in the respective group files, CURVECE.8xg and CURVE84.8xg, available at *www.XXX*.

⁴³I don't understand what this is, even after looking it up. If you think it is important for your application, you probably already know much more than I do. Otherwise, look it up or ask a statistician.

⁴⁴the data used in *Fitting Curves to Data 1* in Volume 1 of this book

21.3.1 Linear: $Y=A+BX$

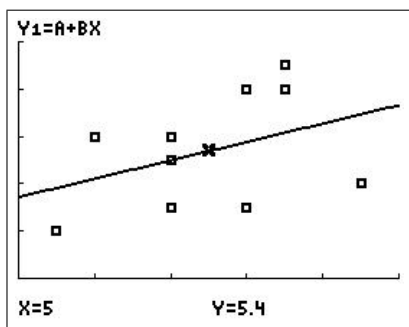
Same as $LinReg(ax+b)$ in *Fitting Curves to Data 1* in Volume 1 of this book. See the discussion there.

```

1. Y=A+BX
   LINEAR

A=3.455555556
B=0.388888889

R2=0.1559
R2c=0.0503
    
```



21.3.2 Reciprocal: $Y=1/(A+BX)$

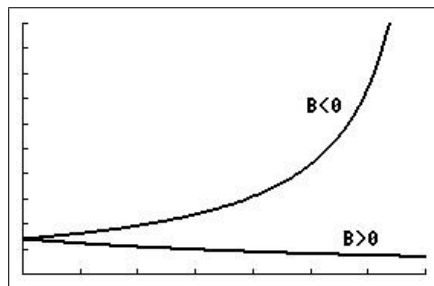
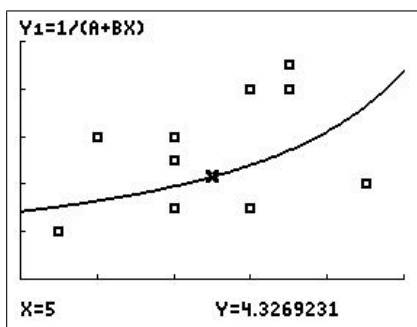
The reciprocal function is one of the simplest functions having an infinite discontinuity (when $A+BX=0$), not seen in the graphs here. Such discontinuities in modelling usually represent a system moving beyond physical bounds (and usually self-destructing). Resonance in a system is one such situation.

```

2. Y=1/(A+BX)
   RECIPROCAL

A=0.3486522634
B=-0.0235082305

R2=0.2135
R2c=0.1152
    
```



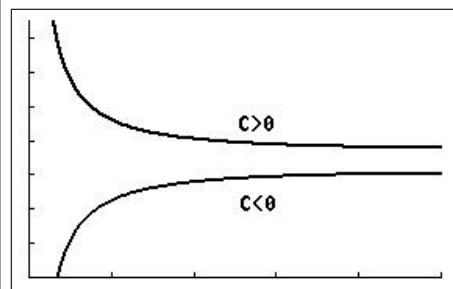
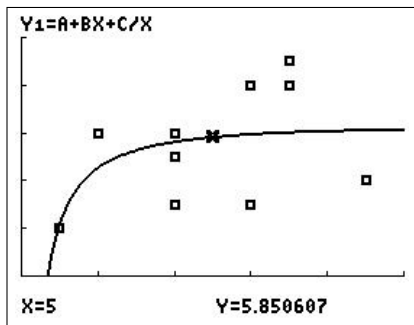
21.3.3 Linear-hyperbolic: $Y=A+BX+C/X$

This function becomes linear for large enough x , and hence tends to $\pm\infty$, but starts with an infinite discontinuity at $x=0$. With appropriate choices for A and B , the curve can have a local maximum or minimum, or can be monotonic.

```

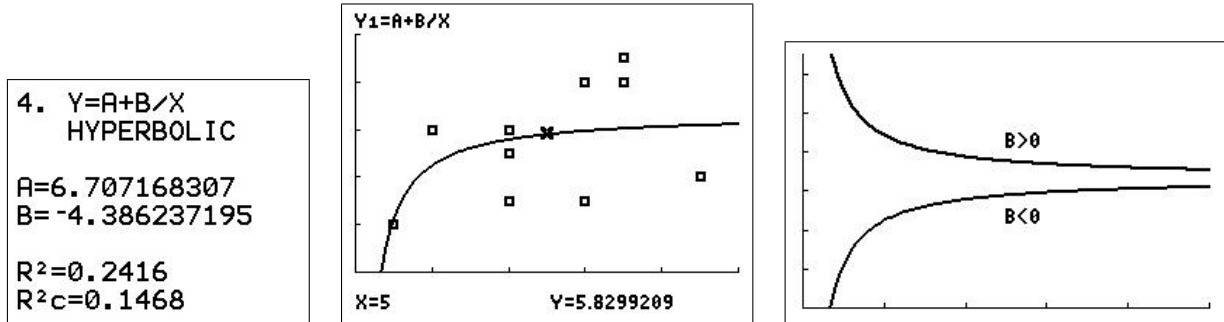
3. Y=A+BX+C/X
   LINEAR-HYPERBOLIC

A=6.911763162
B=-0.0283398015
C=-4.597285845
R2=0.2419
R2c=0.0253
    
```



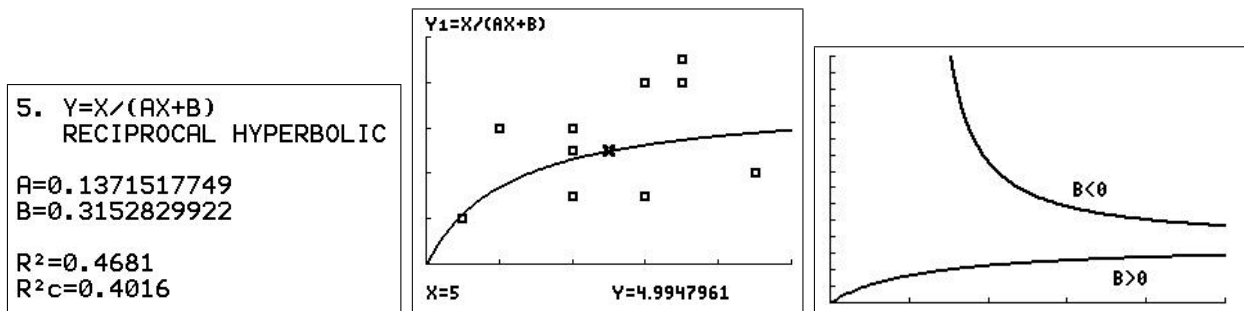
21.3.4 Hyperbolic: $Y=A+B/X$

Has an infinite discontinuity at $x=0$, then tends asymptotically to the constant value A .



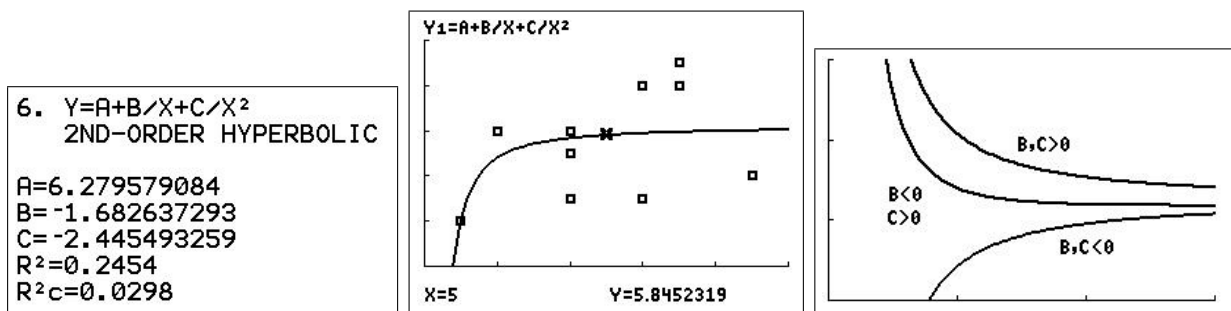
21.3.5 Reciprocal hyperbolic: $Y=X/(AX+B)$

Is zero at $X=0$ (unless $B=0$) and has an infinite discontinuity when $A+BX=0$.
The right-hand graph shows its behaviour when $A > 0$.



21.3.6 Second-order hyperbolic: $Y=A+B/X+C/X^2$

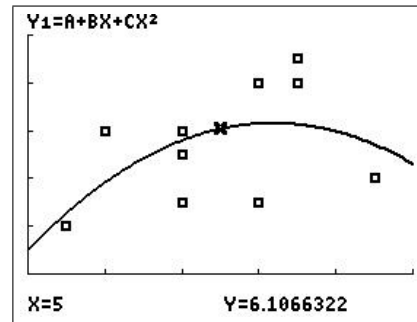
Similar behaviour to the hyperbolic function (Section 21.3.4).



21.3.7 Parabolic: $Y=A+BX+CX^2$

Same as *QuadReg* in *Fitting Curves to Data 1* in Volume 1 of this book. See the discussion there.

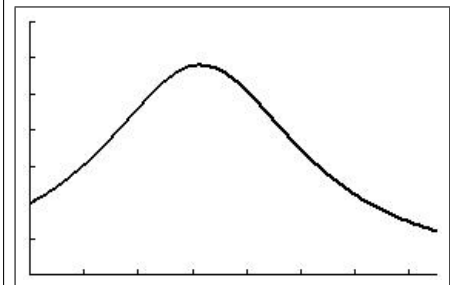
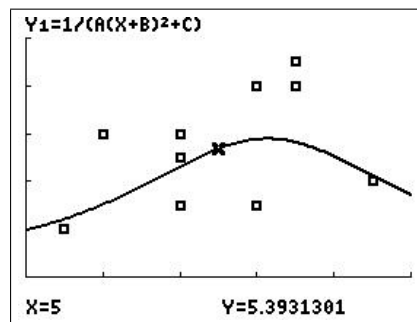
7. $Y=A+BX+CX^2$
PARABOLIC
A=1.036139793
B=1.668387609
C=-0.1308578237
 $R^2=0.2656$
 $R^2c=0.0557$



21.3.8 Cauchy Distribution: $Y=1/(A(X+B)^2+C)$

A well-known continuous probability distribution, also known to physicists as the Lorentz distribution. Can have an infinite discontinuity for positive X if A and C have opposite signs.

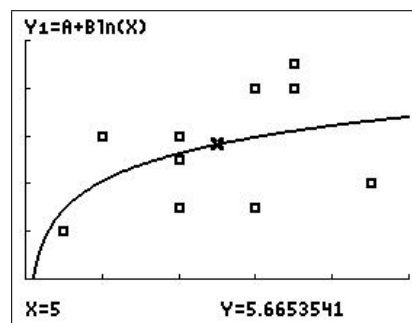
8. $Y=1/(A(X+B)^2+C)$
CAUCHY DISTRIBUTION
A=0.008461117
B=-6.278080793
C=0.171599925
 $R^2=0.3855$
 $R^2c=0.21$



21.3.9 Logarithmic: $Y=A+B\ln(X)$

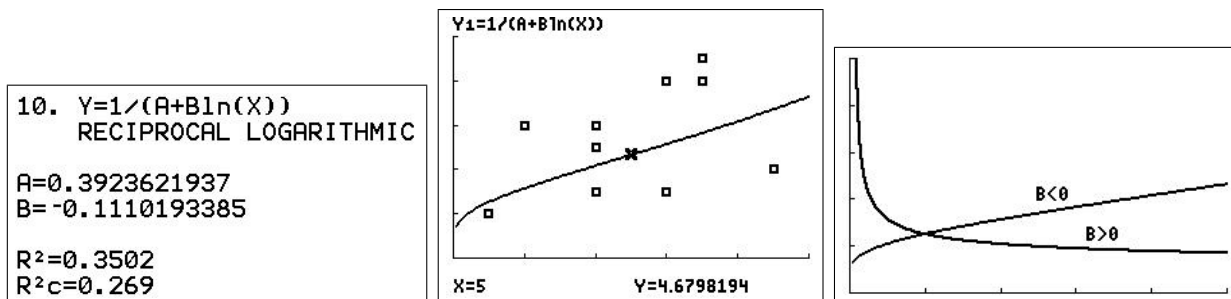
Same as *LnReg* in *Fitting Curves to Data 1* in Volume 1 of this book. See the discussion there.

9. $Y=A+B\ln(X)$
LOGARITHMIC
A=2.94478461
B=1.690384858
 $R^2=0.2166$
 $R^2c=0.1186$



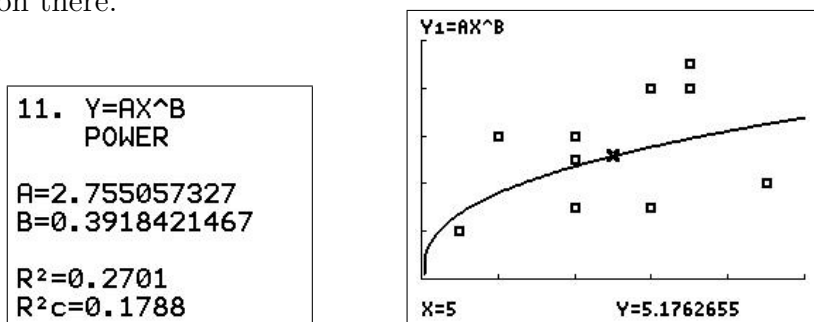
21.3.10 Reciprocal logarithmic: $Y=1/(A+B\ln(X))$

This function is only defined for $X > 0$ because of the $\ln(X)$. $\ln(X)$ tends to $-\infty$ as $x \rightarrow 0$ and ∞ as $x \rightarrow \infty$, so eventually the function tends to 0 in both limits. There is also an infinite discontinuity when $A+B\ln(X)=0$: if A and B have the same sign, this will occur when $0 < X < 1$ ($\ln(X) < 0$); otherwise at some $X > 1$ ($X \approx 34$ here).



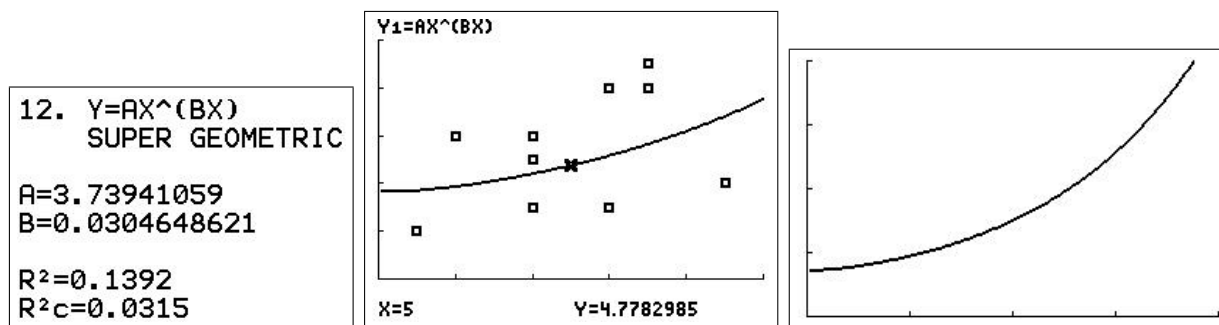
21.3.11 Power: $Y=AX^B$

Same as *PwrReg* in *Fitting Curves to Data 1* in Volume 1 of this book. See the discussion there.



21.3.12 Super exponential: $Y=AX^{(BX)}$

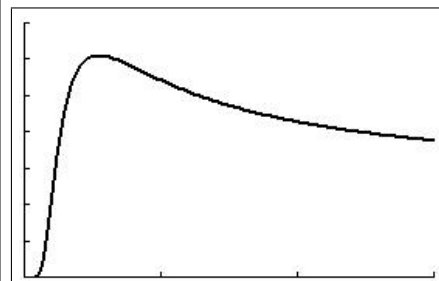
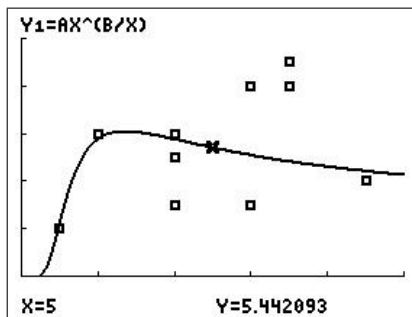
Called Super Geometric in the original publication but has no connection to the Geometric Distribution. Defined for $X > 0$, tends to 0 or ∞ as $X \rightarrow \infty$, depending on the sign of B .



21.3.13 Modified exponential: $Y=AX^{(B/X)}$

Called Modified Geometric in the original publication but also has no connection to the Geometric Distribution. Defined for $X > 0$, tends to A as $X \rightarrow \infty$. Has a local maximum if A and B are positive, a local minimum if A is negative, B positive. Is monotonic if B is negative.

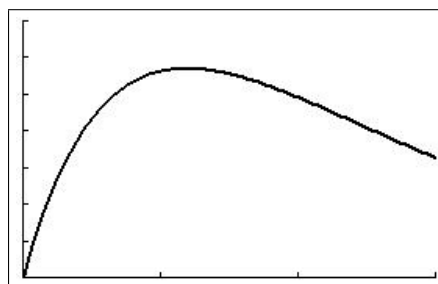
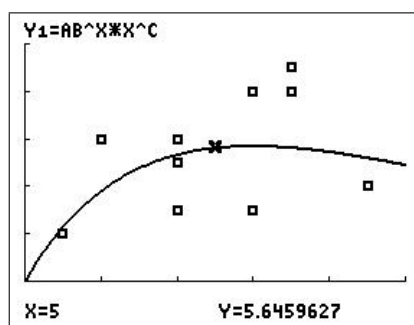
13. $Y=AX^{(B/X)}$
 MODIFIED GEOMETRIC
 $A=2.384139076$
 $B=2.564018285$
 $R^2=0.2856$
 $R^2c=0.1963$



21.3.14 Hoerl Function: $Y=AB^X * X^C$

The exponential dominates for large enough X , so the function tends asymptotically to 0 if $|B| < 1$, with a local maximum or minimum if C is positive. If $|B| > 1$, the function tends to $\pm\infty$.

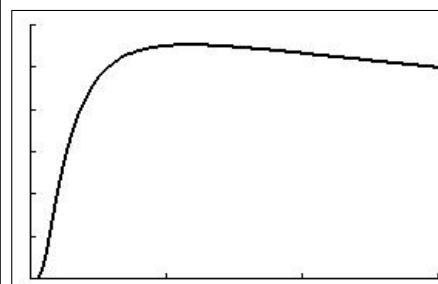
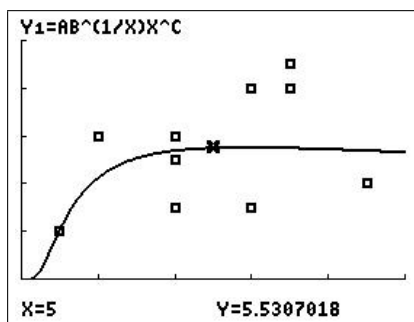
14. $Y=AB^X * X^C$
 HOERL FUNCTION
 $A=2.716042844$
 $B=0.8539540148$
 $C=0.9451463763$
 $R^2=0.3277$
 $R^2c=0.1356$



21.3.15 Modified Hoerl: $Y=AB^{(1/X)} * X^C$

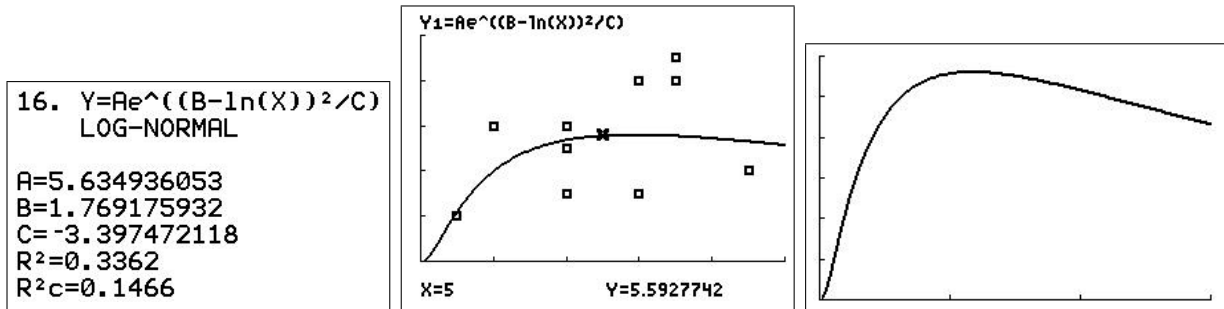
The exponential dominates for large enough X , so the behaviour of the function then depends on whether $|B| < 1$ (tends to 0) or $|B| > 1$ (tends to $\pm\infty$).

15. $Y=AB^{(1/X)} * X^C$
 MODIFIED HOERL
 $A=12.94248275$
 $B=0.1665336571$
 $C=-0.3055034655$
 $R^2=0.3513$
 $R^2c=0.1659$



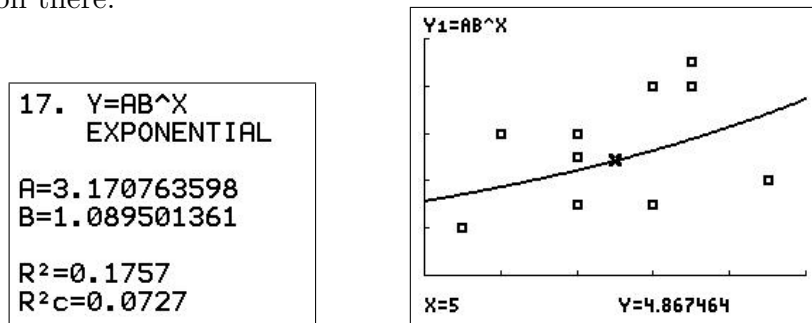
21.3.16 Log-Normal: $Y=Ae^{((B-\ln(X))^2/C)}$

A well-known continuous probability distribution, defined for $X > 0$.



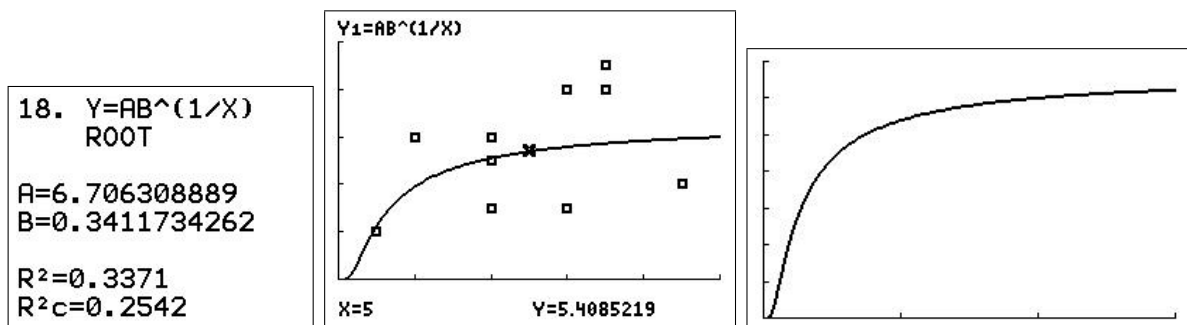
21.3.17 Exponential: $Y=AB^X$

Same as *ExpReg* in *Fitting Curves to Data 1* in Volume 1 of this book. See the discussion there.



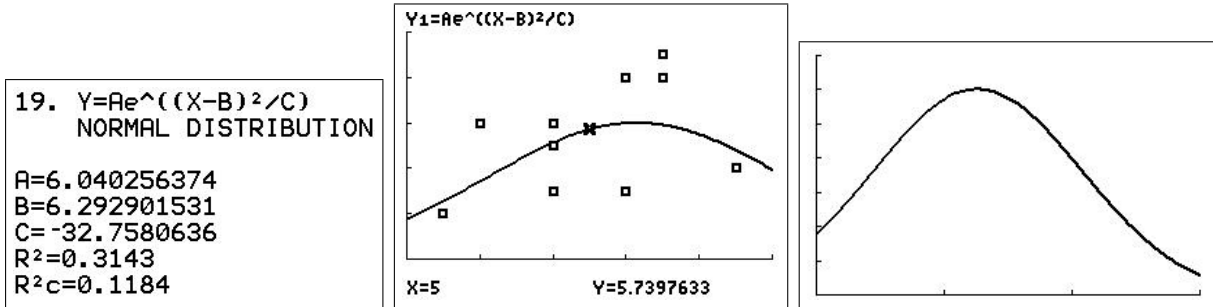
21.3.18 Root: $Y=AB^{(1/X)}$

Similar to the logarithmic distribution but starts at the origin.

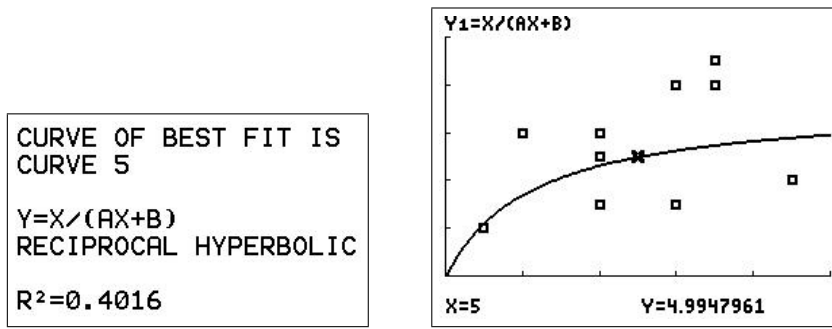


21.3.19 Normal Distribution: $Y=Ae^{((X-B)^2/C)}$

A very well-known continuous probability distribution with a bell-shaped graph.



21.3.20 Best fit



21.4 Time averaging of data

Some data can vary quite rapidly over short time scales such as the numbers of daily confirmed Covid cases in Sydney over 40 days, starting on March 1, 2020, in the table below.⁴⁵ It can be difficult to discern trends in such data, so often the data are averaged over several days, moving-average data, to smooth out the bumps.

Day	Cases	Day	Cases
1	2	21	119
2	3	22	121
3	7	23	176
4	6	24	192
5	4	25	205
6	7	26	185
7	2	27	312
8	7	28	142
9	13	29	124
10	6	30	90
11	16	31	131
12	14	32	125
13	19	33	92
14	20	34	106
15	38	35	73
16	29	36	63
17	64	37	40
18	45	38	44
19	56	39	44
20	88	40	41

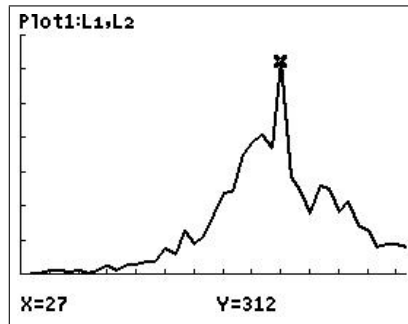
The MOVEAVGE/MVEAVGCE program calculates and plots the data in lists L₁ and L₂ averaged over a period of C days, where you specify C; the first two screens of the program are shown below.

<p>MOVING AVERAGE</p> <p>ASSUMES...</p> <p>TIME IN L₁ DATA IN L₂</p> <p>[ENTER]</p>	<p>AVERAGE OVER A CYCLE OF C PERIODS</p> <p>C=?7</p>
--	--

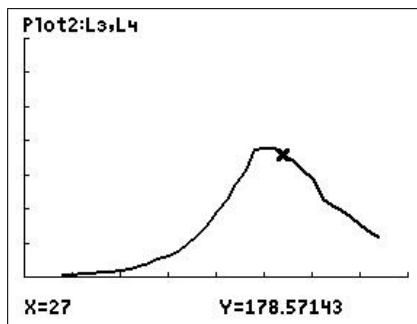
PTO

⁴⁵https://data.nsw.gov.au/data/dataset/aefcde60-3b0c-4bc0-9af-f52778048b29/download/confirmed_cases_table1_location.csv

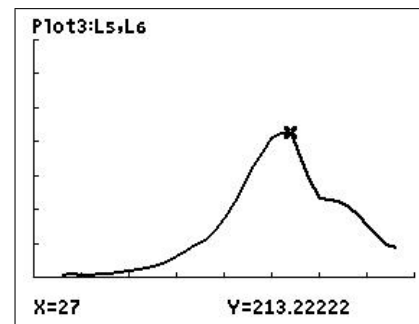
For example, if $C = 7$, the first data point in the average is at $t = 4$; the corresponding number of Covid cases is the average of the first 7 values: the three before $t = 4$; the value at $t = 4$; and the three after $t = 4$, and so on.



raw data: window $[0, 40, 5] \times [0, 350, 50]$

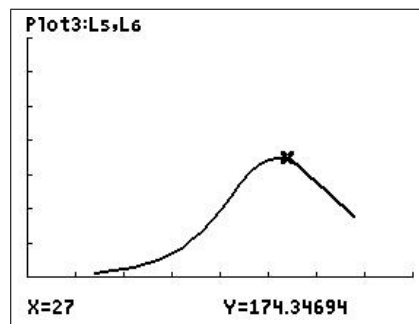


7-day (single) averaged data



3-day (single) averaged data

The program also calculates a double average, the average of the single-average values, for even greater smoothing.



7-day double-averaged data

22 Financial Mathematics 2: TVM Calculations

22.1 Introduction

Financial Mathematics: Part 1 dealt with the basic calculations of compound interest, including the use of sequences on the calculator. Here, to be able to do some useful financial modelling, such as comparing loans, etc, we use the TVM (Time Value of Money) Solver.

The TVM notes are closely based on Chapter 11 of *Mathematics with a Graphics Calculator: Casio cfx-9850G PLUS* by Barry Kissane. This book is a real bible on everything a graphics calculator can do and how to do it on a Casio graphics calculator. Still very relevant but sadly now hard to find.

Texas Instruments also has a book, *Time, Value, Money: Applications on the TI-83*, which covers all the financial calculations you are ever going to want to do, and then more. If you are using calculators from the TI-84 family, this is an invaluable resource.

Section 22.2 here covers calculations which involve compound interest, such as savings with or without regular payments, loans, discounting, effective rate of interest, annuities, sinking funds, leasing and bonds. Other calculations, such as days and dates, cost price/selling price/margins and cash-flow analysis, you will find in the TI book.

The good news is that all the compound-interest calculations are essentially the same: you specify the values of all the variables except one, then find the one you don't know. This is much simplified by using the TVM Solver on the calculator.

There are plenty of TVM exercises and activities with solutions in Sections 22.4 and 22.5.

References

Mathematics with a Graphics Calculator: Casio cfx-9850G PLUS by Barry Kissane, The Mathematical Association of Western Australia, 2003, ISBN 1 876583 24 X.

Time, Value, Money: Applications on the TI-83 by Roseanne and Charles Hoffman, Texas Instruments, 1997, ISBN 1 886309 09 4.

PTO

22.2 Using the TVM Solver

The TVM Solver solves the same basic equations as those encountered in *Financial Mathematics 1* in Volume 1, with the addition of regular payments or repayments; it just makes solving the equations easy.

22.2.1 Selecting the TVM Solver

First press `mode` and select 2 (third line; use the arrow keys and `enter`) to set 2 decimal places for answers in dollar and cents.

```

MATHPRINT CLASSIC
NORMAL SCI ENG
FLOAT 0 1 2 3 4 5 6 7 8 9
RADIAN DEGREE
FUNCTION PARAMETRIC POLAR SEQ
THICK DOT-THICK THIN DOT-THIN
SEQUENTIAL SIMUL
REAL a+bi re^(θi)
FULL HORIZONTAL GRAPH-TABLE
FRACTIONTYPE: n/d Un/d
ANSWERS: AUTO DEC
STAT DIAGNOSTICS: OFF ON
STAT WIZARDS: ON OFF
SET CLOCK 01/04/15 09:07 PM
LANGUAGE: ENGLISH

```

The TVM Solver is in the Finance App: press the `apps` key and select Finance. Most of the other options are variables to use in programs (but see Section 22.2.4).

<pre> APPLICATIONS 1: Finance... 2: Conics 3: Hub 4: SmartPad 5: Transfrm </pre>	<pre> CALC VARS 1: TVM Solver... 2: tvn_Pmt 3: tvn_I% 4: tvn_PV 5: tvn_N 6: tvn_FV 7: npv(8: irr(9: bal(</pre>
--	---

Press `1` to select the TVM Solver.

```

N=0.00
I%=0.00
PV=0.00
PMT=0.00
FV=0.00
P/Y=1.00
C/Y=1.00
PMT: END BEGIN

```

The TVM screen shows that a number of variables are possibly involved in calculations involving compound interest. To use the TVM module, you need to specify the value of each of the variables except one; the calculator works out the missing one for you.

Variables

N: the number of payment periods; if no regular payments, the number of years involved.

I%: the annual rate of interest, given as a percentage.

PV: the present value of the loan or investment. In the case of a loan, the amount borrowed. In the case of an investment, the original deposit.

PMT: the regular payment — either repaying a loan, depositing money for investment or a regular payment from a superannuation or investment account.

FV: the future value of the loan or investment. For a loan, this is usually zero, since the loan must be all paid off.

P/Y: the number of payments made per year. Set to 1 if there are no payments made.

C/Y: the number of compounding periods per year (in case it is different from P/Y).

For each variable, enter the necessary value, followed by `enter` or the up/down-arrow key.

The calculator will set P/Y and C/Y to be the same as soon as you enter P/Y, as these two are usually the same. If they are not the same, you must enter P/Y first, then C/Y.

PMT on the bottom line should be set to END — use the arrow keys and `enter`.

Important Note: Payments made to you (from the bank) are positive, payments made by you (to the bank) are negative.

22.2.2 Compound interest

Most interest calculations involve compound interest, in which interest is paid on the amount of money still owing on a loan or earned on the amount of money in the account (not just on what was originally deposited).

The simplest case of compound interest is depositing some money into an account and leaving it there for some time to accrue interest.

Example 1

An inheritance of \$5000 is placed into an investment account paying 11% interest per annum, compounded annually. How much will there be in the account after 5 years?

As shown in the figure below, N, the number of years, is set to 5, I% to 11, PMT to 0, as there are no regular payments, and P/Y and C/Y to 1. Note that 1 is the minimum value permitted by the calculator for P/Y, even though there are no payments.

PV, the present value, is set to -5000 , negative because this amount is paid to the bank.

To find FV, the future value of this investment (the amount after 5 years), *move the cursor to FV* (it doesn't matter what number is there) and press `solve` (`(alpha)` `enter`).

N=5.00
I%=11.00
PV=-5000.00
PMT=0.00
■ FV=8425.29
P/Y=1.00
C/Y=1.00
PMT:END BEGIN

After 5 years, the investment will have grown to \$8425.29, positive because it is paid to you.

Compound interest can also be calculated without using the TVM Solver. The variables here are related by the standard formula⁴⁶

$$FV = PV \left(1 + \frac{I\%}{100} \right)^N . \quad (1)$$

Evaluating this formula with the parameter values of Example 1, shown below, gives the same answer as the TVM Solver.

<code>5000(1+11/100)^5</code>	8425.29
-------------------------------	----------------

Other TVM calculations can also be done directly using the appropriate formula, but it is usually easier (and sometimes a lot easier) to let the calculator to do them for you. However, you should understand what the calculator is doing.

Example 2

How much is in the account in Example 1 if the interest is now compounded every three months (quarterly)?

Here we just change C/Y to 4. Again, evaluate FV by moving the cursor to FV and pressing `solve` (`alpha` `enter`).

N=5.00
I%=11.00
PV=-5000.00
PMT=0.00
■ FV=8602.14
P/Y=1.00
C/Y=4.00
PMT:END BEGIN

The future value has increased to \$8602.14, so only an extra \$176.85 over the 5 years.

22.2.3 Discounting

Compound-interest calculations can be used to find the future value of an investment at a certain rate of interest. However, sometimes we are interested in the opposite problem: finding the present value of a specific sum of money some time in the future.

One good reason for doing this is to compare various kinds of investments. If all are expressed in present values, a direct comparison can be made. We could also compare investments by calculating their future values, but the use of present values allows for today's money values to be used directly.

The process of finding the present value of a guaranteed future sum of money is called *discounting*, and the interest rate involved called the *discount rate*.

The relationship between PV and FV in Eq. (1) can be rearranged to show how PV can be calculated:

$$PV = \frac{FV}{\left(1 + \frac{I\%}{100} \right)^N} . \quad (2)$$

⁴⁶If this formula is not familiar to your students, you need to work through *Financial Mathematics 1* in Volume 1.

In the TVM module, discounting is performed by entering the appropriate values for FV, I% and N, then calculating PV.

Example 3

What is the present value of \$10,000 paid to you in 10 years' time, assuming a discount rate of 8% per annum?

The TVM set up is shown below after setting the cursor on PV and pressing `[solve]`. Again, it doesn't matter what value is in PV when you press `[solve]`, as this is the value being calculated.

```

N=10.00
I%=8.00
▪PV=-4631.93
PMT=0.00
FV=10000.00
P/Y=1.00
C/Y=1.00
PMT:END BEGIN

```

The present value is \$4631.93, negative in the TVM result because you would have to pay this amount now to have the \$10,000 (FV positive) paid to you in 10 years' time, given an annual interest rate of 8%.

You can check this result using Eq. (2) or by calculating that an investment of \$4631.93 at 8% per annum for 10 years will yield \$10,000.

22.2.4 Effective rate of interest

As seen above, if interest is compounded more often than annually, the amount of interest involved increases, as there is more interest on the interest. To compare interest rates, you therefore need to know how often the interest is compounded. Most interest rates are given as a rate per annum, sometimes called the *nominal rate* (annual compounding). The *effective rate* of interest takes into account the fact that the interest can be compounded more than once a year. The effective rate of interest is useful for comparing different financial offerings.

Essentially we first calculate the total amount of interest earned on an amount of say \$1000 (the amount does not matter) for 1 year under whatever compounding scheme is used. We then calculate the annual interest rate that would have generated this amount if there were annual compounding: this is the effective rate.

Example 4

Consider an investment of \$1000 for 1 year at 9% per annum. After 1 year, if there is annual compounding, the total interest is 9% of \$1000 or \$90, and the future value is \$1090.

If the interest were compounded monthly instead of annually, the interest rate would be 0.75% per month ($9\% \div 12$), and the future value can be determined from Eq. (1). The interest rate here must be the monthly interest rate and the number of payment periods 12N:

$$FV = PV \left(1 + \frac{I\%}{1200} \right)^{12N} = 1000(1+0.0075)^{12} = \$1093.81.$$

PTO

The screens below show these calculations both from the formula and using the TVM Solver. Note that, for monthly compounding for a year, $N = 1$, $P/Y = 1$ (no regular payments), but $C/Y = 12$.

1000×1.09 $1000(1+0.09/12)^{12}$	1090.00 1093.81	N=1.00 I%=9.00 PV=-1000.00 PMT=0.00 FV=1093.81 P/Y=1.00 C/Y=12.00 PMT:END BEGIN
--	------------------------	--

The investment in this case has earned interest of \$93.81, the amount of interest that would be earned with a (nominal) rate of interest of 9.38% compounded annually. This is the effective rate of interest.

You can calculate the effective rate of interest directly using the command \blacktriangleright Eff in the Finance menu. Scroll down this menu (it's actually faster scrolling up) until you come to this command, press \square enter to select it, enter the nominal interest rate and the number of compounding periods in a year, separated by a comma, and press \square enter to evaluate the command.

For the present case, evaluate \blacktriangleright Eff(9, 12) to give the answer 9.38% (to 2 decimal places).

\blacktriangleright Eff(9, 12)	9.38
\blacktriangleright Nom(9.38, 12)	9.00

If you now use the command \blacktriangleright Nom(9.38, 12), the calculator will give the nominal rate corresponding to an effective rate of 9.38%, as shown above.

22.2.5 Analysing a loan

Most loans involve both compound interest and regular payments. Using the calculator we ask questions like: *What if I pay fortnightly rather than monthly? What if the interest rate changes? What if pay a bit more each payment?*, and so on. Here we explore some of those possibilities in an example.

Example 5

I borrow \$9000 to buy a car. The interest rate is 18% per annum, compounded monthly, and I plan to make monthly payments of \$350. How long will it take to pay off the loan?

Here PV is positive because the bank has paid the money to me, PMT is negative because I pay this to the bank and FV is zero because I will pay off the loan. P/Y and C/Y are both 12 because the payment and compounding periods are monthly.

N=32.73 I%=18.00 PV=9000.00 PMT=-350.00 FV=0.00 P/Y=12.00 C/Y=12.00 PMT:END BEGIN
--

Evaluating N , the length of the loan, gives 32.73. The interpretation of this is that 32 payments of \$350 are needed, and that the last payment will then be less than this. The loan will certainly be paid off within 33 months or 2 years 9 months.

Details (and graphs) of the payment-by-payment values — the balance, how much of each payment goes to interest and how much to paying off the principal, the total interest paid and total paid off the principal — can be found by running the AMORTTBL/AMRTBLCE program, which generates an amortisation table — see the Appendix on page 137.

In calculating the total interest paid, we can use the amortisation table or we can just do the calculation

$$\text{total interest} = \text{number of payments} \times \text{payment amount} - \text{loan amount}.$$

Here we have $\text{total interest} = 32.73 \times 350 - 9000 = \2455.50 .

Let's now look at three alternatives to the original loan, shown in the three screens below. In each case, we have calculated N .

▪ N=28.20
I%=18.00
PV=8000.00
PMT= -350.00
FV=0.00
P/Y=12.00
C/Y=12.00
PMT: END BEGIN

▪ N=27.65
I%=18.00
PV=9000.00
PMT= -400.00
FV=0.00
P/Y=12.00
C/Y=12.00
PMT: END BEGIN

▪ N=23.96
I%=18.00
PV=8000.00
PMT= -400.00
FV=0.00
P/Y=12.00
C/Y=12.00
PMT: END BEGIN

The first screen shows that if only \$8000 is borrowed, perhaps after paying a deposit of \$1000, the loan is paid off in a little over 28 months.

The second screen shows that increasing the payment from \$350 to \$400 each month on the original loan of \$9000 will reduce the length of the loan to a little less than 28 months.

The third screen shows that making both these changes will reduce the length of the loan to less than 2 years. The total interest paid here, $23.96 \times 400 - 8000 = \1584 , is a saving of \$871 on the interest paid on the original loan.

Alternatively, we can ask what the payment needs to be to pay off the loan in say 2 years or 18 months.

N=24.00
I%=18.00
PV=9000.00
▪ PMT= -449.32
FV=0.00
P/Y=12.00
C/Y=12.00
PMT: END BEGIN

N=18.00
I%=18.00
PV=9000.00
▪ PMT= -574.25
FV=0.00
P/Y=12.00
C/Y=12.00
PMT: END BEGIN

The loan can be paid off in 2 years with monthly payments of \$449.32, or in 18 months with monthly payments of \$574.25.

The total interest paid is about \$1784 and \$1337, respectively.

22.2.6 Annuities

An annuity involves the payment of a regular sum of money over a period of time at a fixed rate of interest. Although you can regard payments on a loan as a kind of annuity, the term usually refers to other financial arrangements, such as a superannuation scheme (which has regular payouts) or a savings scheme (which has regular deposits). All these essentially involve compound interest, and so can be investigated using the TVM Solver.

Example 6

A family has won a prize of \$200,000 in a lottery and decided to use it as an annuity to provide them with a regular amount of money every 3 months over the next 10 years. They invest the money in an account paying 8.6% interest per annum, compounded annually. How much money will they get every 3 months?

Here there are 4 payments per year for 10 years, giving 40 payment periods altogether, so that $N=40$. The present value PV is $-200,000$ (they are paying it to the bank), and the final value FV is 0, since all the money (and interest) will be used. There are 4 payments per year, so that $P/Y=4$ and $C/Y=1$, as the interest is compounded annually. The only missing variable is the payment PMT.

```
N=40.00
I%=8.60
PV=-200000.00
PMT=7419.22
FV=0.00
P/Y=4.00
C/Y=1.00
PMT:END BEGIN
```

So this annuity provides the family with \$7419.22 every 3 months for 10 years.

22.2.7 Sinking funds

A sinking fund is a further example of an annuity, as it involves regular payments over a certain period, with a fixed rate of compound interest. The idea of a sinking fund is that money is invested in instalments into a fund for some particular purpose, and accumulates over time.

Example 7

You estimate you will need \$10,000 for a long overseas holiday in 3 years' time. The rate of interest available in an investment account is 7%, compounded monthly. How much should you save each month to reach the goal?

In this case, $PV=0$, $FV=10,000$ and $N=3 \times 12=36$.

```
N=36.00
I%=7.00
PV=0.00
PMT=-250.44
FV=10000.00
P/Y=12.00
C/Y=12.00
PMT:END BEGIN
```

The required payment is \$250.44 per month.

22.2.8 To lease or to buy?

The TVM Solver is a powerful tool for comparing alternative financial arrangements over different times. One example of this involves deciding whether it's financially better to purchase something now or lease it over a period of time.

Example 8

A computer store advertises a computer system for sale at \$1999 or offers an alternative involving leasing. The store will lease the same computer system to you for \$75 a month over 2 years at an interest rate of 12% per annum (compounded monthly). At the end of the two-year lease, you will be given an option to purchase the system for \$400. Should you buy the computer system now or lease it first and buy it in 2 years' time?

A quick comparison suggests that there is not much difference between these two choices, as the total payments for the lease-and-purchase option are $24 \times \$75 + \$400 = \$2200$, a bit more expensive than the cash purchase.

To explore these options more carefully, we need to have a way of comparing their monetary value at the same time. One way of doing this is to find the present value of the leasing option, and compare it with the present value of the cash purchase.

The leasing option comprises two parts: the monthly payments and the residual value of \$400. To determine the present value of 24 monthly payments of \$75 each, enter the values as shown below left, and solve for PV.

N=24.00 I%=12.00 PV=1593.25 PMT=-75.00 FV=0.00 P/Y=12.00 C/Y=12.00 PMT:END BEGIN	N=24.00 I%=12.00 PV=315.03 PMT=0.00 FV=-400.00 P/Y=12.00 C/Y=12.00 PMT:END BEGIN
---	---

The present value of the payments is \$1593.25.

To determine the present value of the residual payment of \$400, enter the values as shown above right and evaluate PV.

The present value of the leasing option is $\$1593.25 + \$315.03 = \$1908.28$.

The cash purchase has a present value of \$1999, almost \$100 more than this, so it is a financially better deal to take the lease-and-purchase option in this case.

We could have also compared these two options using their respective future values (in 2 years' time). In the case of the lease, the screen below shows that the monthly payments have a future value of \$2023.01. Added to the \$400 purchase price in 2 years' time (which of course has a future value of \$400), this gives a total future value of \$2423.01.

N=24.00 I%=12.00 PV=0.00 PMT=-75.00 FV=2023.01 P/Y=12.00 C/Y=12.00 PMT:END BEGIN

The cash-purchase option has a future value of \$2538.20:

N=24.00
I%=12.00
PV=-1999.00
PMT=0.00
FV=2538.20
P/Y=12.00
C/Y=12.00
PMT: END BEGIN

Again, leasing is the better option in this case.

22.2.9 Bonds

A bond is an agreement by a debtor to pay a certain amount of money (the *face value*) at a certain future date (the *maturation date*) and also to pay a sum of money periodically, the sum being a fixed percentage (interest rate) of the face value of the bond. Bonds are issued by public authorities, governments and companies as a way of raising money to finance their activities. In many cases, a bond has attached to it small dated coupons that can be used to obtain the periodical payment, so the periodical payments are usually called *coupon payments*, and the interest rate of the coupon payments is called a *coupon rate*. Unlike private bonds issued by a company seeking to raise capital, government bonds may have some associated tax implications (which are not considered here).

After they have been issued, bonds can be bought and sold by investors. Naturally enough then, the main interest in bonds is in determining their present value, deciding how much they should fetch when traded and thus deciding whether or not they are a good investment. Government bonds are usually regarded as more secure investments than private bonds, as there is less risk of default (i.e. there is less risk that the issuing body will not be able to redeem (pay for) the bonds at the maturation date or make the coupon payments). You can use the TVM Solver to analyse bonds.

Example 9

A bond is issued on 1 July 2021 by an electricity authority in order to finance a new power plant. The bond has a face value of \$1000, a maturation date of 1 July 2031 (that is, it matures in 10 years) and a coupon rate of 7%, compounded and paid semi-annually.

You want a rate of return of 9% on your money. Do you buy the bond when it is issued?

In this case, the payments are fixed at $7\% \div 2 = 3.5\%$ of \$1000 = \$35 every six months, and the future value is fixed at the face value of \$1000. The required rate of return is $I\% = 9$, and both P/Y and C/Y are set to 2 for semi-annual payment and compounding. There are 20 compounding periods, 2 each year for 10 years.

To obtain the present value of the bond, solve for PV.

N=20.00
I%=9.00
PV=-869.92
PMT=35.00
FV=1000.00
P/Y=2.00
C/Y=2.00
PMT: END BEGIN

So the bond will return 9% on your money if you pay \$869.92 for it when it is issued. You will get a better rate of return than 9% if you pay less than this.

You can use the same module to calculate the Yield to Maturity (YTM) of a bond. The YTM is the equivalent interest rate you will earn if you retain the bond until it matures and you redeem it, as well as collecting the coupon payments up till then.

Example 10

An investor pays \$800 for an electricity-authority bond as described in the previous example, 5 years before it matures (and just after a coupon payment was made). What is the YTM?

To calculate the YTM, enter the relevant values and solve for I%.

N=10.00
▪ I%=12.50
PV=-800.00
PMT=35.00
FV=1000.00
P/Y=2.00
C/Y=2.00
PMT:END BEGIN

In this case, the investment will yield $12\frac{1}{2}\%$ interest per annum, compounded semi-annually (assuming that the electricity authority is able to redeem the bond when it matures).

22.2.10 A cautionary note

The methods here are generally less complicated than those used in practice, and do not include taxes, fees and other charges. You should seek proper financial advice before making any financial decisions, rather than relying on the calculations outlined here.

22.3 Appendix: Amortisation tables

An amortisation table shows, for each payment period: the balance after this payment; how much of this payment goes to interest and how much to repaying the principle; the total interest paid up to and including this payment; and the total paid towards the principal after this payment.

The TI-84/CE program⁴⁷ AMORTTBL/AMRTBLCE uses the TVM variables *bal* to find the balance and ΣInt to find the cumulative interest; it then calculates the other values.⁴⁸

All values are stored in lists:

- PAYT payment number
- BAL balance after this payment
- INT amount of interest paid in this payment (negative)
- PR amount of principal repaid in this payment (negative)
- CINT cumulative interest paid after this payment (negative)
- CPR cumulative principal paid after this payment (negative).

Press `[stat]` Edit after the program has run to view these lists. Scroll across to see all the lists.

The program also sets up three plots, all with payment number on the x axis:

- balance (Plot1);
- the amount of the payment that goes to paying interest (Plot2);
- the amount of the payment that goes towards repaying the principal (Plot3).

⁴⁷available at *www.XXX*

⁴⁸See *Time, Value, Money: Applications on the TI-83* for more details.

Using the program

First run the TVM Solver to find values for all the variables. Then press **[prgm]**, press the number/letter against the AMORTTBL/AMRTBLCE program and press **[enter]** to run it.

The program will ask if you have run the TVM Solver — respond appropriately. After the calculations have been carried out, the program will display all three plots superimposed. Use the arrow keys to move along and between the plots. Press **[enter]** to move to a screen that tells you where to find the table and plots.

You can then either view the lists by pressing **[stat]** Edit or select the particular plot you want by highlighting it in **[y=]** using **[enter]**. Make sure all functions and other plots are turned off, again using **[enter]**. Press **[zoom]** **[9]** to set appropriate axes for that plot. Press **[trace]** and use the arrow keys to move along the graph. The amounts at each payment are shown at the bottom of the screen.

The following screens show the superimposed three plots generated by the program from Example 5 on page 132, the top few rows of the amortisation table and the three individual plots, *balance*, *amount of payment going to interest* and *amount of payment going to the principal*, all versus *payment number*.

TVM SOLVER RUN?

1: NO

2: YES

3: TIDY UP

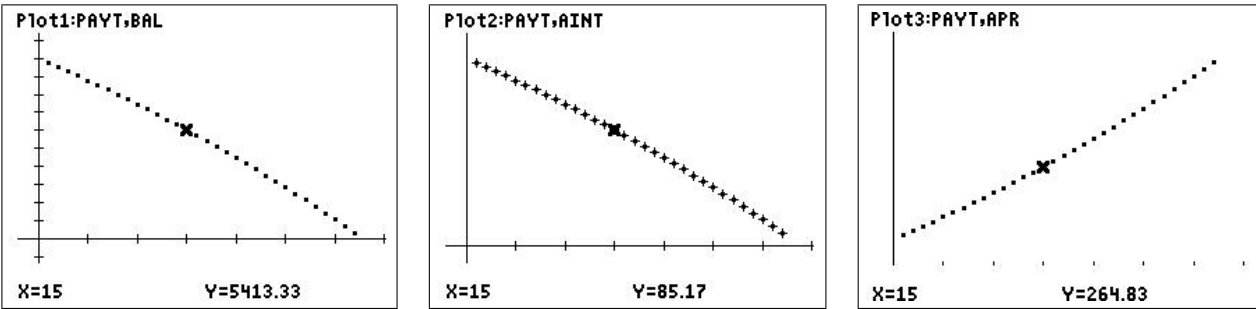
Plot1:PAYT,BAL

PAYT	BAL	INT	PR	CINT
1.00	8785.0	-135.0	-215.0	-135.0
2.00	8566.8	-131.8	-218.2	-266.8
3.00	8345.3	-128.5	-221.5	-395.3
4.00	8120.5	-125.2	-224.8	-520.5
5.00	7892.3	-121.8	-228.2	-642.3
6.00	7660.7	-118.4	-231.6	-760.7
7.00	7425.6	-114.9	-235.1	-875.6
8.00	7186.9	-111.4	-238.6	-986.9
9.00	6944.7	-107.8	-242.2	-1095
10.00	6698.9	-104.2	-245.8	-1199
11.00	6449.4	-100.5	-249.5	-1299

BAL	INT	PR	CINT	CPR
8785.0	-135.0	-215.0	-135.0	-215.0
8566.8	-131.8	-218.2	-266.8	-433.2
8345.3	-128.5	-221.5	-395.3	-654.7
8120.5	-125.2	-224.8	-520.5	-879.5
7892.3	-121.8	-228.2	-642.3	-1108
7660.7	-118.4	-231.6	-760.7	-1339
7425.6	-114.9	-235.1	-875.6	-1574
7186.9	-111.4	-238.6	-986.9	-1813
6944.7	-107.8	-242.2	-1095	-2055
6698.9	-104.2	-245.8	-1199	-2301
6449.4	-100.5	-249.5	-1299	-2551

PAYT(1) = 1

CPR(1) = -215



When you have finished your analysis, re-run AMORTTBL/AMRTBLCE and select **[3]** to delete all the lists and tidy up.

22.4 TVM exercises and solutions

Note that N in the TVM Solver stands for different things in different types of calculations. All results are rounded to two decimal places. Starred questions were not covered here.

- *1. Find the simple interest on a 90-day loan of \$1600 at 5.6% per annum.

Solution No built-in commands. Use the formula.

The daily interest rate is $90/365$ of the annual rate of interest. Therefore the amount of interest is

$$\frac{90}{365} \times 0.056 \times 1600 = 22.09.$$

The interest is \$22.09.

- *2. How many days are there between January 12 2003 and June 3 2003?

Hint: the *dbd* command in the Finance menu.

Solution Finance menu: *dbd* command

Dates are entered in the form *mm.dyyy*.

$$\text{dbd}(1.1203, 6.0303) = 142$$

There are 142 days between January 12 2003 and June 3 2003.

- *3. A store buys a skateboard for \$45. What selling price will give a 30% margin?

Solution No built-in command. Use the formula.

If S is the selling price, $45 + 0.3S = S$, so that $S = 45/0.7 = 64.29$.

The selling price should be \$64.29.

Note that a 30% margin means the return is 30% of the selling price. A 30% profit means a return of 30% of the cost price.

4. (a) How much compound interest will be earned by an investment of \$1600 for 10 years at $9\frac{3}{4}\%$ per annum?

Solution TVM Solver

$N = 10$ with no regular payments, N is the number of *years*

$I\% = 9.75$ annual percentage interest rate

$PV = -1600$ negative because paid to bank

$PMT = 0$ no regular payments

$FV = ?$ doesn't matter what is here — this is what we have to find

$P/Y = 1$ payments per year — must be set to 1 if no payments

$C/Y = 1$ compounding periods per year (assumed to be equal to P/Y)

Then, $FV = 4056.63$ positive because paid to you

The final amount is \$4056.63.

Interest = \$4056.63 - \$1600 = \$2456.63.

- (b) How much more interest will be earned if the interest were compounded quarterly?

Solution Change C/Y to 4 (compounding periods per year).

Then, $FV = 4192.56$.

The final amount is \$4192.56.

Interest = $\$4192.56 - \$1600 = \$2592.56$.

Extra interest = $\$2592.56 - \$2456.63 = \$135.93$.

You could also just subtract one final amount from the other, as the difference is the extra interest.

5. For how long should you leave \$10,000 in the bank at 6% per annum interest, compounded monthly, so that it grows to \$12,000?

Solution TVM Solver

$N = ?$ doesn't matter what is here — this is what we have to find

$I\% = 6$ annual percentage interest rate

$PV = -10000$ negative because paid to bank

$PMT = 0$ no regular payments

$FV = 12000$ the amount we want to receive

$P/Y = 1$ payments per year — must be set to 1 if no payments

$C/Y = 12$ compounding periods per year

Then, $N = 3.046$: because there are no regular payments, N is in years

You would need to leave the money in the bank 3.046 years or 3 years 17 days.

You might want to check that the bank will in fact give you the interest for the 17 days if you withdraw the money then. You may have to leave the money in the bank until the next monthly interest credit. By then you will have \$12,026.64.

6. A credit card company charges 16% per annum interest, compounded monthly. What is the effective rate of interest?

Solution: Here you can use the ►*Eff* command in the Finance menu (Method A). However, this is a bit of a black box. More intuitively, we can do the calculations on an amount of say \$1000 (Method B).

Method A

Finance menu: ►*Eff* command

►*Eff*(16, 12) = 17.23

16 is the annual percentage interest rate.

12 is the number of compounding periods in a year.

The effective interest rate is 17.23% (to 2 decimal places).

Method B

Here say we owe the credit card company an amount of \$1000 over a year (the amount does not matter — you always get the same answer).

If the interest is compounded once a year, the amount of interest to be paid is $0.16 \times 1000 = \$160$.

Now we calculate how much interest is to be paid *if the interest is compounded monthly*.

TVM Solver

$N = 1$ number of *years*, because no regular payments
 $I\% = 16$ annual percentage interest rate
 $PV = 1000$ positive because it is a 'loan' paid to us
 $PMT = 0$ no regular payments
 $FV = ?$ doesn't matter what is here — this is what we have to find
 $P/Y = 1$ payments per year — must be set to 1 if no payments
 $C/Y = 12$ compounding periods per year

Then, $FV = 1172.27$.

The total amount of interest paid this time is $\$1172.27 - \$1000 = \$172.27$.

If this were the result of compounding once a year, the rate of interest would have to have been $172.27/1000 = 0.17227$ or $17.227\% \approx 17.23\%$. This is the effective rate of interest.

7. (a) To finance extensions to her house, Susan borrowed \$7000 at 11% per annum. What monthly payments are needed to repay the loan in 3 years?

Solution TVM Solver

$N = 12 \times 3 = 36$ with regular payments, N is the total number of *payments*
 $I\% = 11$ annual percentage interest rate
 $PV = 7000$ positive because paid to Susan
 $PMT = ?$ doesn't matter what is here — this is what we have to find
 $FV = 0$ loan paid off
 $P/Y = 12$ payments per year — monthly payments here
 $C/Y = 12$ compounding periods per year — assumed equal to P/Y

Then, $PMT = -229.17$, negative because paid to the bank

The monthly payment is \$229.17.

- (b) How much interest will she pay for this loan?

Solution: Total interest is equal to the total amount paid minus the loan amount (principal), here $36 \times 229.17 - 7000 = \1250.12 .

8. Julian's lump-sum superannuation amounts to \$350,000. He invests this sum to receive a regular payment each month for the next 20 years. Assuming the interest rate stays at 6.8% per annum, find the size of each payment.

Solution TVM Solver

$N = 12 \times 20 = 240$ total number of *payments*
 $I\% = 6.8$ annual percentage interest rate
 $PV = -350000$ negative because paid to bank
 $PMT = ?$ doesn't matter what is here — this is what we have to find
 $FV = 0$ capital and interest all used up after 20 years
 $P/Y = 12$ payments per year
 $C/Y = 12$ compounding periods per year — assumed equal to P/Y

Then, $PMT = 2671.69$

The monthly payment to Julian is \$2671.69.

9. A school establishes a sinking fund to save \$90,000 over the next 2 years for a new computer laboratory. How much should be deposited into the fund each month if the interest rate stays at $7\frac{1}{2}\%$ per annum?

Solution TVM Solver

$N = 12 \times 2 = 24$ total number of *payments*

$I\% = 7.5$ annual percentage interest rate

$PV = 0$ no capital

$PMT = ?$ doesn't matter what is here — this is what we have to find

$FV = 90000$ final amount required

$P/Y = 12$ payments per year

$C/Y = 12$ compounding periods per year — assumed equal to P/Y

Then, $PMT = -3487.46$

The monthly payment is \$3487.46.

10. What is worth more? \$1200 today or \$100 per month for 1 year? Assume the interest rate stays constant at 7.1% per annum.

Solution: Find the present value of \$100 per month for a year at 7.1% per annum.

TVM Solver

$N = 12$ total number of *payments*

$I\% = 7.1$ annual percentage interest rate

$PV = ?$ doesn't matter what is here — this is what we have to find

$PMT = -100$ monthly payment to bank

$FV = 0$ you need to put 0 here

$P/Y = 12$ payments per year

$C/Y = 12$ compounding periods per year — assumed equal to P/Y

Then, $PV = 1155.10$

The present value of \$100 per month for 1 year at 7.1% per annum is \$1155.10. This is less than \$1200, so \$1200 today is worth more than \$100 per month for 1 year at 7.1% per annum.

We could have found the future value of both amounts, although this is more work. Proceed as follows.

Set $PV = 0$ and solve for FV to find the future value of \$100 per month for a year at 7.1% per annum to be \$1239.83.

Convert \$1200 today to its future value:

$N = 1$ total number of *years* (no regular payments)

$I\% = 7.1$

$PV = -1200$ present value

$PMT = 0$ no regular payments

$FV = ?$ doesn't matter what is here — this is what we have to find

$P/Y = 1$ payments per year — must be set to 1 if no payments

$C/Y = 12$ compounding periods per year

Then, $FV = 1288.03$, so that the future value of the \$1200 is \$1288.03.

We reach the same conclusion as above: take the \$1200!

22.5 TVM activities and solutions

The activities are to help you use your calculator to learn mathematics.

All results rounded to two decimal places.

1. (a) Find the number of days between 12 January 2003 and 12 January 2004.

Solution: Finance menu, *dbd* command

Dates are entered in the form *mm.ddyy*: $\text{dbd}(1.1203, 1.1204) = 365$

There are 365 days between 12 January 2003 and 12 January 2004.

- (b) Find the number of days between 12 January 2004 and 12 January 2005.

Solution: $\text{dbd}(1.1204, 1.1205) = 366$

There are 366 days between 12 January 2004 and 12 January 2005.

- (c) Explain why the answers to Questions 1a and 1b are different.

Solution: 2004 was a leap year.

2. Banks sometimes advertise that interest is compounded daily on customers' investments in order to attract more business. But how much difference does more-frequent compounding make? To find out, consider an investment of \$100 in an account earning compound interest at a rate of 8% per annum. Compare the interest obtained by compounding: (a) annually; (b) quarterly; (c) monthly; (d) daily.

Solution TVM Solver

$N = 1$ number of *years* (no regular payments)

$I\% = 8$ annual percentage interest rate

$PV = -100$ the amount invested — negative because paid to the bank

$PMT = 0$ no regular payments

$FV = ?$ doesn't matter what is here — we calculate this

$P/Y = 1$ payments per year — must be set to 1 if no payments

$C/Y =$ compounding periods per year: set this value and calculate FV: see table

Compounding	C/Y	FV (\$)	Interest (\$)
annually	1	108.00	8.00
quarterly	4	108.24	8.24
monthly	12	108.30	8.30
daily	365	108.33	8.33

You can also do these calculations directly with the command $\blacktriangleright\text{Eff}(8, C/Y)$ because the amount invested is \$100: interest rate (%) = interest paid (\$).

3. The effective rate of interest associated with more-frequent compounding rapidly approaches $e^r - 1$, where r is the annual rate of interest and e is the base of the exponential function ($e = 2.7182818\dots$). Investigate this claim by taking some values for r and examining the effect of more-frequent compounding (such as weekly, daily, hourly, etc).

Solution: See Exercise 6 above for details on how to calculate effective rate of interest. Here we use various values for $r = I\%/100$, and work out the effective rates of interest for different compounding periods. The results are shown in the table below. In the table, N is the number of compounding periods in a year. Interest rates are rounded to 2 decimal places.

The continuously compounded interest rate is $100(e^r - 1)$.

		Effective interest rate % when compounded ...					
I%	r	yearly $N=1$	monthly $N=12$	weekly $N=52$	daily $N=365$	hourly $N=8760$	continuously $N \rightarrow \infty$
5	0.05	5	5.12	5.13	5.13	5.13	5.13
7.5	0.075	7.5	7.76	7.78	7.79	7.79	7.79
10	0.1	10	10.47	10.51	10.52	10.51	10.52
20	0.2	20	21.94	22.09	22.13	22.14	22.14

Most of the change in the effective interest rate is between annual compounding ($N=1$) and monthly compounding ($N=12$). The change between monthly and continuous compounding $100(e^r - 1)$ is relatively small, but larger for higher interest rates.

4. Jill purchased a new car for \$11,999 by obtaining a 5-year loan from a finance company. The interest on the loan was at a flat rate of 14% per year. Compare the amounts of interest paid on this loan each year with the amounts paid for a loan of the same amount at a rate of 14% reducible compound interest per annum, with monthly payments and compounding.

Solution: The flat rate means she pays 14% of \$11,999, or \$1679.86, each year for 5 years. For the reducing rate:

TVM Solver

$N = 12 \times 5 = 60$ number of *payments*

$I\% = 14$ annual percentage interest rate

$PV = 11999$ the loan amount

$PMT = ?$ we need to find this first

$FV = 0$ loan paid off

$P/Y = 12$ payments per year

$C/Y = 12$ compounding periods per year

Then, $PMT = -279.20$

The monthly repayments are \$279.20.

PTO

The simplest way to find the amount of interest paid each year is to run the AMORT-TBL/AMRTBLCE program. You should get the amounts shown in the table below.

Year	Flat-rate Interest (\$)	Reducing Interest (\$)
1	1679.86	1568.39
2	1679.86	1302.27
3	1679.86	996.40
4	1679.86	644.86
5	1679.86	240.82
Total	8399.30	4752.74

5. A couple are hoping to buy their first home. They expect to obtain a 25-year loan, and their monthly payments cannot exceed \$920 if their living expenses are taken into account. If the interest rate is assumed to be constant at 7% per annum, how much money will they be able to borrow?

Solution TVM Solver

$N = 12 \times 25 = 300$ number of *payments*

$I\% = 7$ annual percentage interest rate

$PV = ?$ doesn't matter what is here — we calculate this

$PMT = -920$ monthly payments to bank

$FV = 0$ loan paid off

$P/Y = 12$ payments per year

$C/Y = 12$ compounding periods per year — assumed equal to P/Y

Then, $PV = 130,167.95$

The couple can only borrow \$130,167. This is why first-home buyers often struggle.

6. A student purchased a new laptop multimedia computer system with software for \$4500 by obtaining a personal loan at 13.4% per annum interest, compounded monthly.
- (a) How much will each payment be if the loan is spread over 3 years?

Solution TVM Solver

$N = 12 \times 3 = 36$ number of *payments*

$I\% = 13.4$ annual percentage interest rate

$PV = 4500$ amount of loan

$PMT = ?$ doesn't matter what is here — we calculate this

$FV = 0$ loan paid off

$P/Y = 12$ payments per year

$C/Y = 12$ compounding periods per year

Then, $PMT = -152.49$

The monthly repayment is \$152.49.

- (b) How much interest will be paid over the 3 years?

Solution

Total interest = total payments – amount of loan = $36 \times 152.49 - 4500 = \989.64 .

- ***(c)** If she arranged for a three-year loan, as in Parts (a) and (b), but then decided to pay the entire loan balance at the end of 2 years, how much would she save?

Solution: To find out the total cost of this scheme, we need to find the balance owing after 2 years and the total payments in the first 2 years.

Run the AMORTTBL/AMRTBLCE program to find the balance after 2 years (\$1703.72), the total interest paid in 2 years (\$863.51) and the total principal paid in 2 years (\$2796.28). The sum of these, \$3659.79, is the total payments made in 2 years. Alternatively calculate $24 \times 152.49 = \$3659.76$ (the difference in the two amounts is due to rounding the payments to 2 decimal places.)

The total payment here is then the total payment for 2 years plus the balance owing after 2 years, that is $\$3659.79 + \$1703.72 = \$5363.51$.

The total payment for the full three-year loan is principal + interest, giving from (b), $\$4500 + \$989.68 = \$5489.68$.

The total saving in Part (c) is then $\$5489.68 - \$5363.51 = \$126.17$.

7. Jeremy decided to invest money in a superannuation account. He deposited \$400 per month into an account earning 9% interest per annum, compounded monthly.

- (a)** How much money will be in the account after 25 years?

Solution TVM Solver

$N = 12 \times 25 = 300$ number of *payments*

$I\% = 9$ annual percentage interest rate

$PV = 0$ nothing in the account to start with

$PMT = -400$ monthly payment to the bank

$FV = ?$ doesn't matter what is here — we calculate this

$P/Y = 12$ payments per year

$C/Y = 12$ compounding periods per year

Then, $FV = 448448.77$

The amount in the account after 25 years is \$448,448.77.

- (b)** At the end of that time, Jeremy arranged to use the accumulated funds in the account to receive a monthly payment for the next 25 years. If the interest rate remains the same, find the size of the payment each month.

Solution

$N = 12 \times 25 = 300$ number of *payments*

$I\% = 9$ annual percentage interest rate

$PV = -448448.77$ accumulated funds

$PMT = ?$ doesn't matter what is here — we calculate this

$FV = 0$ all funds used up after 25 years

$P/Y = 12$ payments per year

$C/Y = 12$ compounding periods per year

Then, $PMT = 3763.37$

The amount paid to Jeremy each month is \$3763.37.

Seems like a good deal! Pay in \$400 per month for 25 years, then receive \$3763.37 per month for the next 25 years. What's the catch?

8. Carole and Joyce have taken out a 25-year home loan of \$120,000 to buy an apartment at an interest rate of $7\frac{1}{2}\%$ per annum, compounded monthly.

(a) Check that their monthly payment for this loan is \$886.79.

Solution TVM Solver

$N = 12 \times 25 = 300$ number of *monthly payments*

$I\% = 7.5$ annual percentage interest rate

$PV = 120000$ amount of loan

$PMT = ?$ doesn't matter what is here — we calculate this

$FV = 0$ loan paid off

$P/Y = 12$ payments per year

$C/Y = 12$ compounding periods per year

Then, $PMT = -886.79$

The monthly repayment is \$886.79.

- (b) What would be the effect on the length of their loan if they were to pay an extra \$50 per month?

Solution: The new payment is $\$886.79 + \$50 = \$936.79$.

Set $PMT = -936.79$.

Then, $N = 258.8$ (months)

The length of the loan would be reduced to 259 months or 21 years 7 months (the last payment would be less than the full monthly payment).

- (c) What would be the effect on the length of their loan of a drop in interest rates to 7% (assuming they pay \$886.79 per month)?

Solution

$I\% = 7$

$PMT = -886.79$

Then, $N = 267.8$ (months)

The length of the loan would be reduced to 268 months or 22 years 4 months (the last payment would be less than the full monthly payment).

- (d) Carole suggests paying \$443.40 every 2 weeks instead of \$886.79 per month, with the interest rate at $7\frac{1}{2}\%$ per annum interest, compounded monthly. How would this plan affect the length of the loan?

Solution TVM Solver

$N = ?$

$I\% = 7.5$

$PV = 120000$

$PMT = -443.40$ fortnightly payment

$FV = 0$

$P/Y = 26$ fortnightly payments per year

$C/Y = 12$ compounding periods per year

Then, $N = 525.5$ (fortnights)

The length of the loan would be reduced to 526 fortnights or 20 years 12 weeks (the last payment would be less than the full fortnightly payment).

- (e) If they were to borrow only \$115,000, yet still pay \$886.79 per month, with interest arrangements still the same, for how long will the loan last?

Solution

$$N = ?$$

$$I\% = 7.5$$

$$PV = 115000 \quad \text{new loan amount}$$

$$PMT = -886.79 \quad \text{monthly payment}$$

$$FV = 0$$

$$P/Y = 12 \quad \text{monthly payments per year}$$

$$C/Y = 12$$

$$\text{Then, } N = 267.0 \text{ (months)}$$

The length of the loan would be reduced to 267 months or 22 years 3 months.

- (f) Calculate how much time Carole and Joyce can save off the original repayment time of 25 years if all of the above strategies are adopted: an interest rate of 7% is obtained, compounded monthly; payments of \$468.40 are made every 2 weeks (an extra \$25 per fortnight); and they only borrow \$115,000 instead of \$120,000.

How much interest would they save?

Solution

$$N = ?$$

$$I\% = 7$$

$$PV = 115000$$

$$PMT = -468.40$$

$$FV = 0$$

$$P/Y = 26$$

$$C/Y = 12$$

$$\text{Then, } N = 401.8 \text{ (fortnights)}$$

The length of the loan would be reduced from 25 years to 402 fortnights or 15 years 24 weeks.

They would pay a total of \$73,203.12 in interest, compared to \$146,037 for the original loan, a saving of about \$72,830.

PTO

9. A *perpetuity* is an annuity that lasts forever.

Find the present value of a perpetual annual payment of \$1000, assuming an interest rate of 6% per annum, compounded monthly. (This is the amount you would need to invest today to provide enough funds for an annual payment of \$1000 forever.)

In TVM mode, explore this situation by choosing a large number for the number of compounding periods N , and set FV to be zero.

Solution TVM Solver

$N = 1000$ number of compounding periods (years)

$I\% = 6$ annual percentage interest rate

$PV = ?$ doesn't matter what is here — we calculate this

$PMT = 1000$ positive because the bank pays this out to the beneficiary

$FV = 0$ nothing left at the end (of time)

$P/Y = 1$ payments per year

$C/Y = 12$ compounding periods per year

Then, $PV = -16213.29$

You would need to invest an amount of about \$16,213.

Changing N to 10,000 years also gives $PV = -16213.29$, so we can say the amount invested is enough to make the payment 'perpetual'.

Here, the amount of interest on the principal of \$16,213.29 is just enough to cover the annual \$1000 payment. The principal actually remains the same from year to year.

We can check that this is the case using the TVM Solver.

$N = 1$ one payment a year

$I\% = 6$ annual percentage interest rate

$PV = -16213.29$ the principal in the bank

$PMT = ?$ doesn't matter what is here — we calculate this

$FV = 16213.29$ principal remains the same at the end of the year

$P/Y = 1$ payments per year

$C/Y = 12$ compounding periods per year

Then, $PMT = 1000$, as we predicted.

PTO

10. You may have heard of the *Rule of 72*, which provides a rough approximation to the amount of time it will take for an investment to double when interest is compounded annually at a rate of $I\%$. The Rule claims

$$\text{doubling time} \approx \frac{72}{I\%}.$$

So, according to the *Rule of 72*, an investment at a rate of 6% , compounded annually, will take about $72/6 = 12$ years to double in value.

Investigate this Rule using the TVM mode of the calculator. Over what range of values for $I\%$ does it provide a useful approximation?

Solution TVM Solver

$N = ?$ doesn't matter what is here — we calculate this

$I\% =$ see table annual percentage interest rate

$PV = -100$ choose any amount

$PMT = 0$ no regular payments

$FV = 200$ double the PV

$P/Y = 1$ payments per year — must be set to 1 if no payments

$C/Y = 1$ compounding periods per year

Set values of $I\%$ and calculate N , the time to double the original amount.

The exact time N , the time according to the Rule of 72 and the percentage error in the Rule of 72 are shown in the table below.

$I\%$	N (yrs)	$72/I\%$	% error
1	69.7	72.0	3.3
2.5	28.1	28.8	2.5
5	14.2	14.4	1.4
7.5	9.6	9.6	0.0
10	7.3	7.2	-1.4
20	3.8	3.6	-5.3

The approximation is good except for $I\% > 20$, but particularly good around $I\% = 7.5$.

23 Complex Numbers

The material here is directly relevant to the topics on Complex Numbers in the Australian Curriculum for Specialist Mathematics.

23.1 Setting complex mode

Press `mode` and select the Cartesian form $a+bi$ (below left, 8th line).

Press `2nd mode` (Quit) to return to the Home screen.

```

MATHPRINT CLASSIC
NORMAL SCI ENG
FLOAT 0 1 2 3 4 5 6 7 8 9
RADIAN DEGREE
FUNCTION PARAMETRIC POLAR SEQ
THICK DOT-THICK THIN DOT-THIN
SEQUENTIAL SIMUL
REAL a+bt re^(θi)
FULL HORIZONTAL GRAPH-TABLE
FRACTION TYPE: n/d Un/d
ANSWERS: AUTO DEC
STAT DIAGNOSTICS: OFF ON
STAT WIZARDS: ON OFF
SET CLOCK 17/02/22 17:50
LANGUAGE: ENGLISH

```

```

1+2i→A
.....1+2i
3-i→B
.....3-i

```

We shall use $z_1 = 1+2i$ and $z_2 = 3-i$ in our examples. i is `2nd` `.`.

Complex numbers can be stored in the same way as ordinary numbers. Store z_1 in memory A: $1+2i$ `sto` A; store z_2 in memory B: $3-i$ `sto` B (above right).

23.2 Basic operations

23.2.1 Addition and subtraction

Just as you would expect.

$$1+2i + 3-i = 4+i \quad \text{or} \quad A+B = 4+i$$

$$1+2i - (3-i) = -2+3i \quad \text{or} \quad A-B = -2+3i$$

```

1+2i+3-i
.....4+i
1+2i-(3-i)
.....-2+3i
(1+2i)(3-i)
.....5+5i
(1+2i)/(3-i)
.....0.1+0.7i

```

```

A+B
.....4+i
A-B
.....-2+3i
AB
.....5+5i
A/B
.....0.1+0.7i

```

23.2.2 Multiplication and division

Again as you would expect. Implied multiplication works too.

$$(1+2i)(3-i) = 5+5i \quad \text{or} \quad AB = 5+5i$$

$$(1+2i)/(3-i) = 0.1+0.7i \quad \text{or} \quad A/B = 0.1+0.7i$$

23.2.3 Conjugation

Finding the complex conjugate.

$$\bar{z}_1 = \text{conj}(1+2i) = 1-2i \quad \text{conj is } \boxed{\text{math}} \text{ CMPLX } \boxed{1}$$

23.2.4 Real part

$$\text{Re}(z_1) = \text{real}(1+2i) = 1 \quad \text{real is } \boxed{\text{math}} \text{ CMPLX } \boxed{2}$$

<code>conj(1+2i)</code>	<code>1-2i</code>
<code>real(1+2i)</code>	<code>1</code>
<code>imag(1+2i)</code>	<code>2</code>
<code>abs(1+2i)</code>	<code>2.236067977</code>

23.2.5 Imaginary part

$$\text{Im}(z_1) = \text{imag}(1+2i) = 2 \quad \text{imag is } \boxed{\text{math}} \text{ CMPLX } \boxed{3}$$

23.2.6 Modulus

Sometimes called length or absolute value.

$$|z_1| = \text{abs}(1+2i) = \sqrt{1^2+2^2} = \sqrt{5} \approx 2.236 \quad \text{abs is } \boxed{\text{math}} \text{ CMPLX } \boxed{5}$$

23.3 Polar form

The polar form (r, θ) of a complex number is represented on the TI-84/CE in exponential form $re^{i\theta}$, where r is the modulus and θ is the (polar) angle. This form is equivalent to the form $r(\cos(\theta)+i\sin(\theta))$, often abbreviated as $r\text{cis}(\theta)$.

23.3.1 Modulus and angle

Use the $\boxed{\text{math}}$ CMPLX operations *abs* and *angle* to extract the modulus and angle, respectively, of a complex number in either Cartesian $(a+ib)$ or polar form.

Similarly, *real* and *imag* can be used to find the real and imaginary parts respectively of a number in polar form, and *conj* its complex conjugate.

<code>abs(1+2i)</code>	<code>2.236067977</code>
<code>angle(1+2i)</code>	<code>1.107148718</code>
<code>real(e^(iπ/4))</code>	<code>0.7071067812</code>
<code>imag(e^(iπ/4))</code>	<code>0.7071067812</code>

23.3.2 Forms of output

Cartesian ($a+bi$) mode

Set $a+bi$ in `mode`.

Input $\sqrt{2}e^{i\pi/4}$ using the `ex` key and press `enter`.

$\sqrt{2}e^{i\pi/4} \rightarrow 1+1i$.

Try the famous calculation $e^{i\pi}$.

To input an angle in degrees, use the degree symbol

`angle` `1`. $\sqrt{2}e^{i45^\circ} \rightarrow 1+1i$.

```

√(2)e^(iπ/4)                1+1i
.....
e^(iπ)                    -1+0i
.....
√(2)e^(i45°)              1+1i

```

Polar ($re^{i\theta}$) mode

Set $re^{i\theta}$ in `mode`. Make sure you are in Radians.

Enter the number $1+i$ and press `enter`.

$1+i \rightarrow 1.414213562e^{0.7853981634i}$ ($=\sqrt{2}e^{i\pi/4}$);

use the right arrow to scroll across.

If you repeat this in Degree mode,

$1+i \rightarrow 1.414213562e^{45i}$,

i.e. the angle is displayed in degrees.

However, **if you do not change back to Radians to input polar form, any input with an angle in degrees will be wrong** (see Section 23.6.1).

```

1+i
1.414213562e^(0.785398163...
1+i
                        1.414213562e^(45i)

```

A better alternative to changing back and forth between Degree and Radian mode when you are using angles in degrees is to stay in Radian mode and use the `CMPXANGD/ CXANGDCE` program⁴⁹ to convert your output to degrees; it rounds all numbers to 5 decimal places. As the program gives both the Cartesian and polar forms, it can also be used to convert one form to the other.

If your calculations produce the decimal equivalent of $\sqrt{2}e^{i\pi/5}$, running the program gives

Cartesian form $1.14412 + i0.83125$

modulus $R = 1.41421$ (stored in memory `R`)

angle θ (rad) = 0.62832 (stored in memory `T`)

angle θ ($^\circ$) = 36 (stored in memory θ).

```

A+iB: (R,B)=
      {1.14412,0.83125}

R: 1.41421
θr: 0.62832
θ°: 36

```

⁴⁹available at www.XXX

23.3.3 Forms of input

In carrying out calculations using complex numbers (page 151), you can input the numbers in any form, Cartesian or polar or mixed. The mode setting only determines the output form of the answer.

$$\begin{aligned} 1+i + \sqrt{(2)}e^{i45^\circ} &= 2.828427125e^{0.7853981634i} && \text{polar form} \\ &= 2+2i && \text{Cartesian form.} \end{aligned}$$

When you are inputting in polar form, it is VITAL that you are in Radian mode (see Section 23.6.1), even if you wish to input an angle θ in degrees.

23.3.4 Conversion between forms

To convert complex numbers from one form to the other, use the conversions in the $\boxed{\text{math}}$ CMPLX menu. This works whether you are in Cartesian or polar mode.

You can also use the CMPXANGD/CXANGDCE program (Section 23.3.2).

$1+i$ $\boxed{\blacktriangleright\text{Polar}}$ $\boxed{\text{enter}}$ gives $1.414213562e^{0.7853981634i}$ in Radian mode and $1.414213562e^{i45}$ in Degree mode.

$\sqrt{(2)}e^{i\pi/4}$ $\boxed{\blacktriangleright\text{Rect}}$ $\boxed{\text{enter}}$ gives $1+1i$ in Radian mode and an error in Degree mode^a (see Section 23.6.1).

^aor the wrong answer on a TI-84Plus

```

1+i▶Polar
1.414213562e^(0.785398163...
1+i▶Polar
.....1.414213562e^(45i.)
√(2)e^(iπ/4)▶Rect
.....1+1i
√(2)e^(iπ/4)▶Rect
Error

```

23.4 Powers and roots

23.4.1 Powers

Integer powers work as you would expect.

$$(1+2i)^4 = -7-24i.$$

$$(1+2i)^{-3} = -0.088+0.016i = -\frac{11}{125} + \frac{2}{125}i.$$

$$(\sqrt{2}e^{i\pi/4})^4 = 4e^{i\pi} = -4.$$

23.4.2 Roots

Unfortunately you only get one root of a complex number when you use square root, cube root, n th root and fractional powers. To find all the roots,⁵⁰ use the CMPXROOT/CXROOTCE program (see Section 23.6.2), which calculates and plots the n n th roots of a given complex number.

⁵⁰A complex number has n n th roots (n an integer).

23.5 Exercises

$$z_1 = 3 + 4i \quad z_2 = 2 + 3i \quad z_3 = \sqrt{(2)}e^{i\pi/4} \quad z_4 = e^{i\pi/2}$$

Find

- | | |
|----------------------|---|
| 1. $z_1 + z_2$ | 11. $\text{Im}(z_2)$ |
| 2. $2z_1 + 3z_2$ | 12. z_1^2 |
| 3. $z_1 - z_2$ | 13. z_1^4 |
| 4. $4z_1 - 2z_2$ | 14. $\sqrt{z_1}$ |
| 5. $z_1 z_2$ | 15. $z_3 z_4$ in polar form |
| 6. z_1 / z_2 | 16. z_3 / z_4 in polar form |
| 7. \bar{z}_1 | 17. z_1 in polar form |
| 8. $z_1 \bar{z}_1$ | 18. z_3 in Cartesian form |
| 9. $ z_1 ^2$ | 19. z_4 in Cartesian form |
| 10. $\text{Re}(z_1)$ | 20. $\sqrt{z_4}$ in polar and Cartesian forms |

Answers in Section 23.7.

23.6 Appendix

23.6.1 Input/output of complex numbers

TI-84CE

	Mode	Input	Output*	Correct?
$a + bi$	Radian	$\sqrt{(2)}e^{i\pi/4}$	$1 + 1i$	YES
		$\sqrt{(2)}e^{i45^\circ}$	$1 + 1i$	YES
	Degree	$\sqrt{(2)}e^{i45}$	ERROR	NO
		$\sqrt{(2)}e^{i45^\circ}$	ERROR	NO
$re^{i\theta}$	Radian	$\sqrt{(2)}e^{i\pi/4}$	$1 + 1i$	YES
		$1 + i$	$\sqrt{(2)}e^{i\pi/4}$	YES
		$\sqrt{(2)}e^{i45^\circ}$	$\sqrt{(2)}e^{i\pi/4}$	YES
	Degree	$1 + i$	$\sqrt{(2)}e^{i45}$	YES
		$\sqrt{(2)}e^{i45}$	ERROR	NO
		$\sqrt{(2)}e^{i45^\circ}$	ERROR	NO

* given in decimal form

TI-84Plus

Where the TI-84CE gives ERROR, the TI-84Plus (OS 2.55MP) gives a **wrong answer**.To input a complex number, **always use Radian mode**.Use the degree symbol ($\boxed{\text{angle}} \boxed{1}$) if the angle is in degrees.

Alternatively, use the CMPXANGD/CXANGDCE program to convert the polar form in radians to the Cartesian form and the polar form in degrees (Section 23.3.2).

23.6.2 CMPXROOT/CXROOTCE program

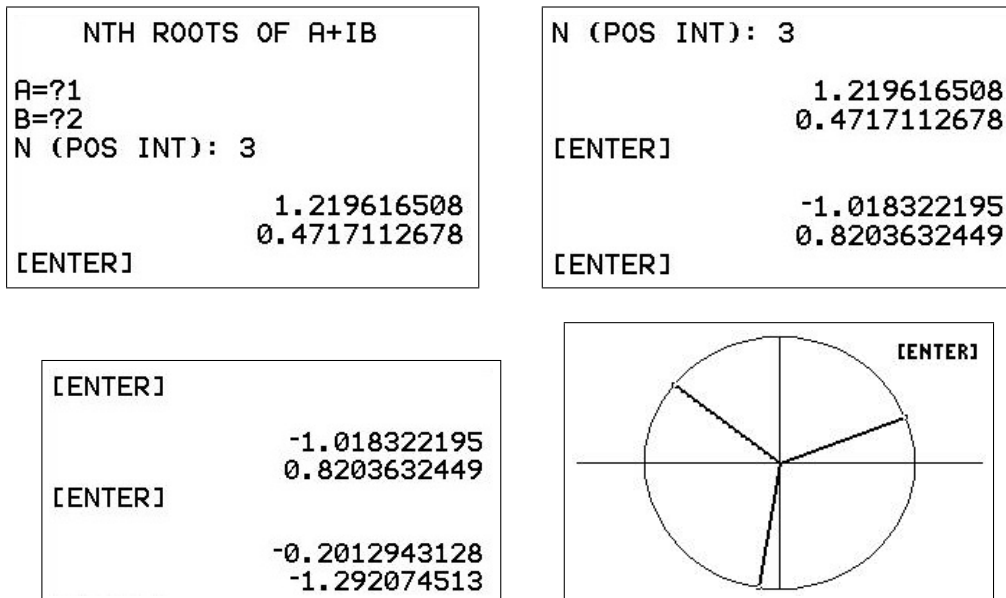
CMPXROOT/CXROOTCE⁵¹ displays x and y of the rectangular (Cartesian) form $x + iy$ and plots the N roots of the complex number $Z = A + iB$ on the Argand diagram. The roots all lie on a circle (their moduli are the same) which is drawn.

Lines are drawn from the origin to each root to show where the root is and to make the symmetry of the roots more obvious. Note that if a root lies on a co-ordinate axis, you won't see it, but it should be obvious from symmetry.

Use: Run the program. Input Z and N (positive integer) when prompted. The program displays each root. Press to continue at each step. The roots are also stored in list L6 for later use.

Move the cursor around the final plot using the arrow keys to see rectangular (x, y) values if RectGC is set in or polar (r, θ) values if PolarGC is set.

Example: The three cube roots ($N = 3$) of $1 + 2i$ are $1.220 + 0.472i$, $-1.018 + 0.820i$ and $-0.201 - 1.292i$, all rounded to three decimal places.



⁵¹available at www.XXX

23.7 Answers to Exercises

1. $5+7i$
2. $12+17i$
3. $1+i$
4. $8+10i$
5. $-6+17i$
6. $18/13-i/13$
▶Frac (`$\frac{1}{1}$`) is useful here
7. $3-4i$
8. 25
9. 25
10. 3
11. 3
12. $-7+24i$
13. $-527-336i$
14. $2+i$ and $-2-i$
15. $\sqrt{2}e^{i3\pi/4}$
16. $\sqrt{2}e^{-i\pi/4}$
17. $5e^{0.927295218i}$
18. $1+i$
19. i
20. $\pm e^{i\pi/4} \equiv \pm(1+i)/\sqrt{2}$

24 Programming

24.1 Introduction

A basic program structure is `INPUT` → `CALCULATION` → `OUTPUT`. The input is usually numbers (it might also be a function), the output can be text (e.g. ‘COMPLEX ROOTS’), numbers, a graph or a table.

The programming language of the TI-84/CE is a subset of the BASIC programming language, containing most of the useful features of that language together with the calculator graphics commands and a large library of scientific functions.

Most of the commands have to be selected from a menu — you cannot type them in letter by letter. `prgm` selected from the Home screen allows you to run, edit and create programs; `prgm` selected while editing a program gives access to most of the programming commands.

The hardest thing to start with is finding the appropriate commands on the keyboard and in menus.⁵² *On pages 176 and 177 of these notes is a list of key strokes for the commands used in the programs here.* The Table of Functions at the back of the TI-84 Guidebook is also very useful.

24.2 Basics

We start with a simple example and successively add features.

Input the radius of a circle, calculate and output its area.

24.2.1 Writing the basic program

Press the `prgm` key and use the left arrow to move to NEW. Press `enter`.

Enter the name of the program (up to 8 characters) — CIRCLE. Note that the calculator is in alpha mode.

Press `enter` and you are now ready to enter the program.

Press `prgm` and have a look at the commands in each of the sub-menus CTL (Control), I/O (Input/Output) and COLOR on the CE.

Here is our basic program. Press `enter` at the end of each line to go to the next line. Press `clear` in a menu to return to the program without selecting a menu item.

Prompt R `prgm` I/O `2` `alpha` `R`

$\pi R^2 \rightarrow A$ `2nd` `^` `alpha` `R` `x^2` `sto` `alpha` `A`

Disp A `prgm` I/O `3` `alpha` `A`

```
PROGRAM:CIRCLE
:Prompt R
:πR²→A
:Disp A
```

When you are happy with the program, press `2nd` `quit` to return to the Home screen.

Press `prgm`, select the CIRCLE program by typing its number or with the cursor and `enter`. This displays *prgmCIRCLE* on the Home screen.

⁵²If you're really stumped, all the commands are in the `catalog` menu (`2nd` `0`): press a letter key (*don't* press `alpha` first) to go to commands starting with that letter. You can scroll upwards from the top or down to the bottom to get to the commands and characters that don't contain letters.

Press `enter` to run the program. Try some R values (R = 1 should give a recognisable area) to make sure the program is running correctly.

```
prgmCIRCLE
R=?1
3.141592654
```

If you press `enter` after the program has finished, you will rerun the program — the calculator executes the last command again, making it easy to enter successive R values.

When a program is running, you will see continuously moving vertical bars (84) or a moving yellow circle (CE) in the top right corner of the screen.

24.2.2 Making corrections

If you have made a mistake in entering the code, the calculator may detect the error and stop with an error message ERR: *message* and a choice of quitting (returning to the Home screen) or going to the error. Select the second option by pressing `2` and the calculator will take you to the problem in the program.

If the program runs OK, but gives the wrong output, press `prgm`, select EDIT, then the program. This allows you to edit the program.

Use the `del` and `ins` keys to edit within a program line. `clear` clears a whole line but leaves the blank line. `del` on a blank line deletes the line.

24.2.3 Making the program more user-friendly

It makes the program easier to use if there isn't clutter on the screen and if the program gives some idea of what it does, what input it requires and what it outputs.

Commands that are new or changed from the previous program are highlighted.

The `ins` key allows you to insert characters in a line.

`ins enter` at the beginning or end of a line gives a blank line.

`2nd alpha` gives alpha LOCK (for letters), `alpha` turns it off.

`ClrHome` `prgm` I/O `8` *clear the Home screen*

`Disp "AREA_OF_A_CIRCLE"` `prgm` I/O `3` `2nd alpha` `+` `A` `R` ...
space `is` `alpha` `0`

`Input "RADIUS:", R` `prgm` I/O `1`; colon `:` is `alpha` `.`

$\pi R^2 \rightarrow A$

`Disp "AREA", A`

Run this version.

```
PROGRAM: CIRCLE
:ClrHome
:Disp "AREA OF A CIRCLE"
:Input "RADIUS: ",R
: $\pi R^2 \rightarrow A$ 
:Disp "AREA",A
```

```
AREA OF A CIRCLE
RADIUS: 1
AREA
3.141592654
```

24.3 Looping

24.3.1 Looping for input

Maybe we'd like to enter a number of values for R without having to rerun the program each time. We do this with a simple loop using a *Label* and a *Goto*.

ClrHome

Disp "AREA OF A CIRCLE"

Lbl 1 `prgm` `9` `1`

Input "RADIUS:␣", R

$\pi R^2 \rightarrow A$

Disp "AREA", A

Goto 1 `prgm` `0` `1`

```
PROGRAM:CIRCLE
:ClrHome
:Disp "AREA OF A CIRCLE"
:Lbl 1
:Input "RADIUS: ",R
: $\pi R^2 \rightarrow A$ 
:Disp "AREA",A
:Goto 1
```

```
AREA OF A CIRCLE
RADIUS: 1
AREA
                                3.141592654
RADIUS: 2
AREA
                                12.56637061
RADIUS:
```

This makes entering a number of radius values easier, but how do we stop the program?

Press `on` and select *Quit*. This stops any program and returns to the Home screen. Use `on` *Goto* to go back to edit the program — useful during its development.

24.3.2 Looping to check input

Another place for a loop is to test whether an input value falls within a specified range. In our example, we expect $R \geq 0$ (although it doesn't matter in this case because we square R). Let's use a loop to test for this. We also introduce the *If* statement here.

ClrHome

Disp "AREA OF A CIRCLE"

Lbl 1

Input "RADIUS:␣ ", R

If R < 0 *If* is `prgm` `1`; < is `2nd` `test` (`math` key) `5`

Goto 1 *Do this step if the If is true, otherwise skip this step*

$\pi R^2 \rightarrow A$

Disp "AREA", A

Goto 1

Check your program with suitable inputs, i.e. at least one negative R.

See Extension 1 on page 170 for another way to use such a loop and Extension 2 on page 170 for a more elegant error check. Make a copy of your program here (see the last chapter in Volume 2 of this book) so you can come back to these extensions later.

```
PROGRAM:CIRCLE
:ClrHome
:Disp "AREA OF A CIRCLE"
:Lbl 1
:Input "RADIUS: ",R
:If R<0
:Goto 1
: $\pi R^2 \rightarrow A$ 
:Disp "AREA",A
:Goto 1
```

24.3.3 Including an error message

Let's say we want to tell the user that he or she has made a mistake in the input. Such diagnostics are always useful, especially for students.

Here we embellish the *If* statement with a *Then* and *End*, and introduce *Pause*, which stops the program until the user presses .

That a program has Paused is indicated by shimmering dots (84) or a rotating white circle (CE) in the top right corner of the screen.

Note that anything in inverted commas is text, and can be whatever you want.

```
ClrHome
```

```
Disp "AREA OF A CIRCLE"
```

```
Lbl 1
```

```
Input "RADIUS: = ", R
```

```
If R < 0
```

```
Then      prgm 2
```

```
Disp "*** MAKE R≥0 ***" * is  used for emphasis here
```

```
Pause    prgm 8
```

```
Goto 1
```

```
End      prgm 7 ends the If Then
```

```
 $\pi R^2 \rightarrow A$ 
```

```
Disp "AREA", A
```

```
Goto 1
```

Don't forget suitable tests for this version of the program.

```
PROGRAM: CIRCLE
:Input "RADIUS: ",R
:If R<0
:Then
:Disp "***MAKE R≥0***"
:Pause
:Goto 1
:End
: $\pi R^2 \rightarrow A$ 
:Disp "AREA",A
```

```
AREA OF A CIRCLE
RADIUS: 1
AREA
3.141592654
RADIUS: -1
***MAKE R≥0***
RADIUS: 2
AREA
12.56637061
RADIUS:
```

24.3.4 Looping through a range of input values

Here we use a *For* loop to carry out the calculations for a specified range of (equally spaced) input values. The *For* command has the format

For (R, F, L, S)

commands

End

Variable R starts with a value of F and *commands* are executed. R is then increased by S and *commands* are executed again. This continues until $R > L$.⁵³

Remove the lines (use `clear` `del`) that test for $R < 0$ to keep the program simple.

ClrHome

Disp "AREA OF A CIRCLE"

Input "FIRST R: ", F *prompt for the initial value of R*

Input "LAST R: ", L *prompt for the final value of R*

Input "R STEP: ", S *prompt for the increment in R*

For (R, F, L, S) `prgm` `4`

$\pi R^2 \rightarrow A$

Disp "R, AREA", R, A

Pause *pause to read answer*

End *end For loop*

Run the program. Press `enter` after each output to continue.

```
PROGRAM:CIRCLE
:Disp "AREA OF A CIRCLE"
:Input "FIRST R: ",F
:Input "LAST R: ",L
:Input "R STEP: ",S
:For(R,F,L,S)
: $\pi R^2 \rightarrow A$ 
:Disp "R, AREA",R,A
:Pause
:End
```

```
FIRST R: 1
LAST R: 5
R STEP: 1
R, AREA
                                     1
                                     3.141592654
R, AREA
                                     2
                                     12.56637061
```

⁵³S can be negative, so that R is decreased. In this case, L must be less than F, and the program stops when $R < L$.

24.4 Displaying results as a table

Rather than displaying the areas one by one as we did in the previous example, we can put them in a table. Then we can scroll up and down to see all the answers.

To make up a table, we need to put the R and A values in lists. Here we use list L₁ for R and list L₂ for A.

Lists L₁–L₆ are built in to the calculator and can be accessed from the keyboard. You can also give lists names of up to five characters.

The variable I in the following program is the list index or counter — we have to increment it by 1 each time we store a value in a list.

ClrHome

SetUpEditor `[stat] [5]` *ensures that lists L₁–L₆ are shown when we press `[stat]` EDIT after running the program*

ClrList L₁, L₂ `[stat] [4] [2nd] [1] [,] [2nd] [2]` *clears the lists*

1 → I *initialise the list index*

Disp "AREA OF A CIRCLE"

Input "FIRST R: ", F

Input "LAST R: ", L

Input "R STEP: ", S

For (R, F, L, S)

R → L₁(I)

πR^2 → L₂(I)

I+1 → I *increment the list index by 1*

End *end of For loop*

Disp "", "PRESS_STAT_EDIT" *tell the user how to view the answers; the inverted commas with no enclosed characters give a blank line in the output*

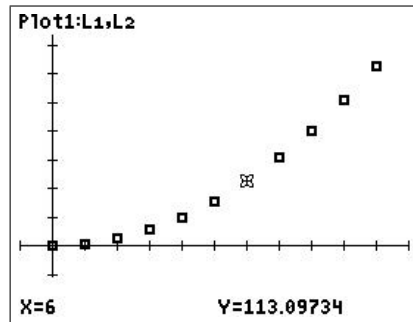
Run the program with R varying from 0 to 10 in steps of 1 and scroll down the L₂ column. Press `[2nd]` `[quit]` to return to the Home screen from the table.

```
PROGRAM: CIRCLE
:Input "FIRST R: ",F
:Input "LAST R: ",L
:Input "R STEP: ",S
:For(R,F,L,S)
:R→L1(I)
: $\pi R^2$ →L2(I)
:I+1→I
:End
:Disp "", "PRESS STAT EDIT"
```

```
AREA OF A CIRCLE
FIRST R: 0
LAST R: 10
R STEP: 1
PRESS STAT EDIT
```

L ₁	L ₂
0	0
1	3.1416
2	12.566
3	28.274
4	50.265
5	78.54
6	113.1
7	153.94
8	201.06
9	254.47
10	314.16

Run the program with R varying from 0 to 10 in steps of 1 to test it. Pressing the left/right arrow keys will move the cursor along the plotted points. Press `enter` to exit from the graph.



24.5.2 Graphing functions from a program

Graphing a function from a program is simple — the function has to be defined in `y=`, either before the program is run or from within the program. Be careful in a program that the correct function is turned on — use `FnOn` and `FnOff` to do this. If you define a function within a program, it is automatically turned on.

There are several commands you can use in a program to display a graph. All of these will cause any functions *and any plots* turned on to be graphed.

- Trace — displays the graph with a cursor you can move along the functions/points plotted, just as you would using `trace` manually. *Trace* pauses the program — press `enter` to continue.
- Input — `prgm` I/O `1` with no arguments. *Input* displays the graph with a cursor that you can move around the screen. *Input* pauses the program — press `enter` to continue.
- DispGraph — `prgm` I/O `4`. *DispGraph* displays the graph without a cursor. *DispGraph* does not pause the program — if it is not the last statement in the program, you will have to follow it with a *Pause*.
- DrawF — `draw` `6`. *DrawF* followed by a function definition either explicitly in terms of X (e.g. $3X+4$) or one of Y_1-Y_0 , draws that function. Use *Input* or *Pause* to view the graph. A function drawn with *DrawF* cannot be Traced. An advantage of *DrawF* is that it allows successive graphs to be superimposed — see Exercise 1.

To continue our example and do some curve fitting, let's fit a quadratic function to our data points and graph it together with the data.

The data-fitting commands are in the `stat` CALC menu. The *QuadReg* command determines the equation of the quadratic function that best fits the data (here a perfect fit), stores it in and turns on Y_1 . The *Trace* command causes Y_1 to be graphed and pauses the program.

```

ClrHome
FnOff
PlotsOff
ClrDraw
1 → Xscl
50 → Yscl
ClrList L1, L2
1 → I
Disp "AREA OF A CIRCLE"
Input "FIRST R: ", F
Input "LAST R: ", L
Input "R STEP: ", S
For (R, F, L, S)
R → L1(I)
 $\pi R^2$  → L2(I)
I+1 → I
End
Plot1 (Scatter, L1, L2, □)
ZoomStat
Trace    view the data points
QuadReg L1, L2, Y1  [stat] CALC [5]; Y1 is [vars] Y-VARS [1] [1]
Trace    view the data points and the fitted curve
Disp "", "PRESS STAT EDIT"

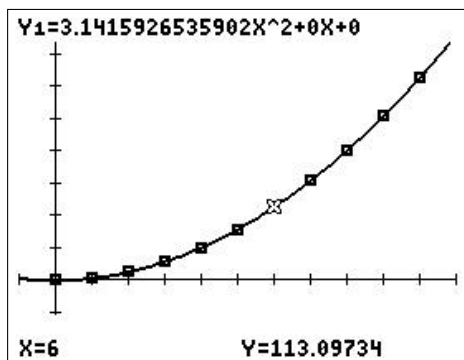
```

```

PROGRAM: CIRCLE
:  $\pi R^2 \rightarrow L_2(I)$ 
: I+1  $\rightarrow$  I
: End
: Plot1(Scatter, L1, L2, □)
: ZoomStat
: Trace
: QuadReg L1, L2, Y1
: Trace
: Disp "", "PRESS STAT EDIT"

```

Run this program with R varying from 0 to 10 in steps of 1 and see if you get the expected fit to the data points. Press `enter` after the points are plotted to see the fitted curve. Then try the up and down arrows as well as the left and right arrows. The line at the top of the screen tells you where the cursor is. Press `enter` to finish the program and look at Y_1 in `y=` to see the equation of the fitted curve.



24.5.3 Putting text on a graph

Sometimes it is useful to annotate graphs so that the user can understand what is being plotted or be prompted for the next key stroke(s).

As an example, in the graph of our last two versions of the program, the calculator pauses in Trace. Possible keys to press are the arrow keys to move along or between the graphs and `enter` to finish the program. We use the *Text* command to indicate this on the graph.

The *Text* command is of the form *Text* (*row*, *column*, "text" or variable). The coordinates *row* and *column* are pixel coordinates rather than the coordinates given by the window. On a TI-84, *row* varies from 0 (top of screen) to 57 (bottom of screen), while *column* varies from 1 (left of screen) to 94 (right of screen). On a TI-84CE, *row* varies from 0 (top of screen) to 150 (bottom of screen), while *column* varies from 0 (left of screen) to 255 (right of screen). It is usually necessary to experiment with the pixel coordinates to get the text in the right place.

Text can also be used to display the value of a variable — put the variable name in the command without inverted commas.

Any combination of text and variable values can be put together in a single *Text* command — just put them one after the other, separated by commas. This is useful if you want to display the name of a parameter and its value, e.g. *Text* (0, 2, "N=", N).

Note the use of a colon to join commands in the program below.

PTO

TI-84

ClrHome: ClrDraw

FnOff : PlotsOff

1 → Xscl: 50 → Yscl

ClrList L1, L2

1 → I

Disp "AREA OF A CIRCLE"

Input "FIRST R: ", F

Input "LAST R: ", L

Input "R STEP: ", S

For (R, F, L, S)

R → L1 (I)

πR^2 → L2 (I)

I + 1 → I

End

Plot1 (Scatter, L1, L2, □)

ZoomStat

Text(8, 13, "[ARROWS] [ENTER]")

Text is ; [is ,]

Trace

QuadReg L1, L2, Y1

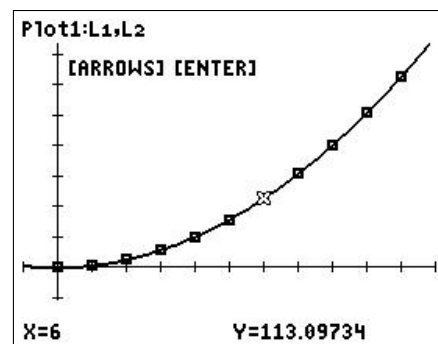
Text(8, 13, "[ARROWS] [ENTER]")

Trace

Disp "", "PRESS STAT EDIT"

TI-84CE: .

```
PROGRAM: CIRCLE
:ZoomStat
:Text(8,30,"[ARROWS] [ENT
ER]")
:Trace
:QuadReg L1,L2,Y1
:Text(8,30,"[ARROWS] [ENT
ER]")
:Trace
:Disp "", "PRESS STAT EDIT"
```



24.6 Simple animation

Animation is achieved by creating the separate frames, storing them in the picture memories Pic1, Pic2, . . . , Pic0, then recalling them one by one. Because of the way the CE recalls Pics, the following program gives a better result on the standard 84 than on the CE.

This program also illustrates the use of *getkey* to record a key press while a program (loop) is running — here we use the Q key to stop the program.⁵⁴

Program ANIMATE

FnOff: PlotsOff: ClrDraw

AxesOff (format); use the cursor and to select

ZStandard *standard axes*

ZSquare *same scale on each axis: circles are circular*

Shade ($-\sqrt{(64-X^2)}$, $\sqrt{(64-X^2)}$) *shade between the two semicircles*

Text (1, 1, "Q:␣QUIT") *on-screen prompt to tell user how to stop the program*

StorePic 1 STO menu *store the first screen in Pic1*

ClrDraw

Circle (0, 0, 8) menu *circle centre (0, 0), radius 8*

Text (1, 1, "Q:␣QUIT")

StorePic 2 *store the second screen in Pic2*

Lbl 1 *start animation loop*

ClrDraw

RecallPic 1 STO menu *recall the first screen*

For (I, 1, 150) *delay loop to view screen*

End *replace these two lines with Wait 0.15 on a CE*

ClrDraw *clear the graphics screen*

RecallPic 2 *recall the second screen*

For (I, 1, 120) *shorter delay loop to allow for the time taken for getkey test*

End *replace these two lines with Wait 0.12 on a CE*

If getKey = 74 I/O menu *getkey detects if a key has been pressed*

Goto 2 *if the Q key (key 74) has been pressed, go to Label 2 (tidy up and quit)*

Goto 1 *if the Q key hasn't been pressed, go back to Label 1 (start of loop)*

Lbl 2 *tidy up*

PTO

⁵⁴ Quit would stop it too, but there would be no 'tidying up' as here.

```

AxesOn
DelVar Pic1      [prgm] (scroll up)  [vars] [4] [1]
DelVar Pic2
ClrHome

```

24.7 Extensions

Extension 1

Follows on from the example on page 160. We could also use $R < 0$ as a way to stop the program, rather than using [on] Quit. Note the message telling the user this after the main heading.

```

ClrHome
Disp "AREA OF A CIRCLE" ,"R<0┘TO┘STOP"
Lbl 1
Input "RADIUS: ", R
If R < 0
  Stop      [prgm] [F] or scroll up and select
 $\pi R^2 \rightarrow A$ 
Disp "AREA", A
Goto 1

```

Extension 2

Follows on from the example on page 160. For a simpler, more elegant check on the input, we can use a *Repeat* loop.

```

ClrHome
Disp "AREA OF A CIRCLE"
Lbl 1
Repeat R ≥ 0 [prgm] [6] repeat down to End until the condition R ≥ 0 is true
Input "RADIUS: ", R
End
 $\pi R^2 \rightarrow A$ 
Disp "AREA", A
Goto 1

```


Extension 4

A program to move a cursor around the screen according to which arrow key is pressed.

```

StoreGDB 0      stores current graphics settings and functions
                 for later restoration: draw STO menu

FnOff: PlotsOff: ClrDraw

AxesOff        turns off the axes: format menu

CoordOff       turns off the display of the cursor coordinates: format menu

28 → A         vertical coordinate of initial cursor position   CE: 75 → A
45 → B         horizontal coordinate of initial cursor position CE: 127 → B
2 → S         distance the cursor will move at each step     CE: 5 → S

Text(A,B, "*" ) draw the cursor at the location (A,B): * is ×

Repeat K = 74   outer Repeat loop: loop until the Q key is pressed
Repeat K ≠ 0   inner Repeat loop: determines when a key has been pressed
getKey → K     store the key coordinate in K

End            end of inner Repeat loop

Text(A,B,"")
               erase the cursor at previous position   CE: 4 (spaces)

If K = 24: B - S → B   subtract S from B if key 24 (left arrow) has been pressed
If K = 25: A - S → A   subtract S from A if key 25 (up arrow) has been pressed
If K = 26: B + S → B   add S to B if key 26 (right arrow) has been pressed
If K = 34: A + S → A   add S to A if key 34 (down arrow) has been pressed

Text(A,B,"*")       draw the cursor at the current location given by A and B

Game commands go here, for example, to test whether the cursor is now in some previously
specified position where something is to be gained or lost, or has reached the edge
of the screen.

End            end of outer Repeat loop

ClrDraw       tidy up and stop

RecallGDB 0    restores the original graphics settings: draw STO menu

DelVar GDB0    delete GDB0: prgm CTRL menu; vars GDB menu

ClrHome

Disp "GAMEOVER"

```

24.8 Exercises

Possible solutions in Section 24.9.

If these exercises seem too hard, think up a suitable calculation and program it.

Exercise 1: Area of a geometric shape

Write a user-friendly program that calculates the area of a triangle, rectangle or circle, given appropriate inputs. *Hint:* see Extension 3.

Exercise 2: Family of curves

Assume that $Y_1 = AX^2 + B$ and that a suitable window has been set.

Your program called FAMILY should

- clear the Home screen
- turn off all functions
- turn off stat plots
- clear the graphics screen
- display a header on the Home screen
- start a loop (*Lbl*)
- input values for A and B
- draw the graph (*DrawF Y1*)
- pause to allow you to look at the graph
- return for another set of values for A and B (end of loop)

FAMILY allows you to see the effect on the function of changing the parameters A and B.

Use a window of $[-5, 5, 1] \times [-10, 10, 2]$. Try $A = 1, -1, 2, 0.5$ with $B = 0$. Use on Quit to stop the program and rerun with $A = 1$ and $B = 0, \pm 2, \pm 5$.

You can put in Y_1 any function containing two parameters A and B, and plot the corresponding family of curves using FAMILY. Try, for example, $Y_1 = A(X-B)^2$.

Exercise 3: Roots of a quadratic equation

Find the real roots of the quadratic equation $Ax^2 + Bx + C = 0$, where A , B and C are inputs. Display a message if the roots are complex (later you might like to extend the program to complex roots). Finally plot the quadratic: specify the x range in the program (X_{\min} , X_{\max}) — the roots should help here — and use *ZoomFit*, which calculates a suitable y range and plots the graph.

To put the quadratic function in Y_1 : " AX^2+BX+C " \rightarrow Y_1 . This works both within a program and from the Home screen (although entering the function using $\overline{y=}$ is clearly an easier option in the second case).

24.9 Possible solutions to the exercises

Exercise 1: Area of a geometric shape

Program AREA

```

Lbl M
ClrHome
Menu ("AREA OF A... ", "TRIANGLE", T,
      "RECTANGLE", R, "CIRCLE", C, "QUIT", Q)

Lbl T
Disp "    TRIANGLE"      CE: 9 (spaces)
Input "BASE:", B
Input "HEIGHT:", H
0.5BH → A
Goto D

Lbl R
Disp "    RECTANGLE"    CE: 8
Input "LENGTH:", L
Input "WIDTH:", W
LW → A
Goto D

Lbl C
Disp "    CIRCLE"       CE: 10
Input "RADIUS:", R
 $\pi R^2 \rightarrow A$ 
Lbl D
Disp "AREA", A
Output (8, 10, "[ENTER]")  CE: Output (10, 20, "[ENTER]")
Pause
Goto M

Lbl Q
ClrHome

```

Exercise 2: Family of curves**Program FAMILY**

```

ClrHome
ClrDraw
FnOff
PlotsOff
Disp "FAMILY_PROGRAM", "PARAMETERS_A,B"
Lbl 1
Prompt A, B
DrawF Y1
Input
Goto 1

```

Exercise 3: Roots of a quadratic equation**Program QUAD**

```

ClrHome
ClrDraw
FnOff
PlotsOff
"AX2+BX+C" → Y1
Disp "QUADRATIC_EQN", "AX2 + BX + C = 0"    CE: 6; 8
Prompt A, B, C
B2 - 4AC → D
If D < 0
Then
Disp "COMPLEX_ROOTS"
Stop
End
(-B + √(D))/(2A) → R
(-B - √(D))/(2A) → S
Disp "ROOTS", R, S
Pause
If R < S
Then
R-2 → Xmin
S+2 → Xmax
Else
S-2 → Xmin
R+2 → Xmax
End
0 → Xscl
ZoomFit

```

Once the program has finished, you can press trace or change the window and re-graph.

24.10 Keystrokes used in the programs

□	<code>2nd</code> <code>statplot</code> (<code>y=</code> key) MARK <code>1</code>
[<code>2nd</code> <code>×</code>
]	<code>2nd</code> <code>−</code>
:	<code>alpha</code> <code>□</code>
"	<code>alpha</code> <code>+</code>
=	<code>2nd</code> <code>test</code> (<code>math</code> key) <code>1</code>
<	<code>2nd</code> <code>test</code> (<code>math</code> key) <code>5</code>
≤	<code>2nd</code> <code>test</code> (<code>math</code> key) <code>6</code>
π	<code>2nd</code> <code>∧</code>
□ (space)	<code>alpha</code> <code>0</code>
√(<code>2nd</code> <code>x²</code>
→	<code>sto</code>
AxesOff	<code>2nd</code> <code>format</code> (<code>zoom</code> key) — use the cursor and <code>enter</code> to select
Circle(<code>2nd</code> <code>draw</code> (<code>prgm</code> key) <code>9</code>
ClrDraw	<code>2nd</code> <code>draw</code> (<code>prgm</code> key) <code>1</code>
ClrHome	<code>prgm</code> I/O <code>8</code>
ClrList	<code>stat</code> <code>4</code>
DelVar	<code>prgm</code> <code>alpha</code> <code>G</code> or scroll up and select
Disp	<code>prgm</code> I/O <code>3</code>
DrawF	<code>2nd</code> <code>draw</code> (<code>prgm</code> key) <code>6</code>
Else	<code>prgm</code> <code>3</code>
End	<code>prgm</code> <code>7</code>
FnOff	<code>vars</code> Y-VARS <code>4</code> <code>2</code>
For(<code>prgm</code> <code>4</code>
getKey	<code>prgm</code> I/O <code>7</code>
Goto	<code>prgm</code> <code>0</code>
If	<code>prgm</code> <code>1</code>
Input	<code>prgm</code> I/O <code>1</code>
L1	<code>2nd</code> <code>1</code>
L2	<code>2nd</code> <code>2</code>

Lbl	<code>prgm</code> <code>9</code>
Menu(<code>prgm</code> <code>alpha</code> <code>C</code> or scroll up and select
Output(<code>prgm</code> I/O <code>6</code>
Pause	<code>prgm</code> <code>8</code>
PlotsOff	<code>2nd</code> <code>statplot</code> (<code>y=</code> key) <code>4</code>
Prompt	<code>prgm</code> I/O <code>2</code>
Plot1(<code>2nd</code> <code>statplot</code> (<code>y=</code> key) <code>1</code>
QuadReg	<code>stat</code> <code>CALC</code> <code>5</code>
RecallPic	<code>2nd</code> <code>draw</code> (<code>prgm</code> key) <code>STO</code> <code>2</code>
Repeat	<code>prgm</code> <code>6</code>
Scatter	<code>2nd</code> <code>statplot</code> (<code>y=</code> key) <code>TYPE</code> <code>1</code>
SetUpEditor	<code>stat</code> <code>5</code>
Shade(<code>2nd</code> <code>draw</code> (<code>prgm</code> key) <code>7</code>
<i>space</i>	<code>alpha</code> <code>0</code>
Stop	<code>prgm</code> <code>F</code> or scroll up and select
StorePic	<code>2nd</code> <code>draw</code> (<code>prgm</code> key) <code>STO</code> <code>1</code>
Text(<code>2nd</code> <code>draw</code> (<code>prgm</code> key) <code>0</code>
Then	<code>prgm</code> <code>2</code>
Xscl	<code>vars</code> <code>1</code> <code>3</code>
Y1	<code>vars</code> <code>Y-VARS</code> <code>1</code> <code>1</code> or <code>alpha</code> <code>f4</code> <code>1</code>
Yscl	<code>vars</code> <code>1</code> <code>6</code>
ZoomFit	<code>zoom</code> <code>0</code>
ZoomStat	<code>zoom</code> <code>9</code>
ZSquare	<code>zoom</code> <code>5</code>
ZStandard	<code>zoom</code> <code>6</code>