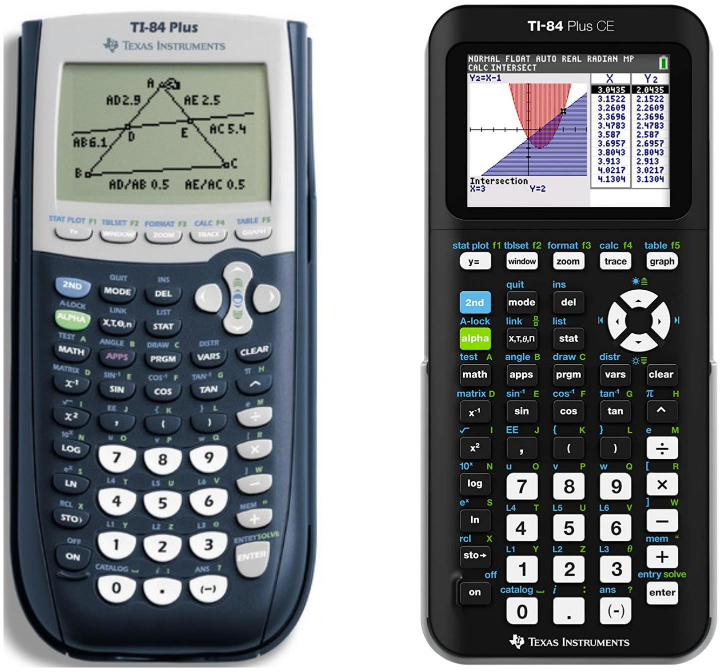
Mathematics on a TI-84/CE

Volume 1 Supplement

Activities for Years 9 and 10



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Contents

The activities *Probably Finding* π, *Reaction Times and Statistics* and *Statistics from Birthdays* are associated with Chapter 8, *Probability and Statistics 1*, of Volume 1 of *Mathematics on a TI-84/CE.* The remaining activities are associated with Chapter 3, *Coordinate Geometry*, of Volume 1.

Calculator versions

Currently (early 2022), TI-84 calculators come in two versions: the TI-84Plus and the more recent TI-84CE. The main difference is that the CE screen has much higher resolution. Calculations, screenshots and figures were done on a TI-84CE in CLASSIC mode.

Some programs have had to be changed for the CE because of the different screen: I usually append 'CE' to the program name to indicate this. All the programs here are available at *www.YYY*.

1 A Classic Problem — The Hare and Tortoise

Year 10, Level 1; Strand: Algebra; Sub-strand: Sketching Other Graphs.

The graphs of distances covered versus time in this classic race are used to answer various questions about the race, such as who won and by how much. A fun exercise in putting questions into maths and solving equations graphically.

A hare and tortoise compete in a one-kilometre race. The distance each competitor has travelled from the starting point is given by a formula. In time t **minutes**, the distance in **metres** travelled by the hare is given by $H(t) = \frac{500}{3}(2\sqrt{t} + \sqrt[3]{t})$, while the distance in **metres** travelled by the tortoise is given by $T(t) = 100t + 250\sqrt{t}$.

Press [y=] and enter the formulas for H and T. You have to use X ([X,T,θ,n]) as the independent variable. The cube root is [math] [4].

Set your [window] so that the two graphs go from the bottom left to the top right of the screen. *Hints*: The race takes about 5 minutes. How far is the race?

If you select *Simul* in the [mode] menu before graphing, you will get a real-time view of the race. Choosing the graph style of a circle with a tail[1] for each competitor makes it even better.

Answer the following questions, writing down the steps you took. Plotting the lines Y3 = 500, Y4 = 1000 and using *intersect* in the [calc] menu will be helpful. On an 84, you may need to increase *Ymax* when using *intersect* so that the function formulas do not obscure the point you are interested in.

1. Who gets to the halfway point first? How long does it takes them? Verify your answer algebraically.

2. What is the time and distance at which the two runners are neck and neck?

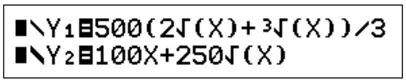
3. Who wins the race, by what time margin and by what distance margin?

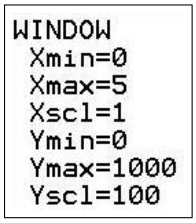
[1] Press [y=], move the cursor to the left of Y1 and press [enter]. On an 84, press [enter] a couple more times. On a CE, use the arrow keys on *Line*. Repeat for Y2.

Notes for Teachers

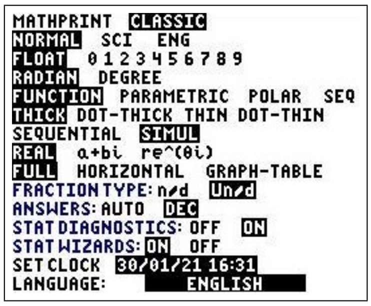
The questions in this version have been written in general terms deliberately for a good class. For a less-advanced class, students may need to be led a little through each question: *What equation do we need to solve to answer this question? What does this mean about the graphs of each side of the equation? How do we solve this equation on the calculator?* and so on.

Press [y=] and put the equation for the hare in Y1 and that for the tortoise in Y2. *Watch brackets here.* You might like to discuss with the class how to write the formulas in a suitable form for the calculator. Time t becomes X on the calculator.

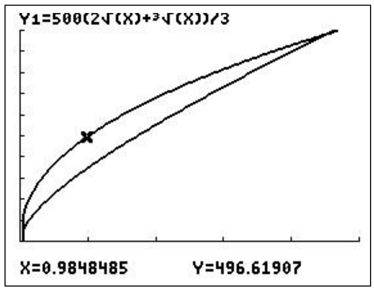




Then set the [window]. Discuss first with the class what each axis represents and suitable scales. The Y axis is distance in metres, so $0 < \mathrm{Y} < 1000$. The winner is then the competitor whose graph first reaches the top of the screen (providing *Simul* is set in [mode]).

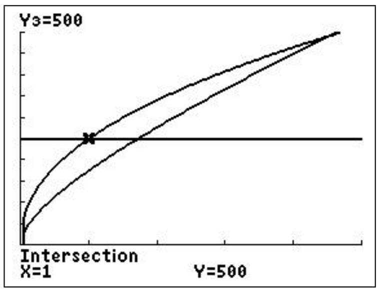


The time (X) scale has to be guessed. The race takes a little less than 5 minutes, so $0 < \mathrm{X} < 5$ gives a good view. Set *Xscl*, the distance between the tick marks on the X axis to 1 and *Yscl* to 100. If either of these is too small, you will get a double line for the axis.



1. From the graph (use [trace] and the up/down arrows to see which graph is which), the hare clearly reaches the halfway point (500 m) first.

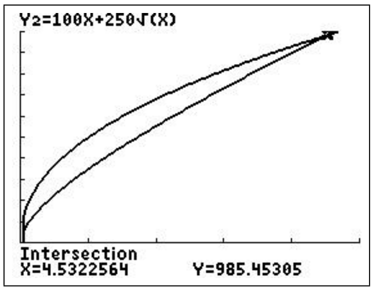
   To find how long the hare took, solve $H(t) = 500$ for t: set Y3 = 500 and find the intersection of Y1 and Y3 using *intersect* in the [calc] menu.
   After you select *intersect*, the cursor will be on Y1: press [enter] to select it. The cursor will move to Y2: press the down-arrow key to move it to Y3 and press [enter] to select it. Move the cursor to somewhere near the intersection and press [enter] to provide a guess.



   The value for t is 1 minute, a value we can confirm algebraically to be exact by substituting $t = 1$ into the equation for the hare. Note that it is easy to **verify** that $t = 1$ is a solution, but tricky to **solve** $H(t) = 500$ algebraically (it turns into a cubic equation).

   The hare reaches the halfway point first in a time of 1 minute.

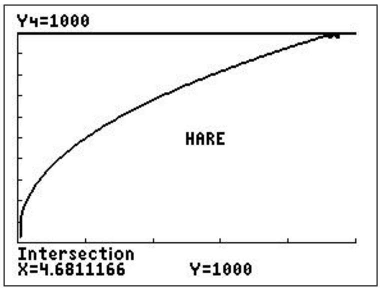
2. To find when they are neck and neck, we have to solve $H(t) = T(t)$, that is find the intersection of Y1 and Y2 (algebraically, this turns into a quartic equation). We obtain, using *intersect*, $t = 4.53$ minutes and distance equal to 985 m (both to 3 significant digits). It might be useful to increase *Ymax* or *Zoom In* (zoom 2) on this part of the graph to see the two curves more clearly.

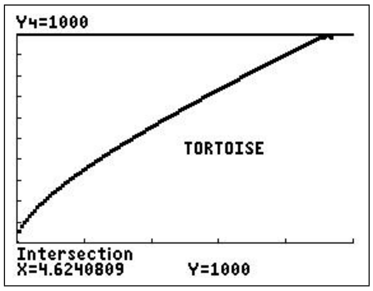


The hare and tortoise are neck and neck after about 4.53 minutes or about 4 minutes 32 seconds, at a distance of about 985 metres from the start.

3. To find the winner, we have to determine the time at which each competitor reaches the finish (1000 m).

Setting $Y_4 = 1000$, we find the hare finishes at $t = 4.681$ minutes (intersection of Y1 and Y4) and the tortoise finishes at $t = 4.624$ minutes (intersection of Y2 and Y4).





To find the distance margin, calculate $H(4.624)$, the position of the hare when the tortoise finishes: $H(4.624) \equiv Y_1(4.624) = 994.45$ m, rounded to 5 significant digits.

The tortoise wins the race by a margin of 0.057 minutes or 3.42 seconds. The distance margin is 5.55 m.

2 Alien Attack

Years 9 & 10, Levels 1 & 2; Strand: Algebra; Sub-strand: Coordinate Geometry.

Authors: Vanessa Moore and Sherry Morton, in Graphing Calculators in Mathematics Grades 7–12, Center of Excellence for Science and Mathematics Education, University of Tennessee. Modified by Peter McIntyre.

Uses one of Newton's equations of motion to explore properties of quadratic equations both numerically and graphically.

When an object is propelled straight upwards, gravity slows it down and eventually pulls it back to Earth. The graph of height vs time is a parabolic curve, even though the path of the object is straight up and down.

The height of the object as a function of time is given by one of Newton's equations of motion,

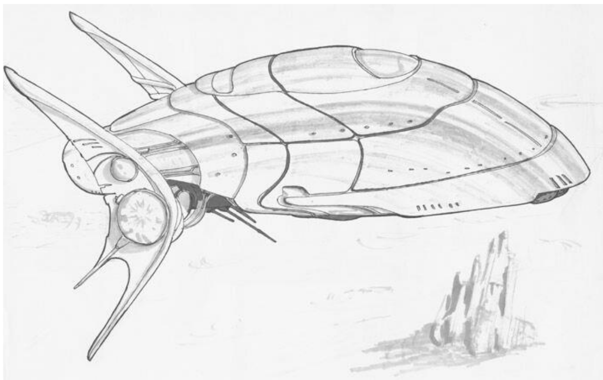
$$h(t) = h_i + v_i t - 0.5gt^2,$$

where h is the height above the ground at time t, h_i is the initial or starting height, v_i is the initial velocity and g is the acceleration due to gravity (a constant).

If we use SI units of height in metres and time in seconds, $g \approx 9.8\,\text{m/s}^2$.

The scene

An alien spaceship is hovering above the city at a height of 100 m. A giant slime blob is housed in a missile-like container. The alien ship launches the container straight up into the air at a velocity of 500 m/s and vanishes immediately into hyperspace.
What happens?



www.sheriftariq.org/markers/elegant_alienship.jpg

To answer this question, you need to put the equation for the height of the container as a function of time into your calculator.

(a) First write the equation for h in the space below, using the given values for h_i, v_i and g. You can probably also work out 0.5×9.8 to shorten the equation.

(b) The calculator uses X for the (independent variable) time and Y for the (dependent variable) height. Write the equation for h in terms of Y and X in the space below.

(c) Now press y= (top left).

Enter your equation into Y1 just as you have written it in (b). Press X,T,θ,n (in the third row) for X.

Make sure you use the blue minus sign $-$, not the white change sign $(-)$.

```
Y1=100+500X-4.9X²
```

Now you will use table to generate a table of values of the height function you have just entered.

(d) To tell the calculator which X values you want in the table, press tblset (2nd window).

Set TblStart = 0 and ΔTbl = 1: the table will start at time X = 0 and increment in steps of 1 s.

If they are not already selected, select *Auto* and *Auto* with the cursor and enter.

```
TABLE SETUP
 TblStart=0
 ΔTbl=1
Indpnt: Auto Ask
Depend: Auto Ask
```

(e) Press table (2nd graph).

You can scroll down in either column, but up past the top of the table only in the X column.

X	Y1
0	100
1	595.1
2	1080.4
3	1555.9
4	2021.6
5	2477.5
6	2923.6
7	3359.9
8	3786.4
9	4203.1
10	4610

X=0

Now answer each of the following questions in the space below the question.

1. What is the height of the container after 25 seconds?

2. Is the container going up or down after 25 seconds? How do you know?

3. What is the maximum height the container reaches?

4. After how many seconds does the container reach its maximum height?

5. How accurate is your value in Question 4? Look at the table and decide between which two X values you are sure the exact answer lies.

The exact time to maximum height is greater than ______ *, but less than* ______ .

6. To obtain a more accurate answer to Question 4, set ΔTbl = 0.1 in tblset. You may want to change TblStart too. Scroll in the Y1 column of the table to see the highlighted Y value with more digits at the bottom of the screen.

Find the time to maximum height, *accurate to 1 decimal place*, and the corresponding maximum height.
Hint: You may need to change ΔTbl more than once.

7. When the container hits the ground, the slime blob will envelop the city. How long does our local superhero[2] ____________________ have to come to the rescue?
Hint: Use the table to answer the question. Tenths of seconds are vital here.

Next we will look at graphs of height vs time to answer some more questions.

You already have the right function to graph because you used it for the table. You need to set a suitable window so the graph appears on your screen. Press [window].

8. What quantity does X represent here? Based on your explorations using [table], what are suitable values for Xmin and Xmax? What does Y represent here? What are suitable values for Ymin and Ymax?

Enter these into your calculator. Set Xscl (the distance between tick marks on the X axis) to 10 and Yscl to 5000. Press [trace].

Change your [window] values if necessary and re-graph so that the graph fills the screen.

9. Sketch your graph below, being sure to label the axes with what they actually represent and giving some idea of scale (one or two values on each axis).

[2] Supply an appropriate name.

10. What is the name of the point on the parabola that corresponds to maximum height? What are the approximate coordinates of this point? (Use the left- and right-arrow keys to determine this.) Does this agree with what you found using the table?

On a TI-84, you will probably find that the cursor coordinates cover the X axis. To fix this, change Ymin in [window] to about −1500 (which minus key?).
Press [trace].

11. Find the approximate time that the container hits the ground using the cursor in trace mode. How accurate is your answer? Does it agree with the answer you found using a table?

The exact time to ground is greater than __________, *but less than* __________.

12. A point at which a graph crosses the x axis (has y value 0) is called an x *intercept* or *zero*.

The time taken for the container to return to the ground is an x intercept of the height graph. In Question 11, you found an approximate value for this.

You will now use the *zero* operation on the TI-84/CE to find the time to ground more accurately.

Press [calc] ([2nd] [trace]) and select *zero*. The calculator asks for a left bound: move the cursor until it lies close to but to the left of the zero.

(a) *How do you know from the Y values when the cursor lies to the left of the zero?*

Press [enter]. Do the same for the right bound.

(b) *How do you know from the Y values when the cursor lies to the right of the zero?*

Finally use the cursor to give a guess for the zero. Here this doesn't have to be very close — the right bound (or any other value within the bounds) will do fine. Press [enter].

(c) *When does the container hit the ground?*
Give your answer to a *sensible* number of decimal places.

13. Does our superhero save the city? Finish the story.

Notes for Teachers

The equation for height as a function of time is

$$h(t) = 100 + 500t - 4.9t^2,$$

with the corresponding calculator equation and table of values shown in the figures.

1. *What is the height of the container after 25 seconds?*

   Scroll down the table to find that $h(25) = 9537.5$, so the container is at a height of 9537.5 m after 25 s.

2. *Is the container going up or down after 25 seconds? How do you know?*

   The container is still going up after 25 s, because the height is increasing with time then.

3. *What is the maximum height the container reaches?*

   Scroll down the table to find that the maximum height the container reaches is apparently 12,855 m.

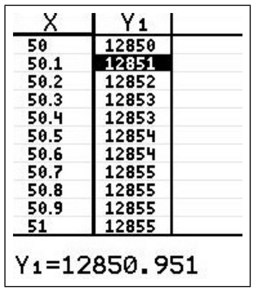
4. *After how many seconds does the container reach its maximum height?*

   According to the table, the container reaches maximum height after 51 s.

5. *How accurate is your value in Question 4? Look at the table and decide between which two X values you are sure the exact answer lies.*

   The exact time to maximum height is greater than 50 s, but less than 52 s.

   The table values are for integer numbers of seconds. We can only say for sure that the maximum time lies between 50 s and 52 s, with a best estimate of 51 s. You could draw a head-up parabola and discuss where the three points in the table near the maximum might lie, in particular that the highest point may not lie right at the vertex.



X	Y1
50	12850
50.1	12851
50.2	12852
50.3	12853
50.4	12853
50.5	12854
50.6	12854
50.7	12855
50.8	12855
50.9	12855
51	12855

Y1=12850.951

6. Using table to obtain more accurate answers.

   By changing ΔTbl in tblset to 0.1, we can see the height every 0.1 s. It's a good idea to set TblStart = 50 so you don't have to scroll through too many values.

   Find the time to maximum height, accurate to one decimal place*, and the corresponding maximum height.*

   The time to maximum height lies between 50.9 s and 51.1 s, so we can only say the answer is 51 s, accurate to 0 decimal places. Changing ΔTbl in tblset to 0.01 (and TblStart to 51.0 say), we find the time to maximum height lies between 51.01 s and 51.03 s, both 51.0 s, accurate to 1 decimal place.

   The corresponding maximum height is 12,855 m, rounded to the nearest metre.

7. *When the container hits the ground, the slime blob will envelop the city. How long does our local superhero ________________________ have to come to the rescue?*
Hint: Use the table to answer the question. Tenths of seconds are vital here.

Use the same method to find when $h = 0$ as you did to find the maximum value of h: with ΔTbl $= 1$, scroll down the table until Y1 changes sign; use ΔTbl $= 0.1$ to find the time accurate to one decimal place.

The time at which the container hits the ground lies between 102.2 s and 102.3 s, with a best estimate of 102.2 s, accurate to the nearest tenth of a second.

Next we will look at graphs of height vs time to answer some more questions.

*Here it would be good to run the SLIME/SLIMECE program on the overhead projector (or have them run it on their calculators) and discuss what you see. See page **??** for details.*

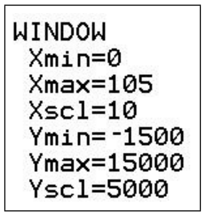
You already have the right function to graph because you used it for the table. You need to set a suitable window so the graph appears on your screen. Press [window].

8. *What quantity does X represent here? Based on your explorations using [table], what are suitable values for Xmin and Xmax? What does Y represent here? What are suitable values for Ymin and Ymax?*

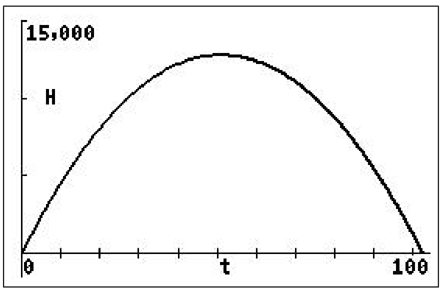
X represents time, so that Xmin $= 0$ is the starting time. From our results using the table, we know that the container hits the ground after about 102 s, so set Xmax $= 105$.

Y represents height above the ground, so that Ymin $= 0$ is ground level. Again from our table results, we know maximum height is about 12,855, so set Ymax $= 15000$ say.

Enter these into your calculator. Set Xscl (the distance between tick marks on the X axis) to 10 and Yscl to 5000. Press [trace].



9. *Sketch your graph below, being sure to label the axes with what they actually represent and giving some idea of scale (one or two values on each axis).*[3]



[3]Note that Ymin has been changed to -1500 here to allow the t label to be drawn.

10. *What is the name of the point on the parabola that corresponds to maximum height? What are the approximate coordinates of this point? (Use the left- and right-arrow keys to determine this.) Does this agree with what you found before?*

The point on the parabola that corresponds to maximum height is called the vertex. Its coordinates are approximately $(51, 12855)$,[4] roughly the values we found from the table for maximum height.

For better students, insert *Finding the equation of the parabola* (page **??**) here.

On a TI-84, you will probably find that the cursor coordinates cover the X axis. To fix this, change Ymin in window to about minus a tenth of Ymax, i.e. to -1500 (change-sign key). Press trace.

11. *Find the approximate time that the container hits the ground using the cursor in trace mode. How accurate is your answer?*

The time to ground is greater than about 101.6 s, but less than about 102.8 s.

These numbers, the X coordinates of adjacent pixels, will vary a little, depending on what values you put in window. The best estimate is 102 s.

12. A point at which a graph crosses the x axis (has y value 0) is called an *x intercept* or *zero*.

The time taken for the container to return to the ground is an x intercept of the height graph. In Question 11, you found an approximate value for this.

We will now use the *zero* operation on the TI-84/CE to find the time to ground more accurately.

Press calc (2nd trace) and select *zero*. The calculator asks for a left bound: move the cursor until it lies close to but to the left of the zero.

(a) *How do you know from the Y values when the cursor lies to the left of the zero?*

To the left of the zero, the Y values are positive.

Press enter. Do the same for the right bound.

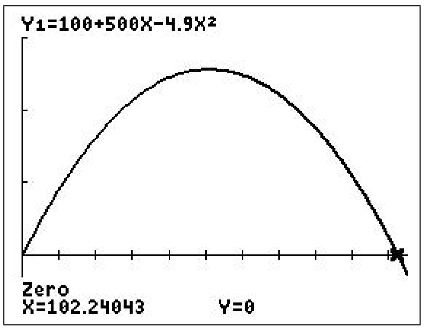
(b) *How do you know from the Y values when the cursor lies to the right of the zero?*

To the right of the zero, the Y values are negative.

Finally use the cursor to give a guess for the zero. Here this doesn't have to be very close — the right bound (or any other value within the bounds) will do fine. Press enter.

(c) *When does the container hit the ground?* Give your answer to a sensible number of decimal places.

The container hits the ground at 102.2 s, rounded to 1 decimal place.



[4] In trace, type 51 and press enter to go to this point.

Further questions

What happens if the initial velocity is changed to

(a) 1500 m/s? *Answer*: max at 153.1 s, 114, 896 m; reaches the ground at 306.2 s.

(b) 3000 m/s? *Answer*: max at 306.1 s, 459, 284 m; reaches the ground at 612.3 s.

(c) 5000 m/s? *Answer*: max at 510.2 s, 1, 275, 610 m; reaches the ground at 1020.4 s.

The SLIME/SLIMECE program

The program shows the action in real time.

- Press [prgm] and press the number against the SLIME/SLIMECE program;[5] press [enter] to run the program. The program plots two graphs simultaneously.
- Press [enter] while the graphs are being drawn if you want to stop the graphing, [enter] again to restart.
- Once the graphs are finished, you can re-live the action in slow motion. Move forward in time with the right arrow, backward in time with the left arrow and between graphs with the up/down arrows. Try this to see corresponding points on the graphs. The cursor coordinates are shown at the bottom of the screen, the relevant ones being time T and height Y.

  Why are the two graphs different? What does each one represent?

  The left-hand graph is a plot of distance versus time. The right-hand graph is the actual trajectory.
- Once you have finished looking at the graphs, press [enter] to stop the program. To rerun it, just press [enter] again.

The SLIME/SLIMECE program is available at *www.YYY*.

[5] or select with the cursor and press [enter].

Finding the equation of the parabola

This follows on after Question 10. Suitable for better students.

If we drew a vertical line through this point, we would have an axis of ______________ of the parabola.

What is the equation of this line?

What is happening to the slime-blob container on the left-hand side of this line? on the right-hand side?

Using this information and a point on the parabola, write the equation of the parabola in the form $y = a(x-b)^2 + c$.

Check your answer first by entering your equation into Y_2 and graphing both functions using trace . Are the two functions the same?
The arrow keys will help here. What does the up/down-arrow key do?

If you go back to just before Question 11 now, turn off Y_2 in [y=] by moving the cursor over its = sign and pressing [enter]. Press [trace] to regraph Y_1.

Second, expand out the brackets in your function here and confirm you obtain the original function. (The coefficients you get doing this may only be approximately correct, because we truncate the numbers we take from either the table or the graph.)

Solutions

The equation of the vertical line is $x \approx 51$. On the left-hand side of the line the container is going up, on the right-hand side going down.

The axis of symmetry of the parabola is $x=b \approx 51$, so that $y \approx a(x-51)^2+c$.

When $x=51$, $y=c\approx 12855$, so that $c\approx 12855$. Therefore, $y \approx a(x-51)^2+12855$.

We have $y(0)=100$. Therefore,

$$\begin{aligned} 100 &= a(0-51)^2 + 12855 \\ &= 2601a + 12855. \\ \therefore\ 2601a &= -12755. \\ \therefore\ a &= -4.9. \end{aligned}$$

Therefore, the equation of the parabola is $y \approx -4.9(x-51)^2+12855$.

The two graphs should be the same. Check by toggling between the two curves using the up- or down-arrow key. Look at the Y values at the bottom of the screen. Try this at several points along the curves (use the left- or right-arrow key to do this).

Expanding out the brackets,

$$\begin{aligned} y &= -4.9(x-51)^2 + 12855 \\ &= -4.9(x^2-102x+2601) + 12855 \\ &= -4.9x^2 + 4.9 \times 102x - 4.9 \times 2601 + 12855 \\ &= -4.9x^2 + 499.8x + 110.1 \\ &\approx -4.9x^2 + 500x + 100, \quad \text{the original equation.} \end{aligned}$$

If we used a better approximation for x, say $x \approx 51.02$ rather than 51, we obtain

$$y \approx -4.9x^2 + 499.996x + 100.102,$$

very close to the original equation.

3 Best Shape for a Can

Year 10, Level 1; Strand: Algebra/Measurement; Sub-strand: Sketching Other Graphs/Volume.

From: Integrating the Graphics Calculator into Years 9 and 10 of the Victorian Mathematics CSF, Teachers Teaching with Technology (T^3), 1998. Modified by Peter McIntyre.

Minimising the surface area of a cylinder (can) for a fixed volume. Numerical and graphical techniques, rather than Calculus, are used to find the minimum. Aspects of mathematical modelling are introduced.

When manufacturers are designing their packaging, they must keep in mind the amount of product that has to fit inside and the amount of material it will take to make the package. Consider the humble soft-drink can. The standard volume is 375 mL or 375 cm^3. Any number of cans can be designed that will hold this volume of liquid, but they will vary in shape and therefore in the amount of material needed to make the can (and therefore cost).

The formula for the volume of a cylinder V in terms of radius r and height h, is

$$V = \pi r^2 h.$$

Rearrange the volume formula to make h the subject and let the volume be 375 cm^3.
What are the units of r and h?

$h =$

Enter this formula as Y_1, with X as radius r.
As a check, enter the volume formula: $Y_2 = \pi X^2 Y_1$.
Y_1 is in the [vars] Y-VARS Function menu.

```
■\Y1■375/(πX²)
■\Y2■πX²Y1
```

Press [tblset] ([2nd] [window]).
Set TblStart = 1 and ΔTbl = 1.
Press [table] ([2nd] [graph]).
Do you get the correct value for the volume in Y_2?

```
TABLE SETUP
 TblStart=1
 ΔTbl=1
Indpnt: Auto Ask
Depend: Auto Ask
```

Write down the formula for the surface area of a cylinder, including the ends.
The surface area determines the amount of material needed to make the can. *Why?* Enter the formula for surface area in Y_2 in terms of X (radius) and Y_1 (height).

SA =

$Y_2 =$

View the table of values again. *What do you notice about the values of the surface area?*

Use [tblset] to set a new TblStart and ΔTbl = 0.1 to find the minimum surface area and the corresponding radius (radius accurate to 1 decimal place).

Minimum SA =

Radius =

Now graph the surface area as a function of radius, and use *minimum* in the calc menu to find the minimum.

Draw your graph here.

Write down your values for the radius, height, surface area and circumference of the can when the surface area is a minimum. *Do these values seem reasonable?*

How does this compare with a can of soft drink? Why the differences? What about other cans?

You might like to read the article *The Best Shape for a Tin Can* by PL Roe, either in *The Mathematical Gazette* 75, 472 (1991) or reprinted in *The College Mathematics Journal* 24, 233 (1993). Search for it online.

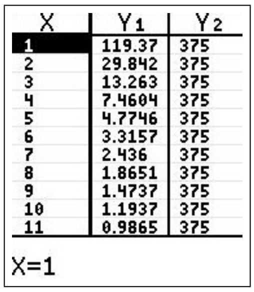
Notes for Teachers

There are a number of similar maximum/minimum activities for all years.

The height of the cylinder is given by $Y_1 = 375/(\pi X^2)$, where X is the radius r. The brackets are essential here because of the order in which the calculator does the different operations.

As a check, enter the volume formula $Y_2 = \pi X^2 Y_1$.

Press [table].



X	Y1	Y2
1	119.37	375
2	29.842	375
3	13.263	375
4	7.4604	375
5	4.7746	375
6	3.3157	375
7	2.436	375
8	1.8651	375
9	1.4737	375
10	1.1937	375
11	0.9865	375

X=1

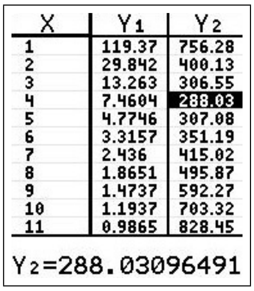
The total volume of the metal used to make the can, assuming the walls are of uniform thickness, is just the surface area times the thickness. Minimum surface area therefore means minimum volume of metal.

The surface area of a cylinder, including the ends, is given by

$$\text{SA} = 2\pi r^2 + 2\pi r h = 2\pi r(r+h).$$

View the table of values again. What do you notice about the values of the surface area?

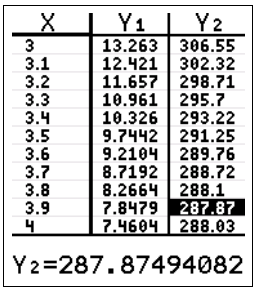
The surface area decreases then increases as the radius increases. There is a (local) minimum.



X	Y1	Y2
1	119.37	756.28
2	29.842	400.13
3	13.263	306.55
4	7.4604	288.03
5	4.7746	307.08
6	3.3157	351.19
7	2.436	415.02
8	1.8651	495.87
9	1.4737	592.27
10	1.1937	703.32
11	0.9865	828.45

Y2=288.03096491

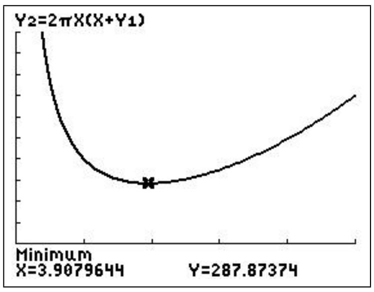
With TblStart = 3 and ΔTbl = 0.1, we find a radius of 3.9 cm for minimum surface area.



X	Y1	Y2
3	13.263	306.55
3.1	12.421	302.32
3.2	11.657	298.71
3.3	10.961	295.7
3.4	10.326	293.22
3.5	9.7442	291.25
3.6	9.2104	289.76
3.7	8.7192	288.72
3.8	8.2664	288.1
3.9	7.8479	287.87
4	7.4604	288.03

Y2=287.87494082

Alternatively, one can view the graph of surface area versus radius and use *minimum* in the [calc] menu to find the minimum.



window $[0, 10, 2] \times [0, 1000, 100]$

Again, $r = 3.9$ cm (accurate to 1 decimal place) for the minimum surface area.

If your students have sufficiently developed calculus skills, they could prove algebraically that the global minimum lies at $r=\sqrt[3]{375/2\pi}\approx 3.9$. More generally, for a given volume V, it is not too hard to show that $h=2r$ (height = diameter) for the least surface area for a given volume.

Is this a reasonable answer for the radius?

Other factors may have to be taken into account such as what circumference is comfortable for the average human hand, the wastage of material when cutting the ends and the cost of making the joins. Some of these could be included in a more-complex model.

The values found here are:

radius $r = \sqrt[3]{\dfrac{375}{2\pi}} \approx 3.9\,\text{cm}$ height $h = \sqrt[3]{\dfrac{1500}{\pi}} \approx 7.8\,\text{cm}$ ratio $\dfrac{h}{r} = 2$

surface area $\approx 288\,\text{cm}^2$ circumference $\approx 24.6\,\text{cm}$

How do this compare with a can of soft drink?

A standard 375 mL soft-drink can has a radius of 3.25 cm and a height of 13 cm: $h/r=4$. Its surface area is about $332\,\text{cm}^2$ and circumference 20.4 cm.

The article *The Best Shape for a Tin Can* by PL Roe, either in *The Mathematical Gazette* 75, 472 (1991) or reprinted in *The College Mathematics Journal* 24, 233 (1993)[6] goes into why there might be differences between the theory here and the actual values. A good example of mathematical modelling.

[6] search for it online.

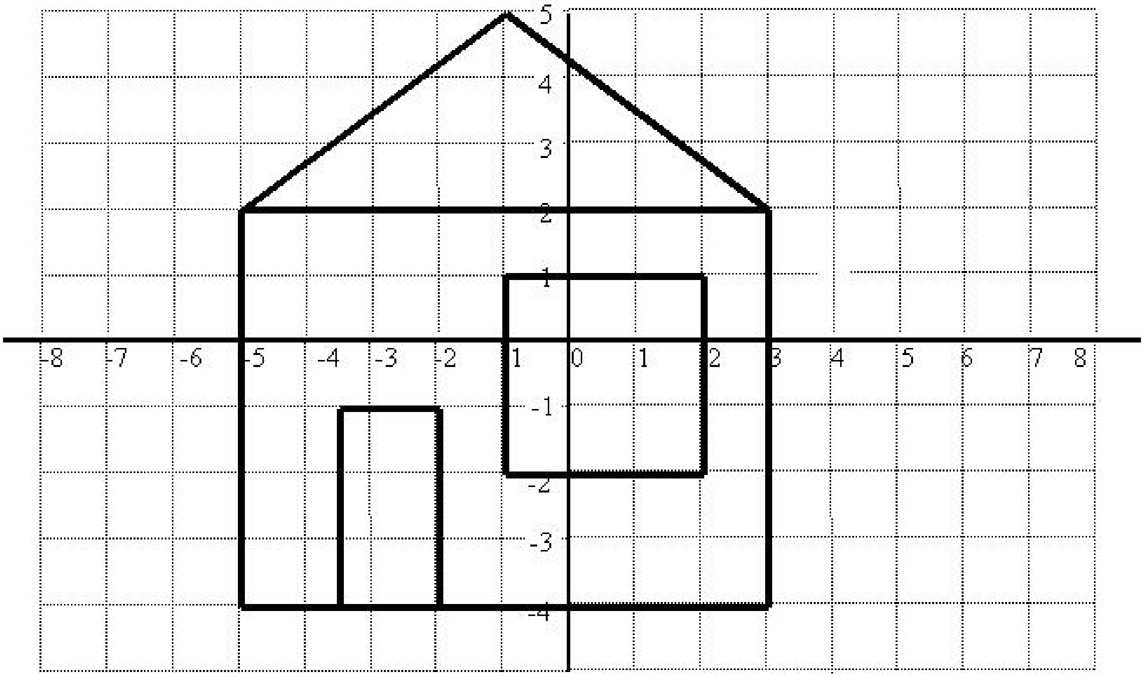
4 Coordinate Geometry Art

Year 9, Levels 1 & 2; Strand: Algebra; Sub-strand: Coordinate Geometry.

*Author: Rex Boggs in The Sub-AToMIC Project, V Geiger et al, QAMT, 1999.
Modified by Peter McIntyre.*

A simple picture consisting of straight-line segments is 'coded' using the coordinates of its vertexes. These are used 'transmit' the picture to someone else. A graphics calculators is used 'decode' and check the 'transmitted' picture.

Consider the beautiful work of art below.



Instructions for drawing this picture

This picture can be drawn by joining the points below in the given order. No line or part of a line is drawn twice. The instruction *lift pencil* means do not join the previous point to the next point.

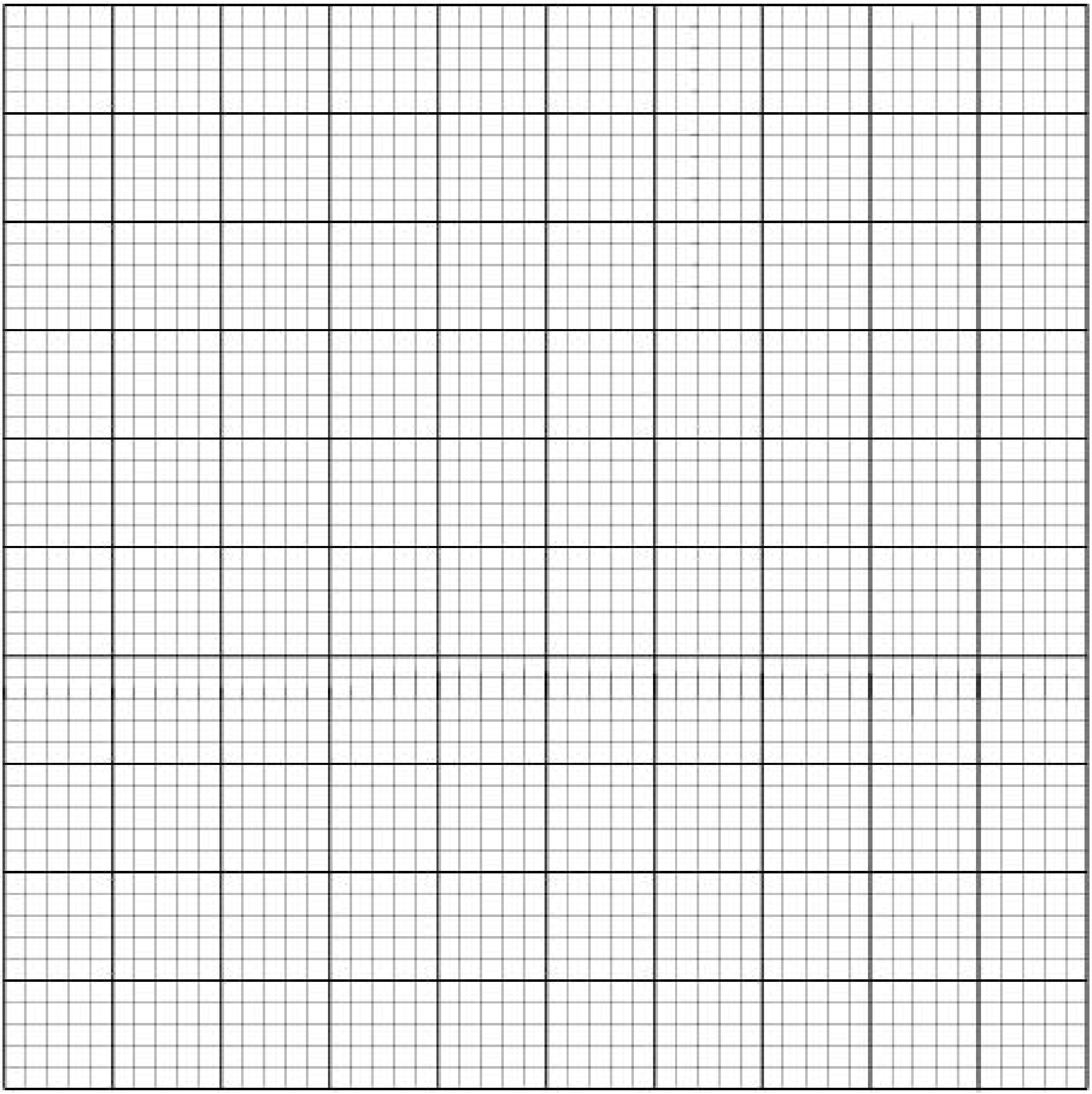
$(-2,-4)$	$(-2,-1)$	$(2,1)$
$(-5,-4)$	$(-3.5,-1)$	$(2,-2)$
$(-5,2)$	$(-3.5,-4)$	$(-1,-2)$
$(-1,5)$	*lift pencil*	$(-1,1)$
$(3,2)$	$(-5,2)$	$(2,1)$
$(3,-4)$	$(3,2)$	
$(-2,-4)$	*lift pencil*	

1. Create your own drawing on grid paper (next page). The rules for this are:

- all line segments must be straight;
- use no more than two *lift pencil* instructions. Sometimes a clever choice of starting point can reduce the number of *lift pencil* instructions.

2. Give a set of instructions as a table of points like that one above so someone else can reproduce your drawing. No line segments are to be drawn twice.

3. Enter your coordinates into a graphics calculator and draw the picture on the screen. Your teacher will show you how. This is a good way to check that your instructions are correct.

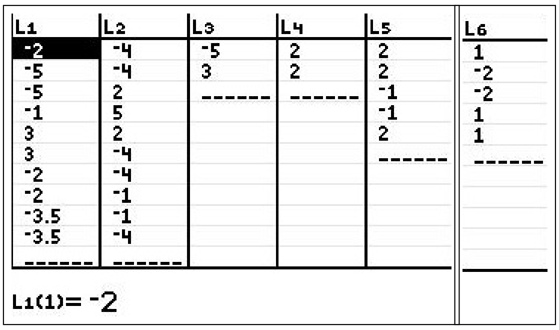


Notes for Teachers

The artwork is displayed on the TI-84/CE as follows.

Press stat and select Edit . . . . Clear all the lists — cursor on heading and clear .

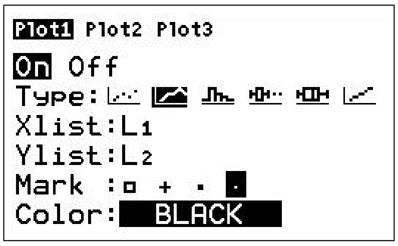
Put the first group of coordinates (up to the first *lift pencil*) in L1 (x coordinates) and L2 (y coordinates), the second group of coordinates in L3 and L4, the third group in L5 and L6. Press enter after each entry, including the last. The TI-84 allows a maximum of three plots at one time, hence no more than two *lift pencil* instructions can be used.

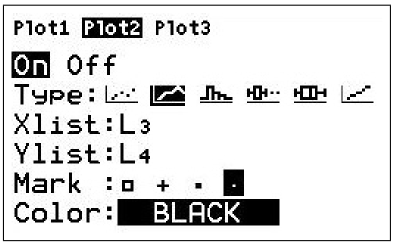


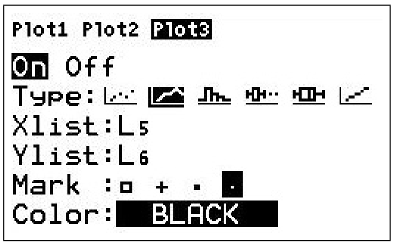
L1	L2	L3	L4	L5	L6
-2	-4	-5	2	2	1
-5	-4	3	2	2	-2
-5	2	------	------	-1	-2
-1	5			-1	1
3	2			2	1
3	-4			------	------
-2	-4				
-2	-1				
-3.5	-1				
-3.5	-4				
------	------				

L1(1)= -2

Press stat plot (2nd y=). Select each of the plots in turn and set up as shown below. Use the scatterplot with joining lines, the second option in *Type*. The list names L1 – L6 are obtained by pressing 2nd , then the corresponding number key. Only the CE has *Color*.







Set a suitable window before graphing (guided by the house figure). Then press zoom 5 to set equal scales on each axis. Turn off any functions in y= . Turning off the axes[7] is also a good idea. Press graph to display the graph. Press trace and use the left/right arrow keys to move between the points; the up/down arrow keys move between the three plots. This will enable you to find out which points, if any, were incorrect.

Don't forget to clear the lists and turn the axes back on when you have finished.

This is a great time to introduce a little bit of network theory. Give the students some pictures that appear to require more than two *lift pencil* instructions, and have them figure out where to start so as to reduce the number of *lift pencil* instructions to a maximum of two. Challenge the students to find the rule for determining the minimum number of *lift pencil* instructions needed to draw the graph.

[7] Press 2nd zoom . **84:** move the cursor to *Axes Off*, press enter then quit ; **CE:** scroll through the *Axes* colours until you reach *off*, then press quit .

5 Graphing Straight Lines

Years 9, 10, Levels 1, 2; Strand: Algebra; Sub-strand: Coordinate Geometry – Straight Lines.

Author: Margie Smith.

Using the graphics calculator to explore the $y = mx + b$ form of a straight line.

Graphing Straight Lines   Worksheet 1

1. Clear all current graphs from your calculator:
[y=] [clear]; down arrow and [clear] as required.

The graphs should appear like this.
↓

2. Press [zoom] [6], then [zoom] [5] for suitable axes.
Change Xscl and Yscl to 2 in [window].

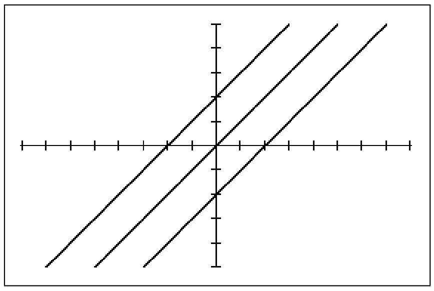
3. Put $y = x$ in Y1: press [y=]; [X,T,θ n] gives X.

Put $y = x + 4$ in Y2.

Put $y = x - 4$ in Y3.

Press [graph].

What can you say about these lines?



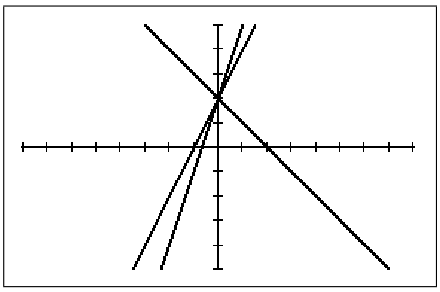
4. Clear the three graphs in [y=] and now draw the graphs of:

$y = 2x + 4$

$y = 3x + 4$

$y = -x + 4$

What can you say about these lines?



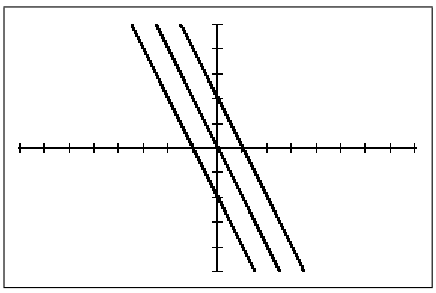
5. Clear the three graphs in Question 4 and now draw the graphs of:

$y = -2x$

$y = -2x + 4$

$y = -2x - 4$

What can you say about these lines?



PTO

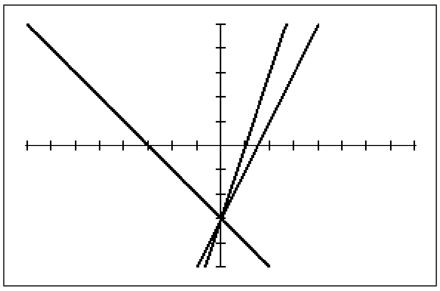
6. Clear the three graphs in Question 5 and now draw the graphs of:

$y = 2x - 6$

$y = 3x - 6$

$y = -x - 6$

What can you say about these lines?

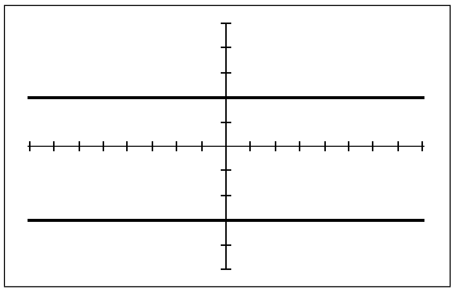


7. Clear the three graphs in Question 6 and now draw the graphs of:

$y = 4$

$y = -6$

What can you say about these lines?

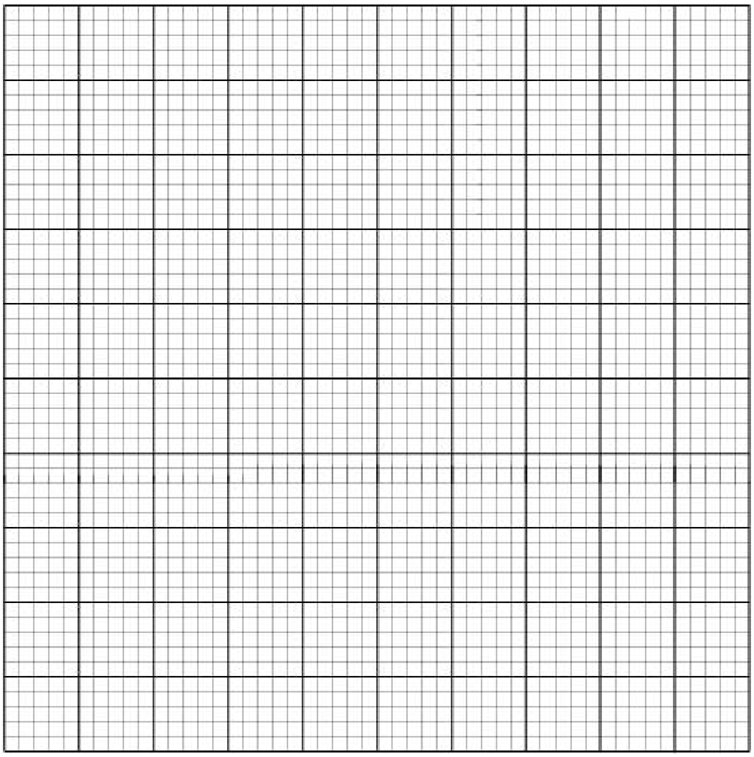


Graphing Straight Lines Worksheet 2

Clear all current graphs from your calculator. Press [zoom] [4] for 'decimal' axes.

1. Graph the function $y = x$ in Y1.

Draw a sketch on the number plane below. The thick lines are 1 unit apart.



2. Graph the function $y = x + 1$ in Y2.

(i) What is the y intercept of $y = x + 1$?

(ii) Draw a sketch of $y = x + 1$ on the number plane above.

3. Graph the function $y = x + 2$ in Y3.

(i) What is the y intercept of $y = x + 2$?

(ii) Draw a sketch of $y = x + 2$ on the number plane above.

4. Graph the function $y = x - 1$ in Y4.

(i) What is the y intercept of $y = x - 1$?

(ii) Draw a sketch of $y = x - 1$ on the number plane above.

5. What are the y intercepts for the following curves?

$y = x + 7$ $y = x + 2.7$ **Check these answers**

$y = x - 2$ $y = x - 3.5$ **using your calculator**

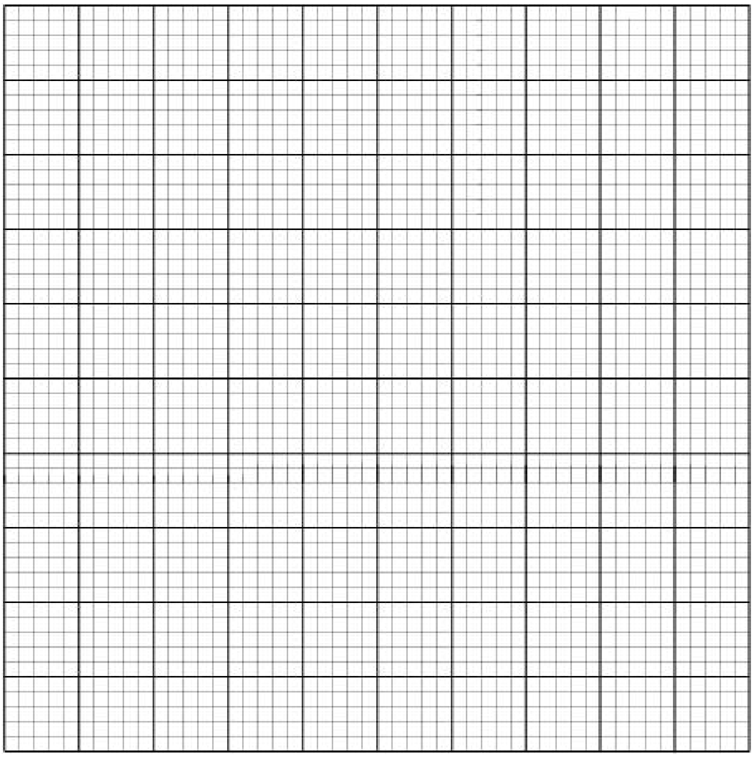
6. Try to generalise your results, i.e. what is the y intercept of $y = x + b$, where b is any number? Test your conjecture with some more graphs.

Graphing Straight Lines Worksheet 3

Clear all current graphs from your calculator.
Change the window of your calculator so that $-5<x<5$. The thick lines are 1 unit apart.

1. Graph the function $y=2x/3$ in Y1.

Draw a sketch on the number plane below.



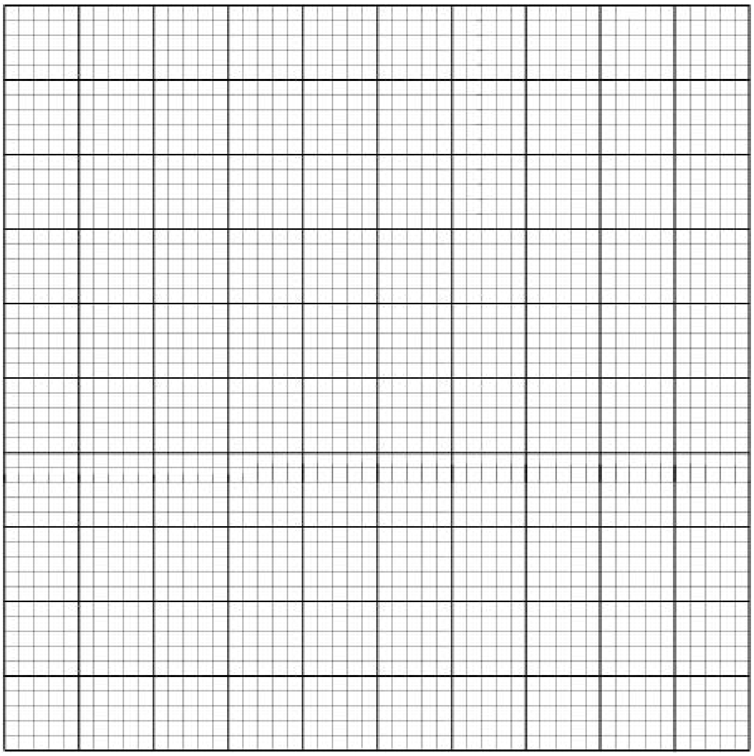
2. Imagine you are a trace dot. Starting at the origin, move in a positive horizontal (x) direction for 3 units ('run'), then in a positive vertical (y) direction ('rise') for 2 units. Where do you end up? Coordinates: $x =$___ $y =$___

3. Draw the triangle on the grid that you have traced by this move.
What sort of triangle is it? ____________

PTO

4. Graph the function $y = x/4$ in Y2.

Draw a sketch on the number plane below.



5. Move from the origin in a positive x direction for 4 units, then in a positive y direction for 1 unit. Is the triangle produced the same type as in Question 3? ________

6. Do you notice a pattern between the x coefficient in the equations and the 'rise' and 'run' of your triangles?

____

7. Graph the following functions on your calculator and see if the results are consistent with your previous findings: $y = 0.4x$; $y = x/3$; $y = 2x$; $y = 3x$.

PTO

8. Complete the tables below for each equation, putting the equations in order, left to right, from the smallest x coefficient to the largest. Choose three x values and calculate the corresponding y values for each equation. Use these to draw each graph on the number plane below.

$y =$

$\boldsymbol{x}$			
$\boldsymbol{y}$			

$y =$

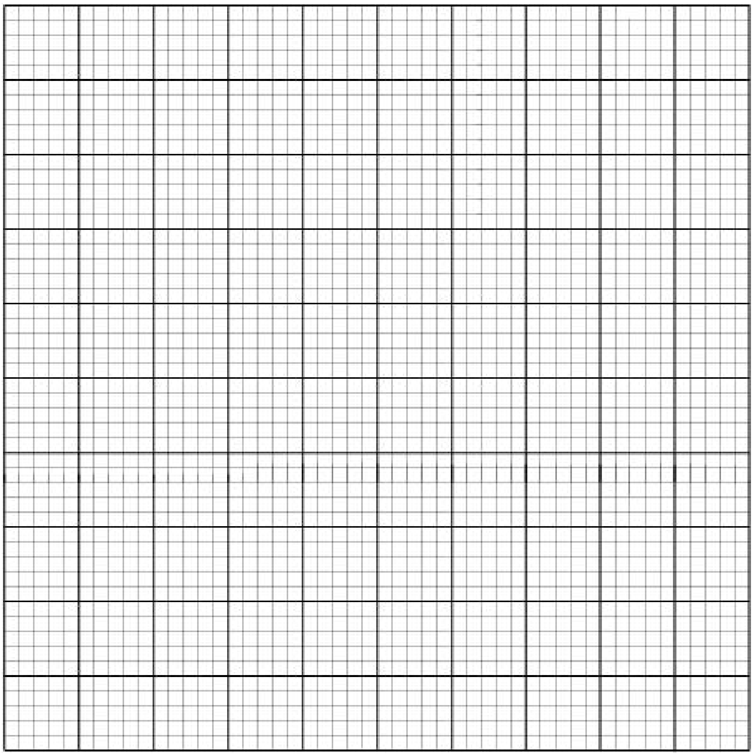
$\boldsymbol{x}$			
$\boldsymbol{y}$			

$y =$

$\boldsymbol{x}$			
$\boldsymbol{y}$			

$y =$

$\boldsymbol{x}$			
$\boldsymbol{y}$			



9. Is there a relationship between the size of the coefficient and the steepness (or slope) of the line?

10. Having completed Worksheets 1, 2 and now 3, can you generalise your results, i.e. say what happens to the graph of $y = mx + b$ when m and b are changed?

6 Guess the Line

Years 9, 10, Levels 1, 2; Strand: Algebra; Sub-strand: Coordinate Geometry – Straight Lines.

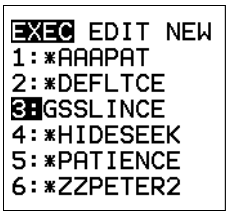
Author: Margie Smith.

Guess the equations of straight lines generated by the calculator. The calculator keeps score.

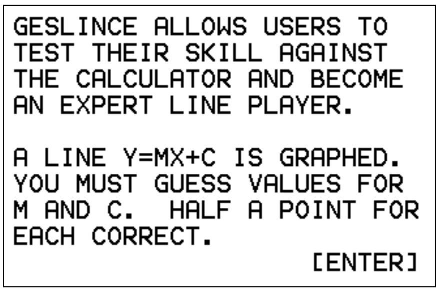
GUESSLIN/GESLINCE program[8]

The program generates random values for M and C in the straight line Y = MX + C, and graphs the line. C is a non-zero integer between −4 and 4. M is a non-zero integer between −3 and 3 divided by C. From the graph, you have to guess values for M and C.

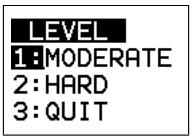
1. Press [prgm] to bring up the program list. Scroll down to highlight the program name and press [enter].

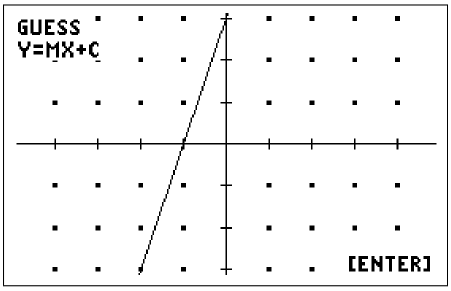


2. To start the program, press [enter] again. Use [enter] to continue through the introduction.



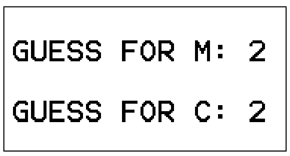
3. Choose a level of difficulty, MODERATE or HARD, and the graph of a straight line is displayed.





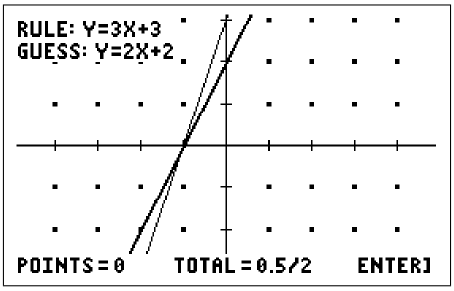
At this point, you should work out the gradient of the line M and the Y-intercept C. *You cannot access the graph again once you proceed!*

4. Press [enter]: the program prompts you to input values for M and C.



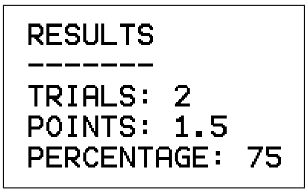
[8] GUESSLIN for the TI-84Plus, GESLINCE for the TI-84CE

5. Press enter again to redraw the line. A bold line is also drawn, to represent the values you inputted for M and C — hopefully the original line will turn bold.



Your values for M and C, and the actual values are displayed on the graph, as are the points you have scored for this line and your total points.

6. Press enter to play again, change the level of difficulty or quit the program. If you quit, your final score is displayed, and perhaps a message.



7 Let's Be Rational

Year 10, Level 1; Strand: Algebra; Sub-strand: Sketching Other Graphs, Polynomials.

Author: Brenda Batten, Ridgeland, South Carolina, USA. Modified by Peter McIntyre.

Understanding the local and global behaviour of rational functions.

The first five letters in *rational* spell *ratio*.

- A rational **number** is the ratio of two **integers**.
- A rational **function** is the ratio of two **polynomials**.
  The simplest rational function is $y = 1/x$.
  It is the ratio of the polynomials $f(x) = 1$ and $g(x) = x$.

There are two types of behaviour we need to investigate in rational functions:

1. when x values are very small in absolute value (*up-close and personal*);

2. when x values are large in absolute value (*like an astronaut*).

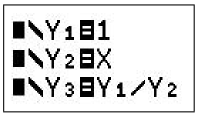
Understanding these two features, the **local and global behaviour** of rational functions, will allow you to appreciate the properties of these functions and to sketch graphs of rational functions ... without the aid of a graphics calculator.

To reach that point, however, the capabilities of a graphics calculator are very useful.

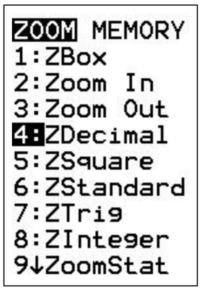
PTO to begin

Local behaviour: Up-close and personal

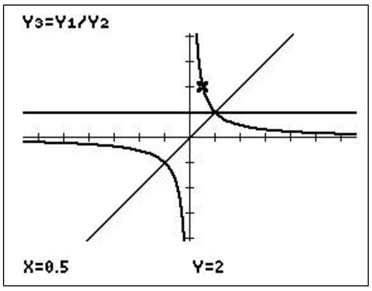
Graphically: Set up your calculator to graph $Y_1 = 1$, $Y_2 = X$ and $Y_3 = Y_1/Y_2$, as shown in the figures below, using [y=], [zoom] [4] and [trace], respectively.



Define functions



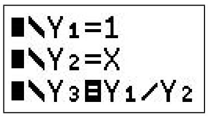
'Friendly' window



Trace along graph

What window does [zoom] Decimal produce? ________________________________

Numerically: Set up your calculator to produce a table of values of Y_3 as shown in the figures below.



Turn off Y_1 and Y_2

[tblset]

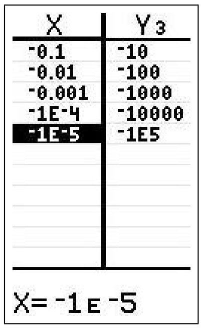
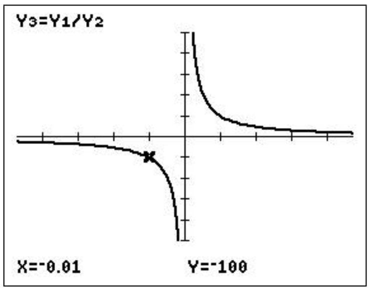


Table values

Use these two calculator modes to answer Questions 1, 2 and 3 below.

1. As the x values approach 0 from the left, what happens numerically to the y values?

2. What happens graphically?
Look at Y3 on the graph on the previous page, then try a window of $[-0.047, 0.047, 0.01] \times [-500, 500, 100]$. Use trace to investigate.

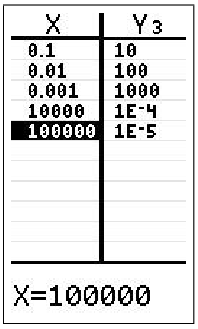


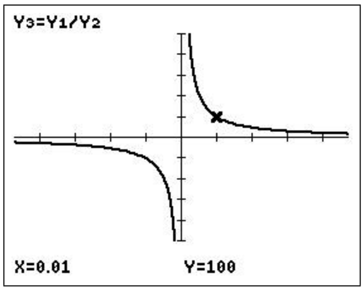
Different window

In words: As x tends to 0 from the left, $f(x)$ tends to ______________

In symbols: As $x \to 0^-$, $f(x) \to$ ______________

3. As the x values approach zero from the right, what happens to the y values?





In words: As x tends to 0 from the right, $f(x)$ tends to ______________

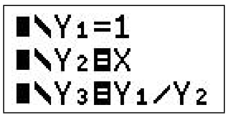
In symbols: As $x \to 0^+$, $f(x) \to$ ______________

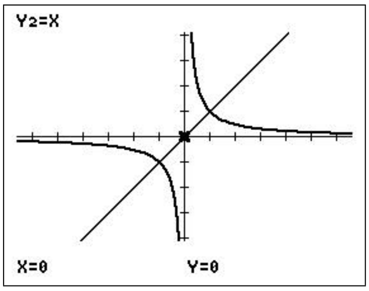
The rational function $f(x) = 1/x$ is undefined at $x = 0$, as the denominator is zero at that point. The value of the numerator is not zero. When both of these conditions are true, there is a ________________________ on the graph.

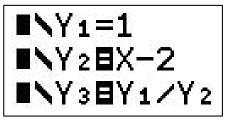
The role of the denominator

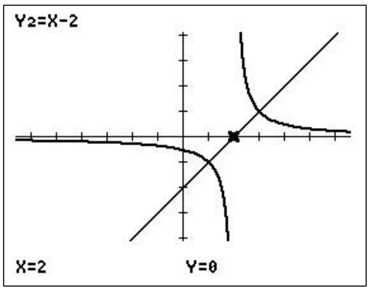
Now look at the functions $f(x)=1/x$ and $g(x)=1/(x-2)$.

Use a zoom Decimal window.





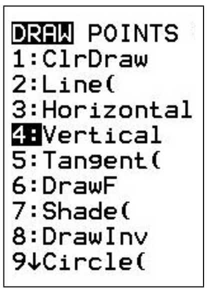


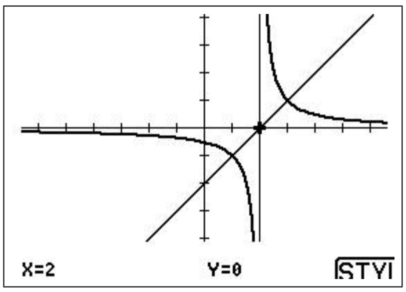


1. For what values of x is the denominator of the rational function in each case equal to 0?

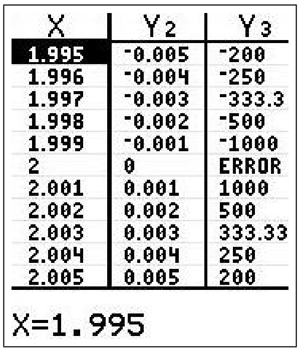
2. What is the equation of the vertical asymptote in each case?

To plot the vertical asymptote, use *Vertical* in the draw menu. On the graph, use the arrow keys to move the line to the correct position and press enter. From the Home screen, select *Vertical*, then type in the X value and press enter. On a CE, you can also choose the line colour and type with the STYLE key.





Now let's investigate numerically the values of the second function, $g(x) = 1/(x-2)$, when x is close to 2.



X	Y2	Y3
1.995	-0.005	-200
1.996	-0.004	-250
1.997	-0.003	-333.3
1.998	-0.002	-500
1.999	-0.001	-1000
2	0	ERROR
2.001	0.001	1000
2.002	0.002	500
2.003	0.003	333.33
2.004	0.004	250
2.005	0.005	200

X=1.995

3. For what values of x is the rational function Y3 undefined?

4. As the x values approach that value from the left, what happens numerically to the y values?

5. As the x values approach that value from the right, what happens numerically to the y values?

As $x \to 2^-$, $g(x) \to$ _____ and, as $x \to 2^+$, $g(x) \to$ _____

The equation of the vertical asymptote is ___________

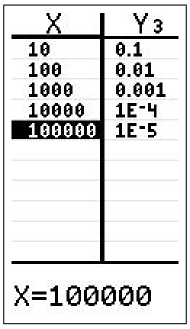
Concept check: What would the equation of the vertical asymptote be if Y2 were changed to $x+3$?

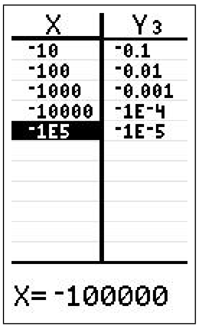
Global behaviour: Like an astronaut

What happens at the extreme 'edges' of the graph of a rational function?

We have investigated the behaviour when values of x get close to the zeros of the denominator. Now we turn our attention the behaviour of the function when the values of x get very large, either positive or negative, that is when the absolute values of x get very large or 'tend to infinity'.

The TI-84 can do the arithmetic for us.





As x values get very large and positive, we say x approaches positive infinity.

Here the function values (Y3) then approach ____________

We write: As $x \to \infty$, $f(x) \to$ ____________

As x values get very large and negative, we say x approaches negative infinity.

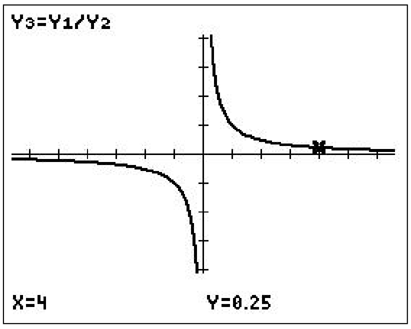
Here the function values then approach ____________

We write: As $x \to -\infty$, $f(x) \to$ ____________

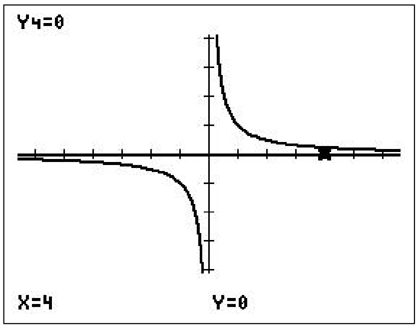
The Rule of Four

We have investigated global behaviour **numerically** and written it **algebraically**. Now investigate global behaviour **graphically**.

Set Y2 = X again, then turn it off so we are just plotting Y3. Use a zoom Decimal window.

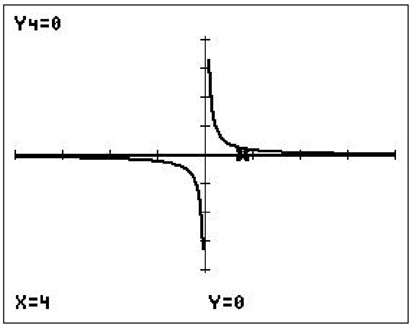



```
Y1=1
Y2=X
Y3=Y1/Y2
Y4=0
```



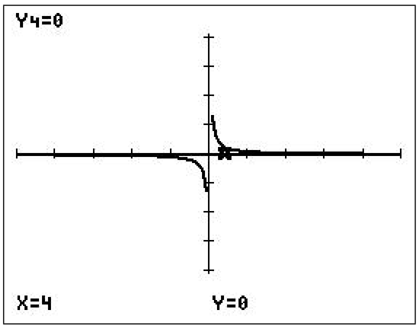
trace   Define Y4   Plot Y3 and Y4

```
WINDOW
 Xmin=-20
 Xmax=20
 Xscl=5
 Ymin=-4.1
 Ymax=4.1
 Yscl=1
```



Broaden X window

```
WINDOW
 Xmin=-50
 Xmax=50
 Xscl=10
 Ymin=-4.1
 Ymax=4.1
 Yscl=1
```

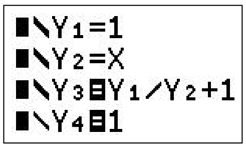


Broaden X window

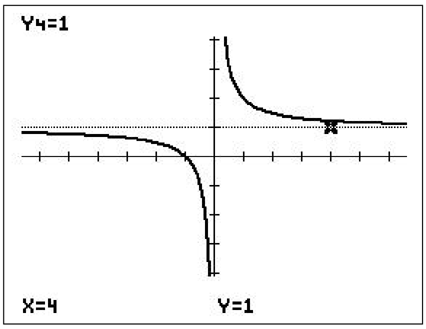
Graphically: (describe what happens as we broaden the window)

There is a horizontal asymptote at ____________

For a different horizontal asymptote, add a constant to the rational function. This will increase the value of every y coordinate by that amount. It shifts the graph vertically. Here the constant is 1.



Y4 = 1 is asymptote



Graphically

X	Y3	Y4
-10	0.9	1
-100	0.99	1
-1000	0.999	1
10	1.1	1
100	1.01	1
1000	1.001	1

Numerically

Verbally: As x tends to $\pm\infty$, $f(x)$ tends to ____________

Exercise: Predict the behaviour of the graph of $f(x) = \dfrac{1}{x-2} + 1$.

1. Which part of the function determines the position of the vertical asymptote?

2. What is the equation of the vertical asymptote? Sketch the vertical asymptote **without** the aid of a graphics calculator.

3. Which part of the function determines the position of the horizontal asymptote?

4. What is the equation of the horizontal asymptote? Sketch the horizontal asymptote **without** the aid of a graphics calculator.

5. Now sketch the graph of the function **without** the aid of a graphics calculator.

6. Confirm your conjecture in 5 using a graphics calculator.

Homework

For each of the following rational functions

1. $y = \dfrac{1}{x+5} - 3$ **2.** $y = \dfrac{1}{x-1} + 1$ **3.** $y = \dfrac{1}{x} - 2$

(a) write the equation of the vertical asymptote.

(b) write the equation of the horizontal asymptote.

(c) sketch the graphs of (a) and (b) **without** the aid of a graphics calculator.

(d) sketch the graph that you predict for the rational function **without** the aid of a graphics calculator.

(e) confirm your conjecture in (d) using a graphics calculator.

(f) combine the terms of each function so that it is written as the ratio of two polynomials.

Notes for Teachers

Local behaviour: Up-close and personal

What window does [zoom] Decimal produce?

On a TI-84, a window $[-4.7, 4.7, 1] \times [-3.1, 3.1, 1]$;
on a TI-84CE a window $[-6.6, 6.6, 1] \times [-4.1, 4.1, 1]$.

Each pixel (trace step on a CE) is 0.1 wide and 0.1 high.

1. As the x values approach 0 from the left, what happens numerically to the y values?

As the x values approach 0 from the left, the y values become more and more negative.

2. What does it mean graphically?

As the x values approach 0 from the left, the graph of the function drops down further and further, and comes closer and closer to the negative y axis.

In words: As x tends to 0 from below, $f(x)$ tends to negative infinity.

In symbols: As $x \to 0^-$, $f(x) \to \underline{-\infty}$.

3. As the x values approach zero from the right, what happens to the y values?

As the x values approach 0 from the left, the y values become larger and larger.

In words: As x tends to 0 from above, $f(x)$ tends to positive infinity.

In symbols: As $x \to 0^+$, $f(x) \to \underline{\infty}$.

The rational function $f(x) = 1/x$ is undefined at $x = 0$, as the denominator is zero at that point. The value of the numerator is not zero. When both of these conditions are true, there is a vertical asymptote on the graph.

The role of the denominator

1. For what values of x is the denominator of the rational function in each case equal to zero?

   $f(x)=0$ when $x=0$. $g(x)=0$ when $x=2$.

2. What is the equation of the vertical asymptote in each case?

   The vertical asymptote for f is (the vertical line) $x=0$.

   The vertical asymptote for g is $x=2$.

Now let's investigate numerically the values of the second function, $g(x)=1/(x-2)$, when x is close to 2.

3. For what values of x is $g(x)$ undefined?

   $g(x)$ is undefined at $x=2$

4. As the x values approach that value from the left, what happens numerically to the y values?

   As the x values approach 2 from the left, the y values approach negative infinity.

5. As the x values approach that value from the right, what happens numerically to the y values?

   As the x values approach 2 from the right, the y values approach positive infinity.

As $x \to 2^-$, $g(x) \to \underline{-\infty}$ and, as $x \to 2^+$, $g(x) \to \underline{+\infty}$.

The equation of the vertical asymptote is $\underline{x=2}$.

Concept check: What would the equation of the vertical asymptote be if Y2 were changed to $x+3$?

The rational function would now be $h(x)=1/(x+3)$: the denominator has a zero at $x=-3$, the numerator is non-zero there (and everywhere else), so that the vertical asymptote is $x=-3$.

Global behaviour: Like an astronaut

What happens at the extreme 'edges' of the graph of a rational function?

We have investigated the behaviour when values of x get close to the zeros of the denominator. Now we turn our attention the behaviour of the function when the values of x get very large, either positive or negative, that is when the absolute values of x get very large or 'tend to infinity'.

As x values get very large and positive, we say x approaches positive infinity.

Here the function values (Y3) then approach zero from above.

We write: As $x \to \infty$, $f(x) \to \underline{0^+}$.

As x values get very large and negative, we say x approaches negative infinity.

Here the function values then approach zero from below.

We write: As $x \to -\infty$, $f(x) \to \underline{0^-}$.

The Rule of Four

We have investigated global behaviour **numerically** and written it **algebraically**. Now investigate global behaviour **graphically** and state it **verbally**.

Graphically: As the x window gets broader and broader, the graph of Y3 becomes indistinguishable from that of the asymptote, illustrating the global behaviour of the function.

For a different horizontal asymptote, add a constant to the rational function. This will increase the value of every y coordinate by that amount. It shifts the graph vertically. Here the constant is 1.

There is a horizontal asymptote at $\underline{y=0}$.

Verbally: As x tends to positive or negative infinity, $f(x)$ tends to $\underline{1}$.

Exercise: Predict the behaviour of the graph of $f(x) = \dfrac{1}{x-2} + 1$.

1. Which part of the function determines the position of the vertical asymptote?

   The denominator (or more specifically, the zero of the denominator) determines the position of the vertical asymptote.

2. What is the equation of the vertical asymptote? Sketch the vertical asymptote **without** the aid of a graphics calculator.

   The equation of the vertical asymptote is $x = 2$.

3. Which part of the function determines the position of the horizontal asymptote?

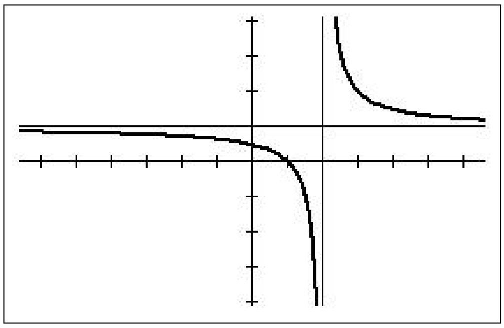
   The number that is added to the rational function determines the position of the horizontal asymptote.

4. What is the equation of the horizontal asymptote?

   The equation of the horizontal asymptote is $y = 1$.

6. Confirm your conjecture in Exercise 5 using a graphics calculator.

   zoom Decimal



Homework

For each of the following rational functions

1. $y = \dfrac{1}{x+5} - 3$ **2.** $y = \dfrac{1}{x-1} + 1$ **3.** $y = \dfrac{1}{x} - 2$

(a) write the equation of the vertical asymptote.

(b) write the equation of the horizontal asymptote.

(c) sketch the graphs of (a) and (b) **without** the aid of a graphics calculator.

(d) sketch the graph that you predict for the rational function **without** the aid of a graphics calculator.

(e) confirm your conjecture in (d) using a graphics calculator.

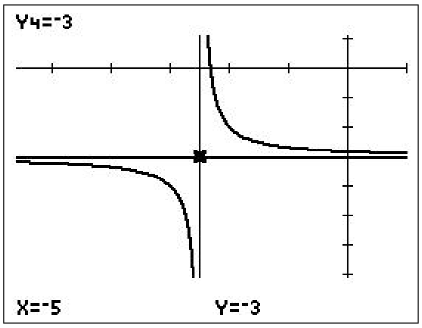
(f) combine the terms of each function so that it is written as the ratio of two polynomials.

Answers

1. **(a)** The equation of the vertical asymptote is $x = -5$.

(b) The equation of the horizontal asymptote is $y = -3$.

(e)

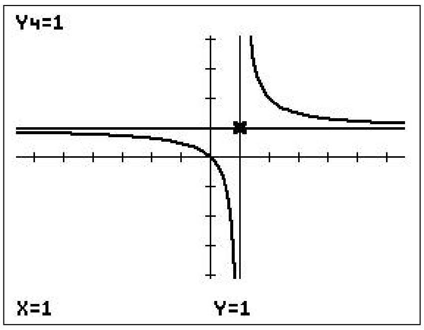


(f) $y = \dfrac{1}{x+5} - 3 = \dfrac{1 - 3(x+5)}{x+5} = \dfrac{-14 - 3x}{x+5}.$

2. **(a)** The equation of the vertical asymptote is $x = 1$.

(b) The equation of the horizontal asymptote is $y = 1$.

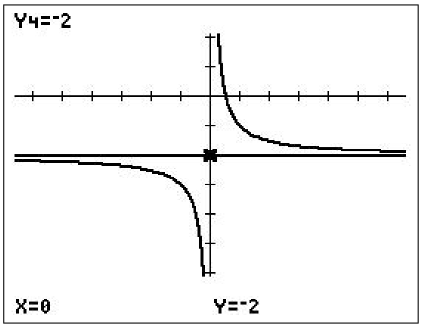
(e)



(f) $y = \dfrac{1}{x-1} + 1 = \dfrac{1 + (x-1)}{x-1} = \dfrac{x}{x-1}.$

3. (a) The equation of the vertical asymptote is $x=0$.

(b) The equation of the horizontal asymptote is $y=-2$.

(e) 

(f) $y = \dfrac{1}{x} - 2 = \dfrac{1-2x}{x}$.

8 Parabolic Aerobics

Year 9, Levels 1 & 2; Strand: Algebra; Sub-strand: Coordinate Geometry.

Based on Parabola Guessing Game, an activity from Activities Integrating the TI-83+ into Algebra by Vicki Fortson Shirley. Modified by Peter McIntyre.

The first activity investigates the effect of changing the numbers A, B and C on the graphs of the family of parabolas $Y=A(X-B)^2+C$. In the second activity, you have to guess the numbers A, B and C for the graph of a mystery parabola generated by the calculator. The calculator checks your answers and keeps score.

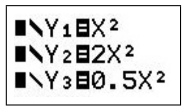
Warm-ups

The graphics keys on the calculator are the top row. The [X,T,θ,n] key (third row) is a quick way to get X, the independent variable that the calculator uses for its graphs.

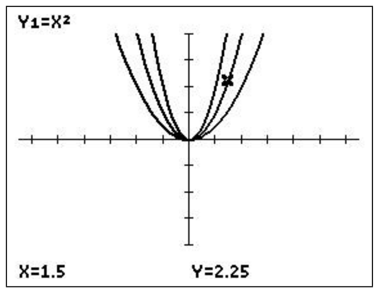
Set a window for plotting the graphs by pressing [zoom] [4].

1. **Stretches and reflections**

   Press [y=]. Set $Y_1 = X^2$, $Y_2 = 2X^2$ and $Y_3 = 0.5X^2$.



   Press [trace]. Use the arrow keys to see which graph is which.



   Let A stand for the number multiplying X^2. You have just plotted graphs for $A = 1$, $A = 2$ and $A = 0.5$.

   Use your graphs to decide what happens when you multiply X^2 by different positive numbers. Test your ideas with some other values of A, that is by graphing $Y = AX^2$ for different values of A. Write down your conclusions in the space below.

   What if A is a negative number? Again, test your ideas by plotting suitable graphs. Write down your conclusions in the space below.

2. Shifts or translations

Set $Y_1 = X^2$, $Y_2 = (X-2)^2$ and $Y_3 = (X+3)^2$. What happens when you vary the number B in the family of graphs $Y = (X-B)^2$? Test your ideas by trying some more values of B. Write down your conclusions in the space below.

Set $Y_1 = X^2$, $Y_2 = X^2+1$ and $Y_3 = X^2 - 2$. What happens when you vary the number C in the family of graphs $Y = X^2+C$? Test your ideas by trying some more values of C. Write down your conclusions in the space below.

3. Summary

In the space below, summarise the effects of changing the numbers A, B and C in the family of graphs $Y = A(X-B)^2+C$.

What parabola is that?

The PARABOLA/PRBOLACE program is available from *www.XXX*.

Run the program by pressing [prgm] (fourth row, middle), pressing the number against the program name and pressing [enter].

The program plots the graph of a mystery parabola $Y = A(X-B)^2+C$ as a solid line, where A, B and C are generated randomly.

Press [enter] to generate the first mystery parabola. The program also plots the basic parabola $Y = X^2$ as a faint line to use for comparison.

Your job is to decide what values of A, B and C the calculator has chosen.

Just to make it a bit easier, A, B and C can take only a restricted number of values.

A: ± 0.5, ± 1 or ± 2.

B, C: 0, ± 1 or ± 2.

When you have decided, press [enter] and input your values. The calculator will then plot the mystery parabola again as a solid line and the parabola with your values of A, B and C as a bold dotted line. *Did you get the right values?*

Press [enter] again to see the calculator values, and [enter] once more for the NEXT... menu.

Here you can generate another mystery parabola, increase or decrease the level of difficulty or quit. You start at Level 0, at which the graph of the mystery parabola remains on screen until you press [enter]. At higher levels, the parabola screen remains on view for a fixed time only; the higher the level, the shorter the time.

When you quit, you will see your final score.

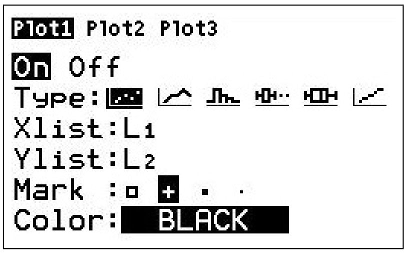
9 Probably Finding π

Year 10, Levels 1 & 2; Strand: Chance and Data; Sub-strand: Probability.

Author: Michael McNally, Lower Canada College, Montreal, Canada.
Modified by Peter McIntyre.

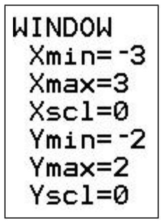
An experimental-probability method for finding π.

1. **Set up the plot screen**

   Press statplot (2nd y=) and select Plot1.
   Set up Plot1 as shown using the cursor and enter.
   L_1 and L_2 are 2nd 1 and 2nd 2, respectively.
   On a CE, you get to choose the colour of the mark too.

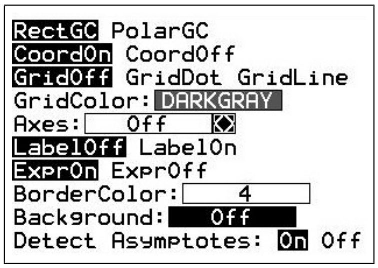
2. **Set a window**

   Press window and set up as shown.

   Then press zoom 5 (Square).

3. **Turn off the axes**

   Press format (2nd zoom).

   Select *AxesOff*.

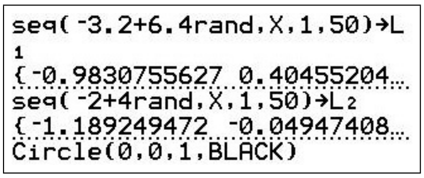
   Press quit (2nd mode) to return to the Home screen.

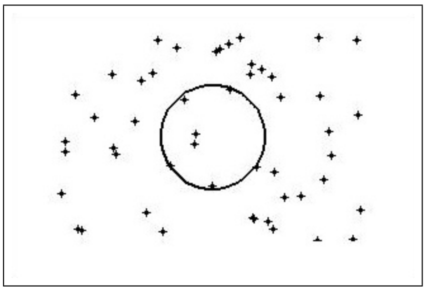
PTO

4. Store the coordinates of 50 random points and draw a unit circle

Type in and execute the following commands.

seq $(-3.2 + 6.4\,\text{rand}, \text{X}, 1, 50) \to$ L1 (**CE** x coordinates)
seq $(-3 + 6\,\text{rand}, \text{X}, 1, 50) \to$ L1 (**84** x coordinates)
seq $(-2 + 4\,\text{rand}, \text{X}, 1, 50) \to$ L2 (y coordinates)
Circle (0, 0, 1, BLACK)





seq: list (2nd stat) OPS menu.
rand: math PROB menu.
The arrow represents the sto key.
Circle: draw menu (2nd prgm).
BLACK: vars COLOR menu (CE only).

How many points lie inside the circle?

If a point lies on the circle, toss a coin to decide if it is in or out.

5. Answer the following questions to find an estimate for π

(a) What is the area of the window?

(b) What is the area of the circle?

(c) What is the *theoretical* probability of a random point in the window landing in the circle? *Hint*: Think areas.

(d) What is the *experimental* probability?
Hint: How many points are there in the window? How many of these lie in the circle?

(e) Assuming the experimental probability and the theoretical probability are approximately equal, solve for an estimate of π.

Pool a series of results (numbers in the circle) to obtain a better estimate for π.

Hint: To do further runs, repeatedly press entry (2nd enter) to recall the commands in the appropriate order and press enter to execute them.

An even better way is to type the two *seq* commands and the *Circle* command all on one line, separated by semi-colons (alpha ·). After a plot, press quit to return to the Home screen and press enter to re-execute all three commands in one go to obtain a new plot. Doing this repeatedly produces lots of data rapidly.

When you have finished, turn your axes on again in format and clear all the lists: 2nd mem 4 enter.

Notes for Teachers

Calculator operations

The x coordinates of each point generated are stored in list L1 by the first *seq* command. The *rand* command generates a random number[9] between (but not equal to) 0 and 1. Therefore, *6.4 rand* generates a random number between 0 and 6.4, and *−3.2+6.4 rand* a random number between −3.2 and 3.2, the range of x values in the CE window.

The corresponding operation on a TI-84Plus gives a random number between −3 and 3, the range of x values in its window.

Similarly, the second *seq* command, containing *−2+4 rand*, generates random numbers between −2 and 2 for the y coordinate and stores them in list L2.

The command *Circle* (x, y, r) draws a circle, centre (x, y) and radius r, on the screen.

Question 5

(a) The area of the window is $6.4\times4=25.6$ ($6\times4=24$ on an 84Plus).

(b) The area of the circle is $\pi\times1^2=\pi$.

(c) The theoretical probability of a random point in the window landing in the circle is the ratio of the area of the circle to the area of the window, i.e. $\pi/25.6$ ($\pi/24$).

(d) The experimental probability is the ratio of the number of points (N say) in the circle to the total number of points in the window, i.e. $N/50$.

(e) If we assume the experimental probability and the theoretical probability are approximately equal, we have

$$\frac{\pi}{25.6} \approx \frac{N}{50} \qquad \left(\frac{\pi}{24} \approx \frac{N}{50}\right)$$

so that

$$\pi \approx \frac{25.6N}{50} = 0.512N \qquad \left(\pi \approx \frac{24N}{50} = 0.48N\right).$$

Counting N then gives us an estimate for π.

It's a good idea to pool the data (number of points inside the circle) from all the students to obtain (hopefully) a better estimate for π than individual students will obtain; put these values in a list, say L1, on your calculator. It is easier to record integers when pooling the data than all the individual estimates for π. Find the mean number of points that land in the circle[10] and multiply it by 0.512 (0.48) to find the mean estimate for π.

[9] If the calculators have all been reset before this activity, successive *rand* commands will generate the same sequence of random numbers on each calculator, so that students will have identical results. If this is the case, ask each student to store his/her favourite number between 1 and 100 in *rand* (key strokes: *number* [sto] [math] PROB [1] [enter]), so that they each start with different random-number seeds and therefore obtain different sequences of random numbers.

[10] [list] [MATH] menu

You can gain some idea of the expected accuracy in your estimate for π by using the mean $N \pm$ one standard deviation.[11] Does the actual value of π lie in this range?

You could discuss why more data should give a better estimate (experimental probability $\rightarrow$ theoretical probability as the number of data points $\rightarrow \infty$).

The hint on repeated plots allows individual students to generate lots of data of their own if they wish.

Extension

How would you get the calculator to decide whether a point (x, y) lies inside the circle?

A point (x, y) lies inside the circle if $\sqrt{x^2+y^2} < 1$ or, equivalently, if $x^2+y^2 < 1$.

Write a calculator or computer program to generate the random points, calculate the number lying inside the circle and hence estimate π automatically.

Further Research

Find out about Buffon's needle problem, a 'hands-on' forerunner of this problem.

[11] [list] [MATH] menu

10 Reaction Times and Statistics

Years 9 &10, Levels 1 & 2; Strand: Chance and Data; Sub-strand: Statistics.

Authors: Jamie Alford and Dan Graeme, Queensland University; Peter McIntyre, UNSW. Modified by Margie Smith and Peter McIntyre.

Programs are used to measure reaction times in various scenarios, including driving a car. The data are displayed as box-and-whisker plots for subsequent analysis.

Introduction

There are three activities here, based around a common theme of reaction times. In each activity, a calculator program is used to measure and record reaction times under different conditions.

1. Use the REACTHND/RCTHNDCE program to measure reaction times using first one hand, then the other. Analyse the resulting box-and-whisker plots, one for each hand, to draw conclusions from your data.

2. Use the REACT/REACTCE program to measure reaction times under three different conditions:

   (a) intense concentration and complete silence;

   (b) relaxed in relative quiet;

   (c) distracted in a noisy environment.

   These conditions measure a person's reaction time under different conditions. The data are plotted as box-and-whisker plots and are also available in lists for further analysis.

3. Use the CARSTOP/CARSTPCE program to simulate driving behind a car whose brake lights suddenly flash on. The speed of your car and the distance behind the car in front can be varied. Scenarios such as those in Activity 2 can be used. The reaction times are plotted as a box-and-whisker plot, with the data available in a list for further analysis.

   This activity is designed to appeal to, and maybe even instruct, students approaching driving age. The outcomes, crashing or stopping safely, reflect reasonably accurately what would actually happen, assuming both cars stop in the same distance after the brakes are applied.

Real-World Statistics #1

Introduction

You are a statistician employed by Microsoft. You have been commissioned to assist in the design of a new control pad for the XBOX® III. The data you are tasked to compile are the reaction times of the 'better' and 'worse' hands of users.

You are required to measure and analyse the reaction times, and to comment on the differences between the two hands. To help with your research, you are provided with a graphics-calculator program to measure reaction times and a set of steps to work through.

Method

Use the REACTHND/RCTHNDCE program to measure your reaction times using *at least* 15 trials with each hand. Sketch your boxplots and complete the quartile analysis below — use the arrow keys to find the numbers from the boxplots.

	Left hand	**Right hand**
Mean		
minX		
Q1		
Median		
Q3		
maxX		

Analysis

Compare your results with those of others in your group. Detail any differences in the data obtained from different individuals.

From the data, describe the differences between your 'better' hand and your 'worse' hand. How accurate are these terms?

Further Activities

1. What, from your findings, should Microsoft consider when designing the control pad?

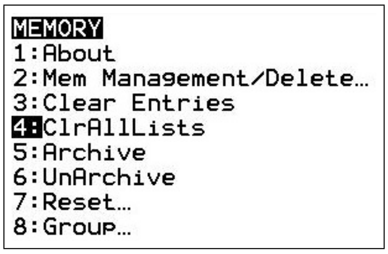
2. Present a rough design of the new control pad. This should be light on artistic focus, but high on thinking. Microsoft has paid designers. You are the statistician. Your job is to give a rough outline of the control pad **with a high level of justification**. Your design should be a one-third-page diagram, with the other two-thirds of the page used for justification and conclusion.

3. Reflect on the activity. Explain your work as a statistician, including formal descriptions of the types of tasks you performed and why you believe they were relevant. You should also address what you think the purpose behind statistical work is.

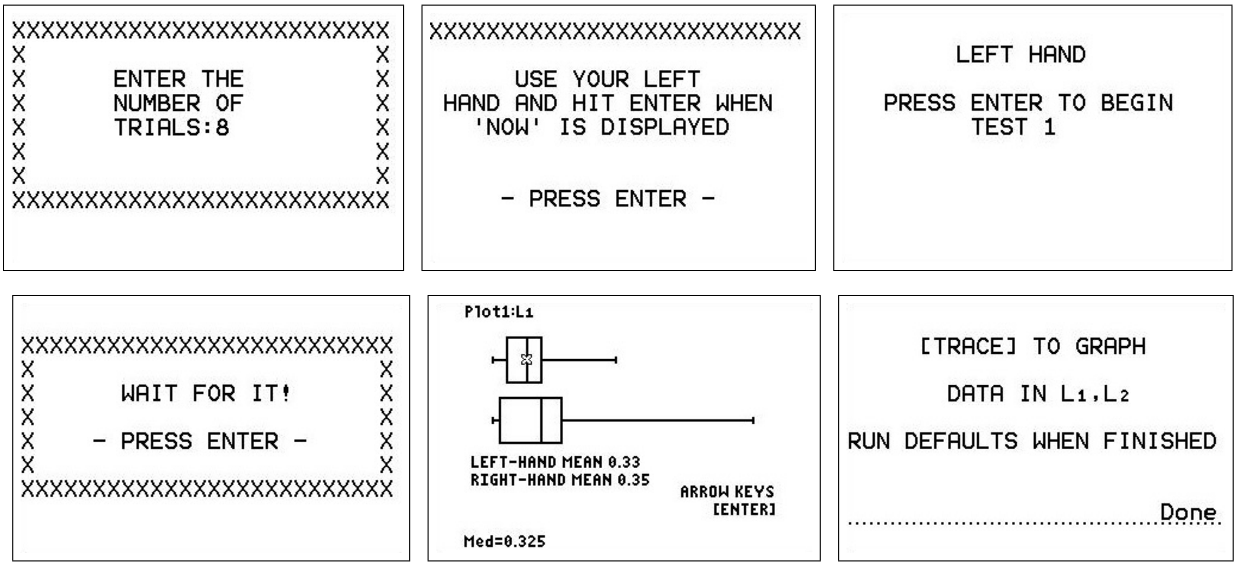
Notes for Teachers

This activity uses a calculator program REACTHND/RCTHNDCE to create box plots using reaction-time data collected from the students. The scenario may need updating to current technology.

Before you begin this activity, have the students clear all lists in their calculators: press 2nd + (mem), then 4 to select ClrAllLists and enter to carry out the process.



To run the REACTHND/RCTHNDCE program, press prgm, then the number against the program name and enter to start it. Below are some of the screens that students will see in collecting the data.



DEFAULTS in the last screen refers to the DEFAULTS/DEFLTCE program. If you do not have this program, turn off the StatPlots in y= with enter.

Note: Students should rest their finger on the full-stop key instead of having it hover over the enter key.

The means are displayed on the screen. Using the arrow keys when the box plots are graphed will give students the following information for each hand.

- minX: minimum — smallest value in the data
- Q1: lower quartile — cutoff point for the bottom 25% of the data
- Median: cutoff point for the bottom 50% of the data
- Q3: upper quartile — cutoff point for the top 25% of the data
- maxX: maximum — largest value in the data

The program REACT/REACTCE for the next activity works in exactly the same way as REACTHND/RCTHNDCE here.

Real-World Statistics #2

Introduction

You are a statistician employed by a car company ROLLER. In their current advertising campaign, they claim that their cars have the least road noise of any current car on the market and therefore their cars are the safest, as drivers are not distracted as they are in other noisier makes of car.

You have been commissioned to compile data to measure the effect of noise on the reaction times of drivers. The data are required to be collected under each of the following conditions.

Scenario 1: Total concentration and complete silence.

Scenario 2: Relaxed in relative quiet.

Scenario 3: Distracted in a noisy environment.

Method

Use the graphics-calculator program REACT/REACTCE to measure reaction times for each of the three scenarios. The program leads you through the steps. Note that the reaction time just pressing a calculator key is likely to be significantly shorter than when you move your foot onto the brake pedal and push it. The simulation will be more realistic if you rest your finger on the [y=] key while waiting for NOW to be displayed.

Do *at least* 15 trials for each scenario. After you have done all the trials, the program will present the results as boxplots, one for each scenario.

Use the arrow keys to move around the boxplots.

Results

Sketch your boxplots and complete the quartile analysis below — use the arrow keys to find all the numbers (except the means) from the boxplots.

Press [enter] to obtain the means.

	Scenario 1	**Scenario 2**	**Scenario 3**
minX			
Q1			
Median			
Q3			
maxX			
Mean			

Press enter to finish the program. As the final screen tells you, the data are available in lists $L_1 - L_3$ for further analysis.[12]

Analysis

From your results, describe the differences between your reaction times in the three scenarios. Explain what these mean in plain English — don't just say the means are different, or something similar.

From your findings, should ROLLER continue with their current advertising campaign? Why or why not?

Compare your results and conclusions with those of other 'drivers'. Summarise any similarities and differences.

[12]Press stat 1 to see the data. You can replot the boxplots by pressing y= and using the cursor and enter to highlight Plot1, Plot2 and Plot3 at the top of the screen. Press trace to replot. If you have changed the window since the boxplots were plotted by the program, press zoom 9 to set an appropriate window.

Further analysis

On one calculator, enter the *median* from each participant for Scenario 1 into L1, for Scenario 2 into L2 and for Scenario 3 into L3. Turn on Plot1, Plot2 and Plot3 in [y=] and press [zoom] [9] to plot the boxplots of these medians.

Sketch the boxplots of the class medians and fill in the quartile analysis below.

	Scenario 1	**Scenario 2**	**Scenario 3**
minX			
Q1			
Median			
Q3			
maxX			
Mean			

Now do the same for the *means*. Sketch the boxplots of the class means and fill in the quartile analysis below.

	Scenario 1	**Scenario 2**	**Scenario 3**
minX			
small Q1			
Median			
Q3			
maxX			
Mean			

What is the difference between the boxplots of the median and the boxplots of the mean of the group data?

Which of these would ROLLER use to support their ads? Why?

Real-World Statistics #3

Introduction

You have just begun to drive a car. One of the important and basic things you have to learn is how far behind the car in front should you drive so that you can stop safely if it brakes.

The calculator program CARSTOP/CARSTPCE simulates this situation. You select a speed to travel at and the distance behind the car in front. After a random interval, the brake lights of the car in front will come on, indicating that it is braking hard to stop in minimum distance. Will you stop in time or will you crash into the car in front?

When you see the brake lights come on, you have to move your finger from the [y=] key (top left) to the [enter] key (bottom right) and press it. This simulates taking your foot off the accelerator pedal, moving it to the brake pedal and pushing it hard.[13] Each time you do this, the calculator records your reaction time, as well as indicating the outcome. The program assumes that both cars stop in the same distance once the brakes are applied.

Method

Run the CARSTOP/CARSTPCE program.

Choose a speed S in kilometres per hour (for example, city 60 km/h, main road 80 km/h, highway 100 km/h). Choose a distance behind D — specified in metres and car lengths. To set speed or distance, move the cursor (highlight) to either S or D with the left/right arrow keys. Use the up/down arrow keys to increase/decrease the value.

When you have set the values, press [y=] to start driving and keep your finger there.

Press [enter] to apply your brakes when the brake lights come on. There is a short time interval (as in real life) before the outcome is known.

The instructions at the top right of the screen tell you which keys are operative at each step.

Run 15 trials at each speed, changing the distance behind if necessary, for each of the three scenarios below (you will have to re-run the program for each scenario). These scenarios simulate different driving conditions.

Scenario 1: Total concentration and complete silence.

Scenario 2: Relaxed in relative quiet, e.g. radio or music playing.

Scenario 3: Distracted in a noisy environment, e.g. people talking to you.

When you have completed your trials for a scenario, press the [Q] key ([alpha] not necessary), select QUIT and PLOT BOXPLOT.

[13] The real-life situation is still likely to take longer than the calculator simulation.

Results

Sketch the boxplot for each scenario and complete the quartile analysis below — use the arrow keys to find all the numbers (except the mean) from each boxplot.

	Scenario 1	Scenario 2	Scenario 3
minX			
Q1			
Median			
Q3			
maxX			
Mean			

Press enter to finish the program. As the final screen tells you, the data for the scenario are available in list L1 for further analysis.[14]

Analysis

From your results, describe the differences between your reaction times in the three scenarios. Explain what these mean in plain English — don't just say the means are different, or something similar.

What do these results suggest regarding the distance you should drive behind the car in front in each of the scenarios?

[14]Press stat 1 to see the data. You can replot the boxplot by pressing y= and using the cursor and enter to highlight Plot1 at the top of the screen. Press trace to replot. If you have changed the window since the boxplot was plotted by the program, press zoom 9 to set an appropriate window.

Given that a safe distance is one at which you can stop before running into the car in front 100% of the time, fill in the three relevant columns in the table below. You may need to do some more trials for this.

	Safe distance (m)			
Speed km/h	**Scenario 1**	**Scenario 2**	**Scenario 3**	**Three-second criterion**
60				
80				
100				

Can you come up with a rule of thumb that relates a safe distance in metres or car lengths to speed in km/h? Check using the program that your rule of thumb makes sense for higher and lower speeds.

A time rule of thumb often taught is that there should be a three-second gap between cars. Calculate corresponding distance values to add to your table above. Compare with your values for the three scenarios and comment.

What distance rule of thumb (car lengths) corresponds (approximately) to the recommended three-second time rule of thumb?

How would your reaction time using the program compare, do you think, to your reaction time in taking your foot off the accelerator pedal and pressing the brake pedal down hard. How would this change your rule of thumb?

11 Simultaneous Equations

Year 9, Levels 1 & 2; Strand: Algebra; Sub-strand: Solve Simultaneous Linear Equations.

Authors: Margie Smith, Peter McIntyre.

Solving simple simultaneous equations numerically (with a table), graphically and algebraically.

Simultaneous Equations 1

Example: Solve the simultaneous equations

$$x - y = -1 \quad (1)$$

$$x + y = 3. \quad (2)$$

This means we have to find pairs of values, x and y, that satisfy both equations simultaneously, i.e. such that $x - y = -1$ <u>and</u> $x + y = 3$. Each pair of values, x, y, is called a solution of the simultaneous equations.

To use a table or graph to find the solutions, first re-write the equations with y as subject,

$$y = x + 1 \quad (3)$$

$$y = 3 - x. \quad (4)$$

Using a table

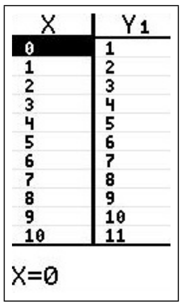
Here we use the calculator to generate a table of values of x and the corresponding y values from each of Equations (3) and (4).

Press [y=] and set Y1=X+1 (Equation (3)).
For X, press the [X,T,n,θ] key.

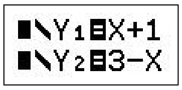
We'd like the table to start at X = 0 and increment by 1. Press [tblset] ([2nd] [window]) and set TblStart to 0 and ΔTbl to 1.

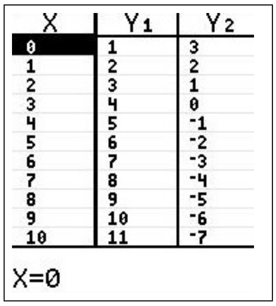
Now press [table] ([2nd] [graph]).

For each pair of values X, Y1 on the screen, verify that the pair satisfies both Equation (1), i.e. verify that X−Y1=−1, **and** Equation (3), i.e. verify that Y1=X+1. Try scrolling up and down in both the X and Y1 columns.



Now enter the second equation: press [y=] and put Equation (4) into Y2.





X	Y1	Y2
0	1	3
1	2	2
2	3	1
3	4	0
4	5	-1
5	6	-2
6	7	-3
7	8	-4
8	9	-5
9	10	-6
10	11	-7

X=0

Press [table].

For each pair of values X, Y2 on the screen, verify that it satisfies both Equation (2), i.e. verify that X+Y2=3, **and** Equation (4), i.e. verify that Y2=3−X.

Now we wish to find all pairs of values X, Y that satisfy Equations (3) **and** (4) (or equivalently Equations (1) and (2)) **simultaneously**.

In the table, all pairs (X, Y1) satisfy Equation (3) and all pairs (X, Y2) satisfy Equation (4).

If, for some X, we have Y1 = Y2, then the pair (X, Y1) = (X, Y2) satisfies both equations simultaneously, and is therefore a solution of the simultaneous Equations (3) and (4).

Look at your table to find this solution — there's only one solution in this case, but scroll up and down the table just to make sure.

Finally check your solution back in Equations (1) and (2).

Exercises

Use a table to find the solutions to the following simultaneous equations. You may need to scroll up or down the table to do this. Check that your answers satisfy both equations.

Be aware of the difference between the white change-sign key [(-)] *and the blue minus key* [−].

1. $$\begin{aligned} y &= 2x+3 \\ y &= 3x+1. \end{aligned}$$

2. $$\begin{aligned} 4x-y &= -5 \\ 2x+y &= -7. \end{aligned}$$

3. $$\begin{aligned} 2x-y &= -8 \\ x+y &= -1. \end{aligned}$$

Simultaneous Equations 2

Example: Solve the simultaneous equations (as in *Simultaneous Equations 1*).

$$x - y = -1 \quad (1)$$

$$x + y = 3 \quad (2)$$

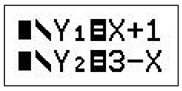
or, equivalently

$$y = x + 1 \quad (3)$$

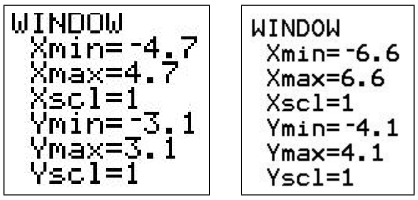
$$y = 3 - x. \quad (4)$$

Using a graph

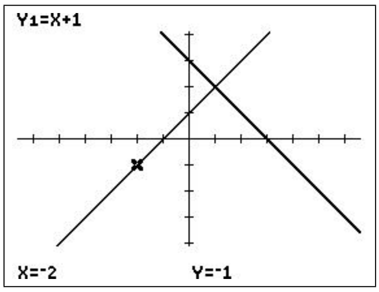
Press y= and make sure you have Equations (3) and (4) entered into your calculator.



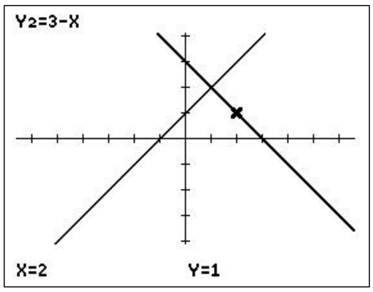
Press zoom 4 (ZDecimal) to set suitable axes (shown in the figure: 84Plus on the left, 84CE on the right) and plot the graphs of the functions in Equations (3) and (4).



Press trace and move the cursor along the graph of Y1 using the left- and right-arrow keys. The coordinates at the bottom of the screen are points on this graph. Verify, for a couple of points, that they satisfy Equation (1), i.e. that X−Y=−1.



Press the down-arrow key to move the cursor to Y2. Again, verify, for a couple of points, that they satisfy Equation (2), i.e. that X+Y=3. Move the cursor to X=0 and press the down-arrow key several times to toggle between Y1 and Y2. *What happens to the Y value?*

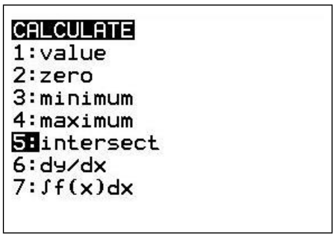


Which point satisfies both equations? Move the cursor there and press the down-arrow key several times to verify that the point does lie on both graphs. This should be the same solution you found in *Simultaneous Equations 1*.

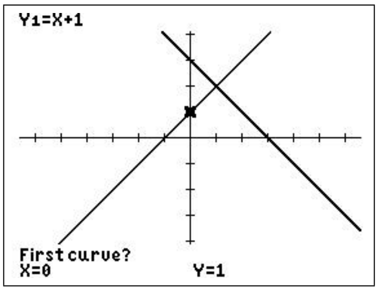
In this case we were lucky. We could move the cursor exactly to the point we wanted. *What if it had turned out that the solution was at X = 0.25?* Try to move the cursor there along either curve.

We can use a built-in calculator operation *intersect* to find the intersection of any two graphs on the calculator screen. Try it here for practice, so you know how to use it when it is necessary.

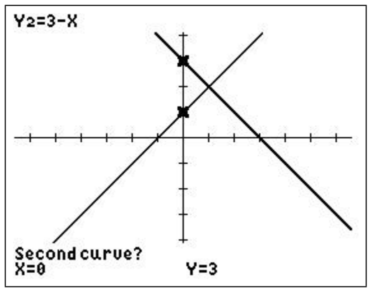
1. Press calc (2nd trace) to bring up the CALCULATE menu.



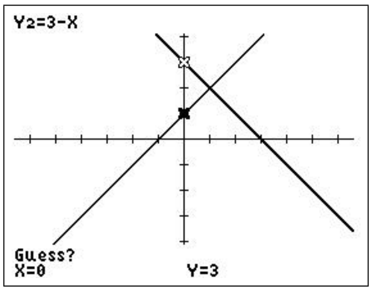
2. Press 5 to select *intersect*. The calculator puts a flashing cursor on the graph of Y1 and asks you to choose the first curve to intersect. Press enter to confirm that this is one of the graphs you want to intersect.



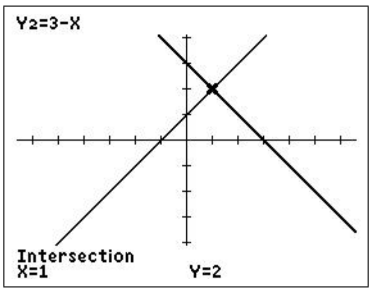
3. The calculator next moves the flashing cursor onto the graph of Y2 and asks you to choose the second curve. Press enter to confirm that this is the other graph you want to intersect.



4. The calculator next asks you for a 'guess' to the intersection point. (This is because there may be more than one intersection point on screen — the guess picks out the one you want.) Press enter to select whatever point you are on as the guess — it's not crucial here.



5. The calculator now gives (an approximation to) the intersection point. *Did it get the right point here? How accurate was its answer in this case?*



Exercises

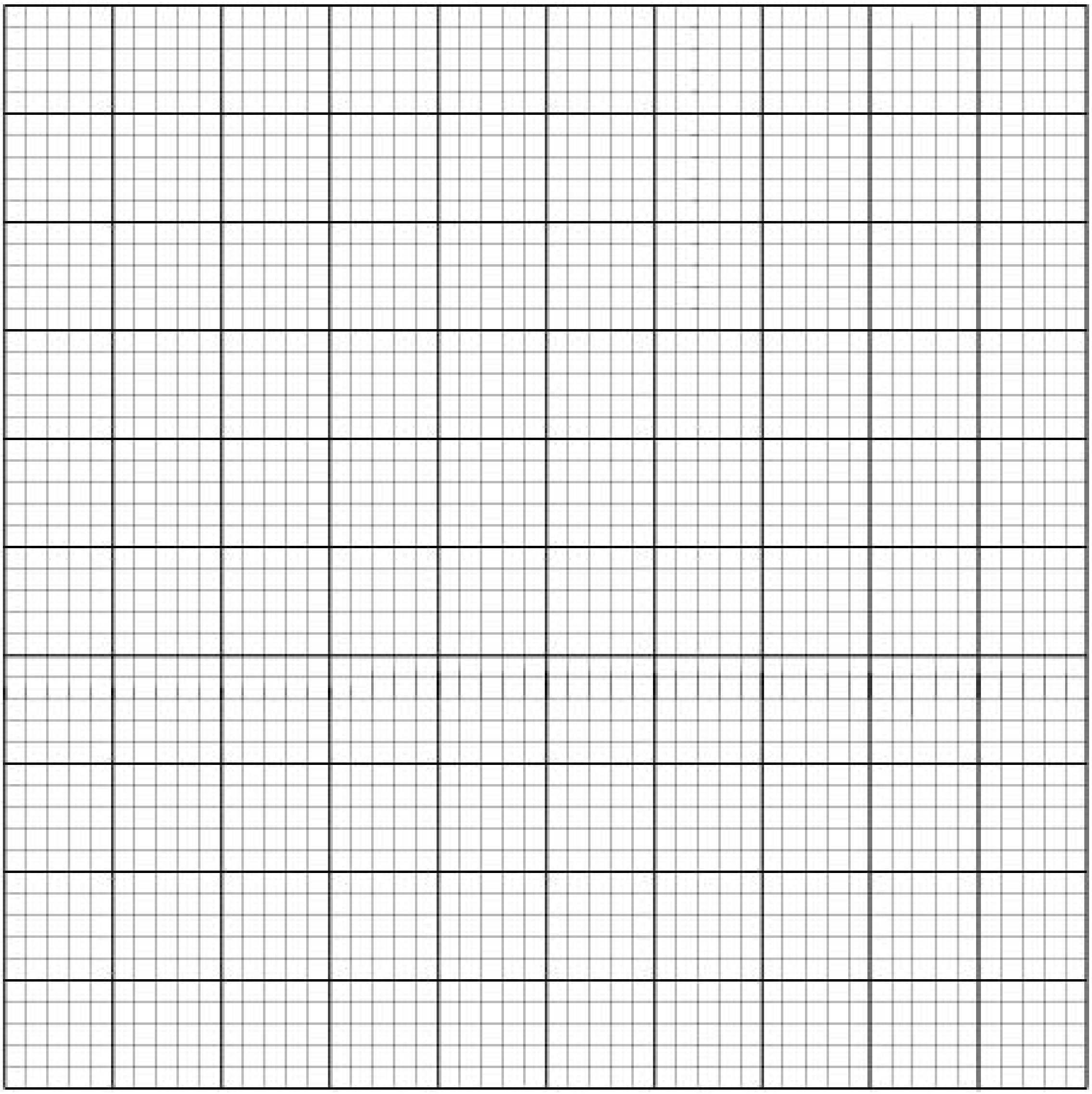
Find the solutions to the following pairs of simultaneous equations by graphing each pair of lines accurately on graph paper and finding the point of intersection. Label each axis and put on scales. Label the point of intersection.

Check your answer using a calculator graph and *intersect* if necessary. You may need to change the window to see the point of intersection. Check that your answers satisfy both equations.

1. $$\begin{aligned} y &= 2x+3 \\ y &= 3x+1. \end{aligned}$$

2. $$\begin{aligned} 4x-y &= -5 \\ 2x+y &= -7. \end{aligned}$$

3. $$\begin{aligned} 2x-y &= -8 \\ x+y &= -1. \end{aligned}$$



Simultaneous Equations 3

Example: Solve the simultaneous equations (as in *Simultaneous Equations 1, 2*)

$$x - y = -1 \qquad (1)$$

$$x + y = 3 \qquad (2)$$

or, equivalently

$$y = x + 1 \qquad (3)$$

$$y = 3 - x. \qquad (4)$$

Solving algebraically

These simple simultaneous equations are easy to solve algebraically. Between the two equations, we can eliminate one of the variables and solve the resulting equation for the other variable.

Using Equations (1) and (2): if we add the equations (sum of LHS = sum of RHS), we have

$$x - y + x + y = -1 + 3.$$

$$\therefore\ 2x = 2.$$

$$\therefore\ x = 1.$$

Putting $x = 1$ back into Equation (1) $\big($or Equation (2)$\big)$ gives $y = 2$.

Therefore, the solution to the simultaneous equations is $x = 1$, $y = 2$, as we found previously.

If we use Equations (3) and (4), we subtract Equation (4) from Equation (3) to give $0 = 2x - 2$, so that $x = 1$. Substituting into Equation (3) $\big($or Equation (4)$\big)$ gives $y = 2$. Alternatively, we could have added Equations (3) and (4) to find y first.

Exercises

Find the solutions to the following simultaneous equations algebraically Write down what operations you do to achieve this, e.g. (2)−(1), (1)+(2), etc. Check that your answers satisfy both equations.

1. $$\begin{aligned} y &= 2x+3 \\ y &= 3x+1. \end{aligned}$$

2. $$\begin{aligned} 4x-y &= -5 \\ 2x+y &= -7. \end{aligned}$$

3. $$\begin{aligned} 2x-y &= -8 \\ x+y &= -1. \end{aligned}$$

In the following exercises, solve the simultaneous equations first using a table, then a graph and finally algebraically.

4. $$\begin{aligned} 2x-y &= -3 \\ x-y &= 1. \end{aligned}$$

5. $$\begin{aligned} x+y &= -1 \\ x-y &= -5. \end{aligned}$$

6. $$\begin{aligned} x-y &= 6 \\ 2x+y &= 6. \end{aligned}$$

7. $$\begin{aligned} 2x-y &= 1 \\ x-y &= -5. \end{aligned}$$

8. $$\begin{aligned} x+y &= 3 \\ x-y &= 6. \end{aligned}$$

9. $$\begin{aligned} 2x+y &= 6 \\ y-4x &= -9. \end{aligned}$$

10. $$\begin{aligned} 3x+y-1 &= 0 \\ 10x-y-25 &= 0. \end{aligned}$$

12 Speeding — A Study in Linear Functions

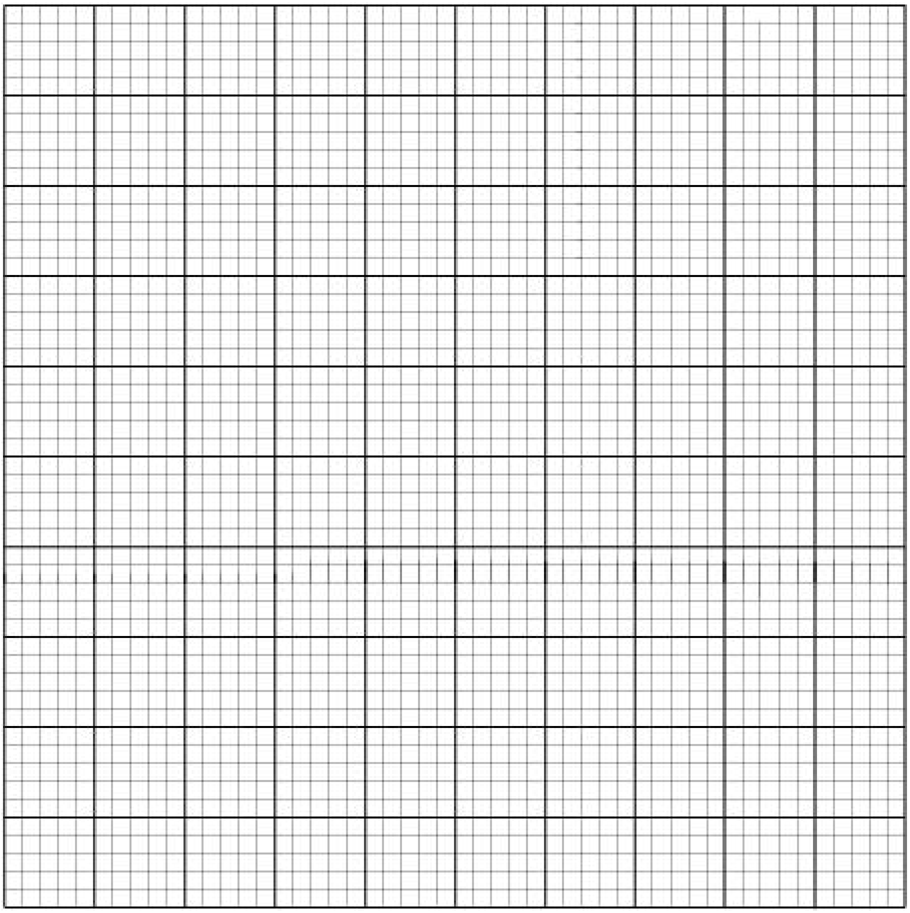
Year 9, Levels 1 & 2; Strand: Algebra; Sub-strand: Coordinate Geometry.

Authors: Bonnie Peterson and Eddie Keel in Graphing Calculators in Mathematics Grades 7–12, Center of Excellence for Science and Mathematics Education (CESME) at The University of Tenessee at Martin, USA. Modified by Peter McIntyre.

Students learn and apply basic knowledge of linear functions to problems involving speeding tickets.

Activity 1: Review

1. Plot the following points: $A\,(2,5)$; $B\,(-1,3)$; $C\,(0,2)$; and $D\,(4,-3)$.

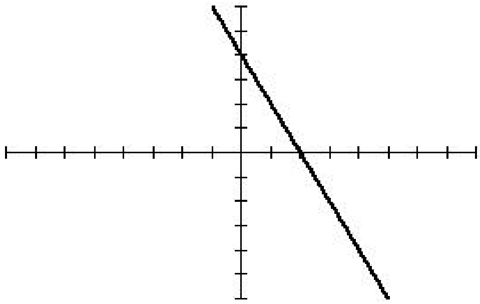


2. From the graph, using a ruler, work out the slope of the line through the points $(2,5)$ and $(-1,3)$.

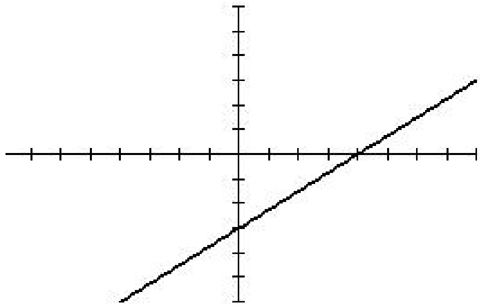
3. Calculate the slope of the line through the points $(-2,5)$ and $(-4,0)$.

4. Calculate the slopes of the two lines below. The tick marks are 1 unit apart.

(a)

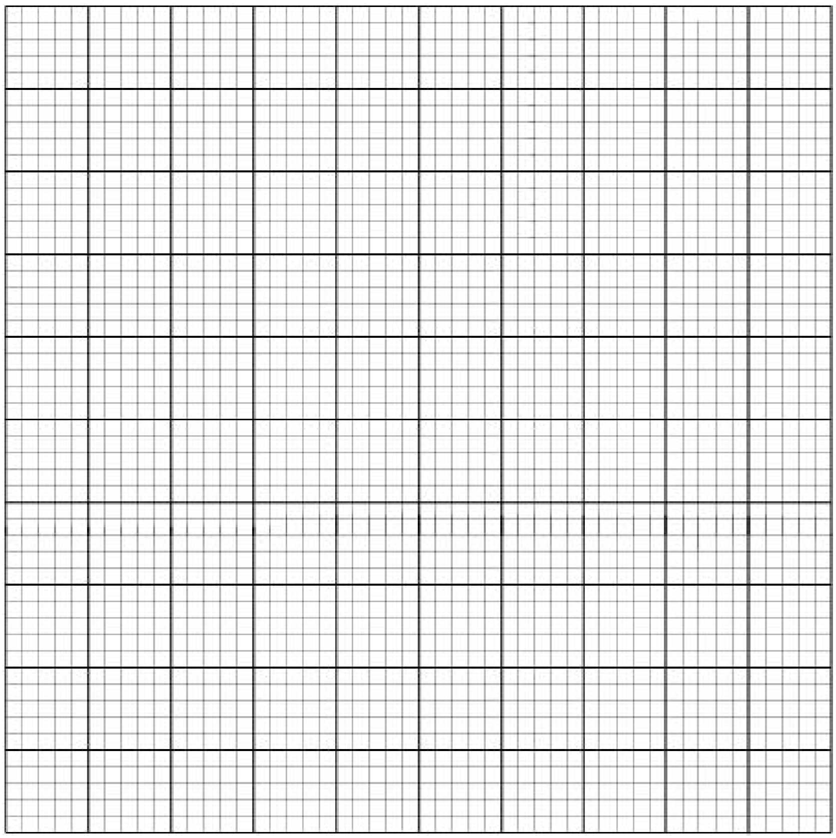


(b)



5. Without using a graphics calculator, graph the following lines:

(a) $y=2x+3$; **(b)** $y=\frac{2}{3}x+1$; **(c)** $2x+3y=6$.



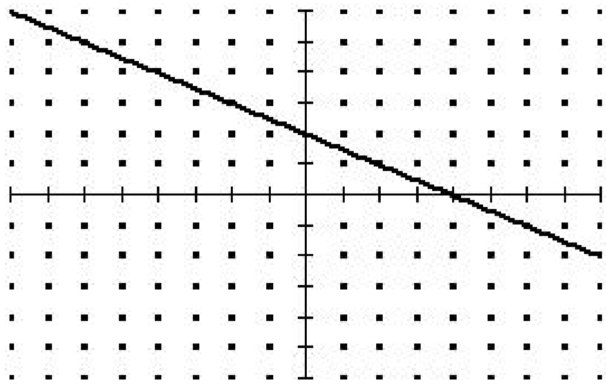
6. Draw the following lines on your graphics calculator using a suitable window:

(a) $y=3x-4$; **(b)** $y=\frac{4}{3}x+2$; **(c)** $2x+3y=6$.

See page **??** for instructions.

7. Find the equation of the straight line through the points $(1,5)$ and $(9,2)$.

8. Find the equation of the line in the following graph using the slope-intercept form of the line $y = mx + b$. The grid points are 1 unit apart.



9. Using linear regression on your calculator, find the equation of the line through the points $(-1, -3)$, $(1, 1)$, $(2, 3)$, $(12, 23)$. Check that each point lies on the line.

See page **??** for instructions.

10. Using your graphics calculator, complete the following table for $f(x) = x^2 + 5x^6$.

See page **??** for instructions.

x	$f(x)$
0	
1	
2	
3	
4	
5	
6	
7	
8	

Instructions

The instructions and figures are for a TI-84CE. Those for a TI-84 are very similar but the key labels are in capital letters.

Graphing lines *Problem 6 of Activity 1*

- Press the [y=] key.

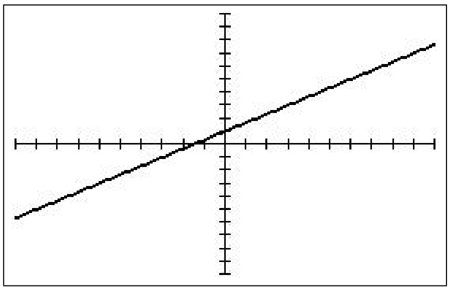
```
\Y1=
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
\Y8=
\Y9=
```

- Type in the equation. Use the [X, T, θ, n] key for the independent variable X. The equation in the figure is $y = \frac{2}{3}x + 1$.

```
\Y1=(2/3)X+1
```

- Press the [zoom] key; press [6] for *ZStandard*.

```
ZOOM MEMORY
1:ZBox
2:Zoom In
3:Zoom Out
4:ZDecimal
5:ZSquare
6:ZStandard
7:ZTrig
8:ZInteger
9↓ZoomStat
```

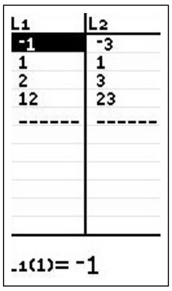


window $[-10, 10, 1] \times [-10, 10, 1]$

- If the graph on your calculator is not acceptable or does not show up, press the [window] key and change the *Xmin*, *Xmax*, *Ymin* and *Ymax* settings.

Regression *Problem 9 of Activity 1*

- Press stat.
- Press 1 for *Edit*.
- If the names for the columns are L1, L2, L3, etc, skip the next step.
- Press stat 5 for *SetUpEditor*. Press enter. Press stat 1 to return to *Edit*.
- If there are numbers in the lists already, you need to clear them. Use the up arrow to highlight the heading, say L1, press clear, then enter. Repeat as necessary for the other lists.
- Enter the x coordinates of the points in the list L1, the y coordinates in L2. Move to the beginning of L1, enter the first x coordinate and press enter. Repeat for all the x coordinates.

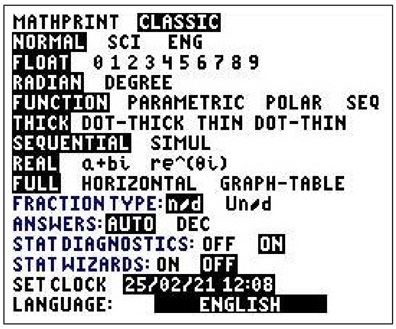


- Next move to the beginning of L2, enter the first y coordinate and press enter. Repeat for all the y coordinates.

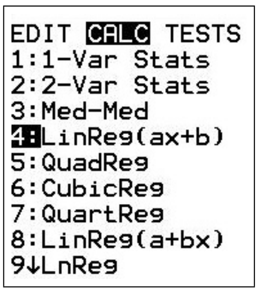
  Check that the two lists are of the same length and that the numbers you entered are correct.

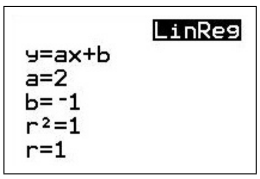
  Press quit (2nd mode) to return to the Home screen.

- To make sure the correlation coefficients are displayed, press mode. Make sure STAT DIAGNOSTICS are ON.



- To run the linear regression, press stat, use the right arrow to move to the CALC menu and press 4 for *LinReg*$(ax+b)$. Press enter to execute it.[15] The results of the calculation will appear on the screen. $r^2 = 1$ means a perfect fit.

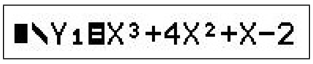




[15] The command assumes that the relevant lists (defaults) are L1 and L2.

Making a table *Problem 10 of Activity 1*

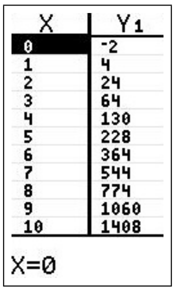
- Press the [y=] key.
- Enter the equation you wish to use.



- Press [tblset] ([2nd] [window]). *TblStart* is the starting X value and Δ*Tbl* is the increment. Change these to what you want.

- Press [table] ([2nd] [graph]).

  Scroll down in the X or Y1 column, up in the X column.



X	Y1
0	-2
1	4
2	24
3	64
4	130
5	228
6	364
7	544
8	774
9	1060
10	1408

X=0

Activity 2: An introduction to speeding tickets

You are driving along the highway, when you are stopped by a policeman. To your disgust, you are given a speeding ticket. The cost of a ticket T (in dollars) on a road with a speed limit of 100 km/h is determined by the following function:[16]

$$T = 120 + 1(S - 110)$$

S is the speed you are travelling at in kilometres per hour.

1. What would be the cost of your speeding ticket if you were travelling at 130 km/h? Explain how you arrived at your answer.

2. Find T if $S = 90$.

3. Explain why you would not get a traffic ticket if you were travelling at 90 km/h, even though your answer in Q2 indicates your speeding ticket is a certain amount.

4. For what values of S would you not use this formula? Why?

5. Your teacher received a $135 speeding ticket on the same road. How fast was he or she going? Explain how you arrived at your answer.

[16] The 1 before the brackets is here for a reason: revealed in Activity 3.

6. The local government has hired you to establish a way for the police to determine quickly the cost of a traffic ticket. You decide a table would be the best solution to this problem. Using your graphics calculator, make a speeding-ticket table for speeds between 110 km/h and 120 km/h, the first part of the full table.

Speed (km/h)	**Cost of ticket** ($)
110	
111	
112	
113	
114	
115	
116	
117	
118	
119	
120	

7. Suburban streets have a speed limit of 50 km/h. The local government has hired you as a consultant to establish a formula that determines the cost of a speeding ticket on these streets. They have given you this stipulation:

The minimum ticket should be $50.

Create a fair and reasonable function using S and T. Explain how you arrived at your function, and why it is fair and reasonable.

Activity 3: When is a ticket the same?

Michelago in the ACT and Bredbo in NSW (both on the Monaro Highway on the way from Canberra to Cooma) are having a special blitz on motorists speeding to and from the snowfields on the highway near each town (speed limit 100 km/h).

The local police use the following functions to determine the cost of a speeding ticket:

Michelago: $T = 130 + 2(S - 110)$

Bredbo: $T = 10 + 10(S - 110)$

1. Use a graphics calculator to complete the table below, listing the costs of speeding tickets near Michelago and near Bredbo for speeds between 120 km/h and 130 km/h.

At what speed is the cost of a speeding ticket the same in both places? Circle this speed on your table.

Michelago

Speed (km/h)	Cost of ticket ($)
120	
121	
122	
123	
124	
125	
126	
127	
128	
129	
130	

Bredbo

Speed (km/h)	Cost of ticket ($)
120	
121	
122	
123	
124	
125	
126	
127	
128	
129	
130	

2. For every additional km/h you go over the speed limit near Michelago, what happens to the cost of a speeding ticket? *Hint*: Use the table in Q1.

3. Write $T = 130 + 2(S - 110)$ in slope-intercept form ($T = mS + b$). What is the slope of the equation? What does the slope mean? What are its units?

4. For every additional km/h you go over the speed limit near Bredbo, what happens to the cost of a speeding ticket?

5. Write $T=10+10(S-110)$ in slope-intercept form. What is the slope of the equation? What does the slope mean? What are its units?

6. Explain how slope and the change in the cost of a speeding ticket are related.

7. Sketch the speeding-ticket functions on the same plot with the help of your graphics calculator. What window did you use and why?

8. Describe how the graphs are similar. Why do you think they are similar?

9. Describe how the graphs are different.

10. Find the speed at which the cost of a speeding ticket is the same for both Michelago and Bredbo. There are at least three ways to attack this problem.

(a) **Algebraic method:** Solve the equations for Michelago and Bredbo simultaneously. Show all working.

(b) **Graphical method:** Graph the two functions on your calculator. Where do they intersect? Use *intersect* in the calc menu if you know how.

Justify your answer — draw the graphs, showing the point of intersection.

(c) **Numerical method:** You already did this in Question 1. Use your answer there as a check.

Activity 4: Piecewise and other functions

Some places use a schedule similar to the following to work out the cost of a speeding ticket.

Traffic offence	**Fine** ($)
Speeding — Exceed limit	50
Speeding — Limit + 10 km/h	70
Speeding — Limit + 20 km/h	90
Speeding — Limit + 30 km/h	120
Speeding — Limit + 40 km/h	160

This is a piecewise function.

1. Draw the graph that represents the speeding fines above for a speed limit of 80 km/h.

2. What would be the cost of a ticket if your speed in a 80 km/h zone is

92 km/h?

110 km/h?

115 km/h?

130 km/h?

Explain how you arrived at these answers.

3. Write down this function in standard functional form.

$$T(S) = \begin{cases} 50 & 80 < S < 90 \\ & \\ & \\ & \end{cases}$$

Another example of speeding fines in an 80 km/h zone is given below.

Speed (km/h)	**Cost of ticket** ($)
90 – 99	50
100 – 109	100
110 – 119	120
120 – 129	140
130 or greater	180

This is another piecewise function.

4. Explain the pattern for speeds from 100 km/h to 129 km/h.

5. Why do you think the $50 fine does not fit the pattern?

6. Why does the $180 fine not fit the pattern?

7. Why do you think there is no listing for speeds from 81 km/h to 89 km/h?

8. You have been hired by the government, who wants to deter people from speeding. You have been asked to create a function that is not linear and will particularly punish high speeds.

Create a function and explain why your function would accomplish this goal. Use tables and graphs to support your claim.

Notes for Teachers

Activity 2: An introduction to speeding tickets

You are driving along the highway, when you are stopped by a policeman. To your disgust, you are given a speeding ticket. The cost of a ticket T (in dollars) on a road with a speed limit of 100 km/h is determined by the following function:

$$T = 120 + 1(S - 110).$$

S is the speed you are travelling at in kilometres per hour.

1. What would be the cost of your speeding ticket if you were travelling at 130 km/h? Explain how you arrived at your answer.

The cost of the ticket T is $120 + 1(130 - 110) = \$140$.

2. Find T if $S = 90$.

$T = 120 + 1(90 - 110) = 100$.

3. Explain why you would not get a traffic ticket if you were travelling at 90 km/h, even though your answer in Q2 indicates your speeding ticket is a certain amount.

The speed 90 km/h is less than the speed limit.

4. For what values of S would you not use this formula? Why?

You would not use this formula for speeds less than or equal to the speed limit, i.e. for $S \leqslant 100$. The formula appears to indicate that it might not actually be used for speeds less than 110 km/h, perhaps to allow for speedometer error, but this is not certain.

5. Your teacher received a \$135 speeding ticket on the same road. How fast was he or she going? Explain how you arrived at your answer.

We have to find S such that $T = 135$, that is solve

$$120 + 1(S - 110) = 135$$

for S. This gives $S = 110 + 135 - 120 = 125$.

The teacher was travelling at 125 km/h.

6. The local Government has hired you to establish a way for the police to determine quickly the cost of a traffic ticket. You decide a table would be the best solution to this problem. Using your graphics calculator, make a speeding-ticket table for speeds between 110 km/h and 120 km/h, the first part of the full table.

Speed (km/h)	**Cost of ticket** ($)
110	120
111	121
112	122
113	123
114	124
115	125
116	126
117	127
118	128
119	129
120	130

7. Suburban streets have a speed limit of 50 km/h. Suppose the local government has hired you as a consultant to establish a formula that determines the cost of a speeding ticket on these streets. They have given you this stipulation:

The minimum ticket should be $50.

Create a fair and reasonable function using S and T. Explain how you arrived at your function, and why it is fair and reasonable.

Any sensible answer here. Class discussion?

Activity 3: When is a ticket the same?

Michelago in the ACT and Bredbo in NSW (both on the Monaro Highway on the way to Cooma) are having a special blitz on motorists speeding to and from the snowfields on the highway near each town (speed limit 100 km/h).

The local police use the following functions to determine the cost of a speeding ticket:

$$\textbf{Michelago:}\quad T=130+2(S-110)$$

$$\textbf{Bredbo:}\quad T=10+10(S-110)$$

1. Use a graphics calculator to complete the table below, listing the costs of speeding tickets near Michelago and near Bredbo for speeds between 120 km/h and 130 km/h.

At what speed is the cost of a speeding ticket the same in both places? Circle this speed on your table.

Michelago

Speed (km/h)	Cost of ticket ($)
120	150
121	152
122	154
123	156
124	158
125	**160**
126	162
127	164
128	166
129	168
130	170

Bredbo

Speed (km/h)	Cost of ticket ($)
120	110
121	120
122	130
123	140
124	150
125	**160**
126	170
127	180
128	190
129	200
130	210

The cost of a speeding ticket is the same in both places at a speed of 125 km/h.

2. For every additional km/h you go over the speed limit near Michelago, what happens to the cost of a speeding ticket? *Hint*: Use the table in Question 1.

For every additional km/h you go over the speed limit near Michelago, the cost of a speeding ticket goes up by $2.

3. Write $T=130+2(S-110)$ in slope-intercept form ($T=mS+b$). What is the slope of the equation? What does the slope mean? What are its units?

In slope-intercept form, $T=2S-90$. The slope is 2. This is the number of dollars that the cost of a speeding ticket increases for an increase of 1 km/h in the speed. The units are dollars per km/h.

4. For every additional km/h you go over the speed limit near Bredbo, what happens to the cost of a speeding ticket?

For every additional km/h you go over the speed limit near Bredbo, the cost of a speeding ticket goes up by $10.

5. Write $T = 10 + 10(S - 110)$ in slope-intercept form. What is the slope of the equation? What does the slope mean? What are its units?

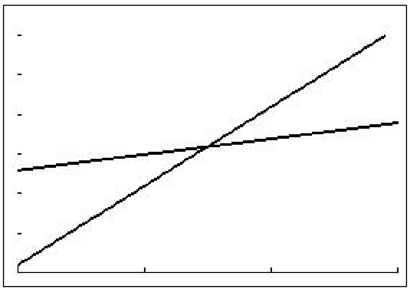
In slope-intercept form, $T = 10S - 1090$. The slope is 10. This is the number of dollars that the cost of a speeding ticket increases for an increase of 1 km/h in the speed. The units are dollars per km/h.

6. Explain how slope and the change in the cost of a speeding ticket are related.

The slope is equal to the change in the cost of a speeding ticket with an increase in speed of 1 km/h.

7. Sketch the speeding-ticket functions on the same plot with the help of your graphics calculator. What window did you use and why?

A window of $[110, 140, 10] \times [0, 300, 50]$ to cover the range of speeds (X axis) and the range of fines (Y axis).



8. Describe how the graphs are similar. Why do you think they are similar?

The graphs are both straight lines. Both functions are linear functions.

9. Describe how the graphs are different.

The graphs have different slopes.

10. Find the speed at which the cost of a speeding ticket is the same for both Michelago and Bredbo. There are at least three ways to attack this problem.

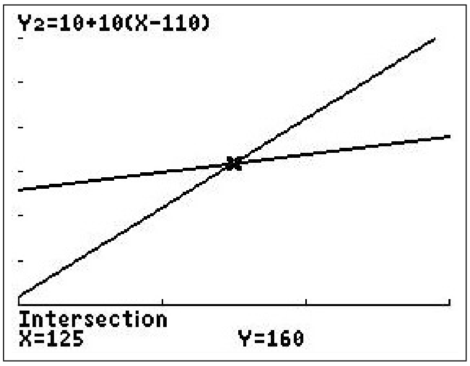
(a) Algebraic method: Solve the equations for Michelago and Bredbo simultaneously. Show all working.

Equating the two expressions for T gives

$$
\begin{aligned}
130 + 2(S-110) &= 10 + 10(S-110). \\
\therefore 8(S-110) &= 120. \\
\therefore S - 110 &= \frac{120}{8} \\
&= 15. \\
\therefore S &= 110 + 15 = 125.
\end{aligned}
$$

The cost of a speeding ticket is the same for both Michelago and Bredbo at a speed of 125 km/h.

(b) Graphical method: Graph the two functions on your calculator. Where do they intersect? Justify your answer — show the graphs, the point of intersection and the window. Use *intersect* in the calc menu if you know how.



window $[110, 140, 10] \times [0, 300, 50]$

(c) Numerical method: You already did this in Question 1.

Activity 4: Piecewise and other functions

Some places use a schedule similar to the following to work out the cost of a speeding ticket.

Traffic offence	**Cost** ($)
Speeding — Exceed limit	50
Speeding — Limit + 10 km/h	70
Speeding — Limit + 20 km/h	90
Speeding — Limit + 30 km/h	120
Speeding — Limit + 40 km/h	160

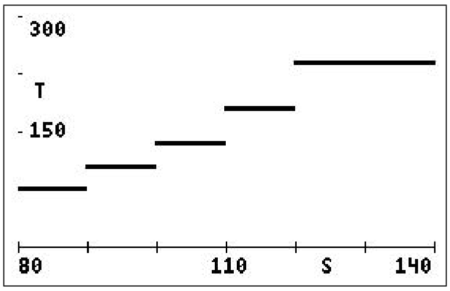
This is a piecewise function.

1. Draw the graph that represents the speeding fines above for a speed limit of 80 km/h.

Note that students are not expected to use their graphics calculator to do this. The procedure using a graphics calculator is shown below to give the graphs and show teachers how to do it if they wish. The function uses logical operators such as (X > 80), which gives 1 if true, 0 if not. Functions must be set to plot dotted lines.

```
Y1=50(X<90)+70(X≥90)(X<1
00)+90(X≥100)(X<110)+120(X
≥110)(X<120)+160(X≥120)
```

```
WINDOW
 Xmin=80
 Xmax=140
 Xscl=10
 Ymin=-25
 Ymax=200
 Yscl=50
```



2. What would be the cost of a ticket if your speed in a 80 km/h zone is

92 km/h?	$70
110 km/h?	$120
115 km/h?	$120
130 km/h?	$160

Explain how you arrived at these answers.

Work this out by looking at the table above.

If you, the teacher, have plotted the graph on your graphics calculator as shown above, press [trace], type in the speed and press [enter].

3. Write down this function in standard functional form.

$$T(S) = \begin{cases} 50 & 80 < S < 90 \\ 70 & 90 \leq S < 100 \\ 90 & 100 \leq S < 110 \\ 120 & 110 \leq S < 120 \\ 160 & 120 \leq S \end{cases}$$

Another example of speeding fines in an 80 km/h zone is given below.

Speed (km/h)	**Cost of ticket** ($)
90 – 99	50
100 – 109	100
110 – 119	120
120 – 129	140
130 or greater	180

This is another piecewise function.

4. Explain the pattern for speeds from 100 km/h to 129 km/h.

The fine increases by $20 each time we reach a speed ending in 0.

5. Why do you think the $50 fine does not fit the pattern?

The fine is relatively smaller for speeds exceeding the speed limit by only a small amount.

6. Why does the $180 fine not fit the pattern?

The fine is relatively larger for speeds exceeding the speed limit by a very large amount.

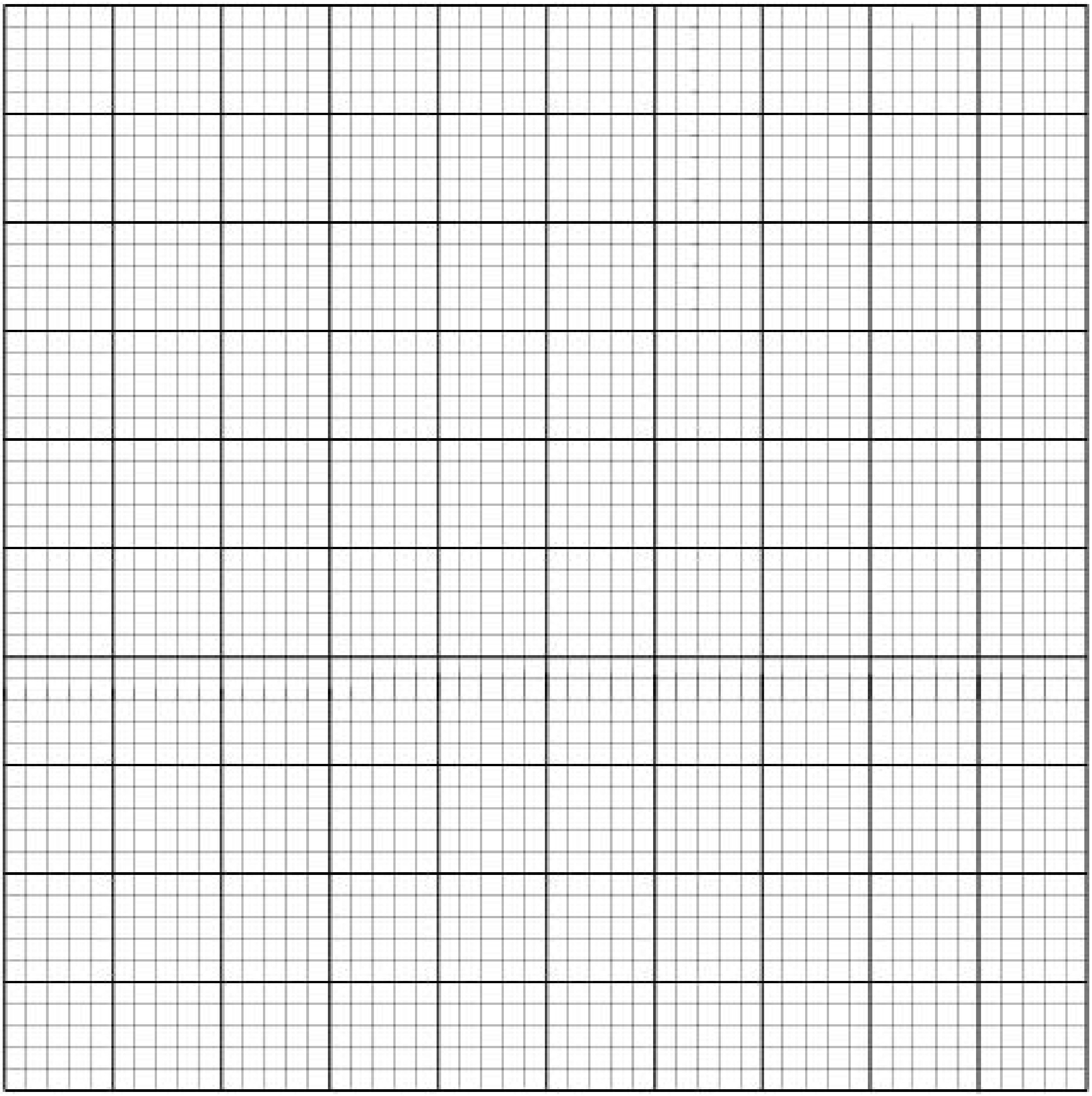
7. Why do you think there is no listing for speeds from 81 km/h to 89 km/h?

This allows for inaccuracy in speedometer readings (supposedly less than 10%).

8. You have been hired by the government, who wants to deter people from speeding. You have been asked to create a function that is not linear and will particularly punish high speeds.

Create a function and explain why your function would accomplish this goal. Use tables and graphs to support your claim.

Anything they like here, as long as they can justify it. If they can come up with a formula, graphics calculators can be used. Otherwise you might give them some graph paper (over the page) on which to draw their function, so they can read off values and make up a table. Working in small groups might be a good idea here.

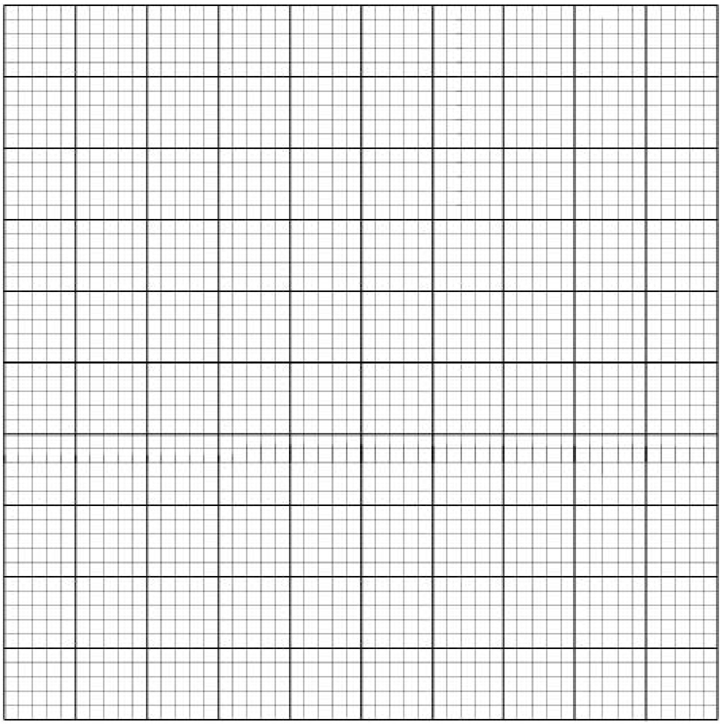


13 Starburst

Year 9, Levels 1 & 2; Strand: Algebra; Sub-strand: Coordinate Geometry.

A study of straight lines: slope and intercept.

On your graphics calculator, draw the graphs of $y = 2x$ and $y = -2x$ for $-5 < x < 5$ and $-5 < y < 5$. Sketch the two graphs on the grid below. Explain the similarities and differences.



....

....

....

....

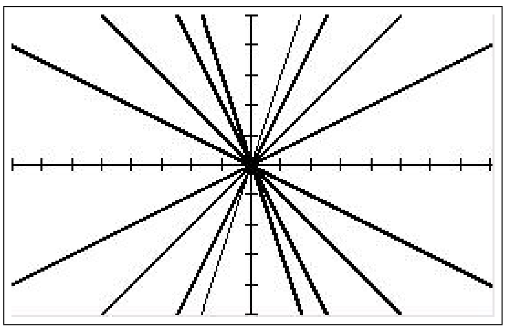
....

....

....

Now draw $y = 3x$ and $y = -3x$ on the grid above.

Once you have drawn those lines, suggest how you could make the picture like the one below. Try your idea out on your graphics calculator and see if you were correct.



How I would make this picture

....

....

....

....

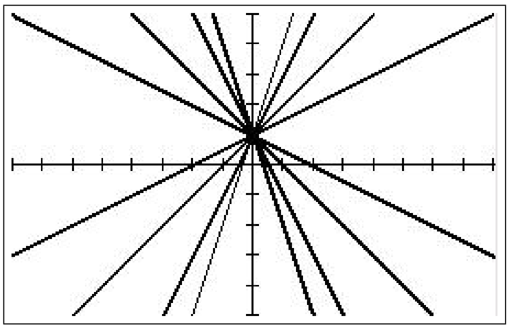
How would you describe this picture to your friend over the phone so that they could draw it on their graphics calculator?

....

....

....

Now try to draw the star pattern below. You may first need to work out how you can shift the straight-line graphs up the y axis.



How I shifted the straight lines up the y axis

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....

Explain what would happen if we drew the graphs of $y = 3x - 2$ and $y = -3x - 2$.

....

....

....

The equation $y = mx + c$ gives a straight-line graph. Explain in your own words what changing m does to the straight line. What about changing c?

If you are not sure, look back on what you have already done or try graphing some more straight lines on your calculator to see their orientation and location.

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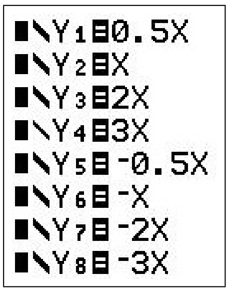
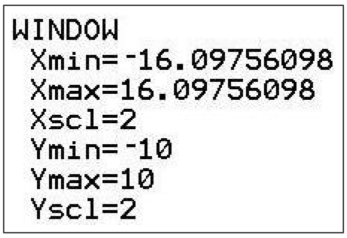
Notes for Teachers

The equations for the first starburst are $y = \pm 0.5x$, $y = \pm x$, $y = \pm 2x$ and $y = \pm 3x$.





These were generated on a TI-84CE graphics calculator, selecting first the standard axes (zoom 6) $[-10, 10, 1] \times [-10, 10, 1]$, then the square option (zoom 5), which makes the scales on each axis the same. I then set the tick marks on the axes to be every 2 units. The right-hand figure above shows the resulting WINDOW.

The equations for the second starburst are just those of the first, but with 2 added to the right-hand side of each equation.

14 Statistics from Birthdays

Year 9, Levels 1 & 2; Strand: Chance and Data; Sub-strand: Statistics.

Author: Stephen Arnold in Integrating Technology in the Middle School, T^3 Publication, 2003. Modified by Peter McIntyre.

Class data on day and month of birth are used to provide an introduction to data presentation on a graphics calculator.

Preparation

Press y=. Turn off any functions selected here (highlighted equal sign) by moving the cursor over the = sign and pressing enter. If any other plots are highlighted at the top, turn them off here too.

To enter the data, press stat 1 (Edit...). This takes you to the list editor.

If you don't see headers for lists L_1, L_2 and L_3, press stat 5 (SetUpEditor) and enter to restore the default lists. Press stat 1 to go back to the list editor.

If there are numbers in a list, move the cursor up to the list heading, press clear and move the cursor back down to the top of the column.

Collecting the data

Go round the class and get every student to say the day of the month on which he or she was born. Enter these numbers into list L_1 as you go, pressing enter after each number, including the last.

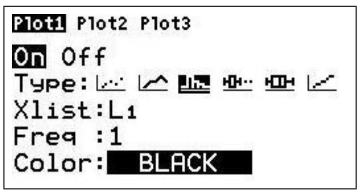
Move the cursor to L_2 and enter, in the same order, the month in which each student was born. Make sure there is the same number of values in each column.

Plotting the data

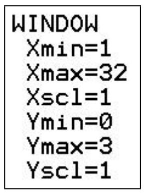
Histogram

What would you expect if you plot a histogram of the birth days, particularly if the class is large? Would you expect more births on some days of the month than on others?[17]

Press 2nd y= (statplot) 1 to set up your plot in Plot1.
Use the cursor and enter to set up the screen as shown.
The key for L_1 is 2nd 1.
The color option is only on the CE.

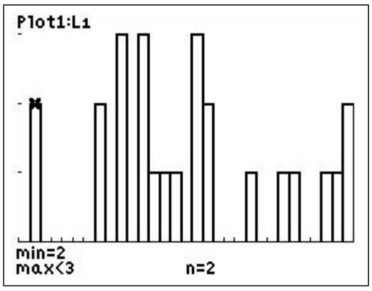


Next set the window for the plot. Press window and set the values as shown. The value for Xscl sets the width of each bin.



[17]The answer is actually yes because the days 29, 30 and 31 don't occur in all months.

Now press trace. You may need to adjust Ymax and Ymin for your data. Use the left- and right-arrow keys to move around the plot. Your plot should be similar to, but not the same as, the figure here.



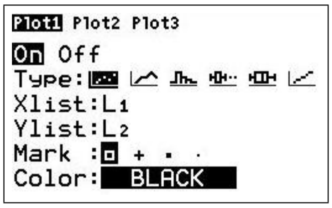
Did you see what you expected? If not, why not?

Now plot a histogram of month of birth. Explain your graph.

Scatter plot

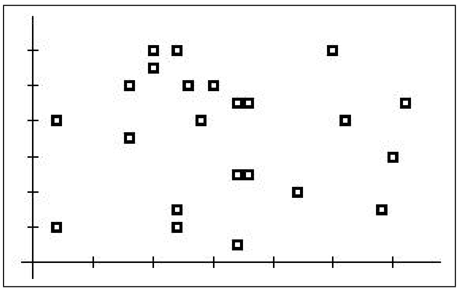
Here we will plot birth day (L_1) along the X axis and birth month (L_2) along the Y axis. *What do you expect to see this time?*

Press 2nd y= (statplot) 1 and set up Plot1 as shown.



For statistics plots other than histograms, there is an automatic way to set the window: zoom 9 (ZoomStat) .[18] This could be used here initially to avoid having to worry about the window settings. *ZoomStat* sets the window and plots the graph.[19]

To do it manually, press window and set appropriate values. Discuss with students what the window should be. Press graph to display the plot.



Press trace to see the coordinates of the points. Move around with the left- and right-arrow keys. Your plot will be different to that above, but the conclusions should be the same.

Did you see what you expected? If not, why not?

[18] *ZoomStat* works for histograms too, but it chooses the bin width: this may not be what you want.

[19] You may want to increase Xscl to say 5 so there aren't so many tick marks on the X axis. Press window, change Xscl, then press graph. For orientation purposes, if Xmin > 0, set Xmin $= 0$, so the Y axis is shown.

Sorting the data

We will now sort L_1 and L_2 separately in ascending order.

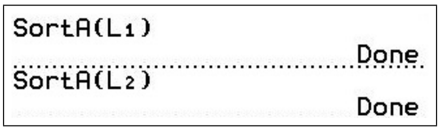
What do you expect to see this time when we plot a scatter plot of month versus day?

Press [2nd] [stat] to get the [list] menu.

Move the cursor to OPS and press [1] or [enter] to select *SortA*.

Press [2nd] [1] for L_1 and [enter] to execute.

Repeat for L_2.

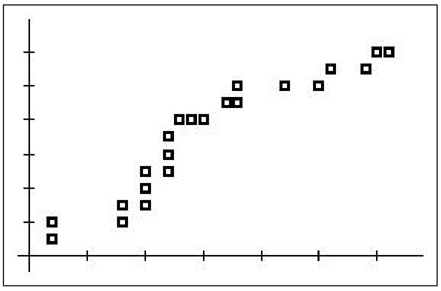


What should you see now when you press [*graph*] *?*

Hopefully a graph in which the points lie approximately on a straight line.

What's the equation of the line?

What do you predict it to be?



Notes for Teachers

When trying out this activity beforehand, you can use the random-number generator to generate some data. The command *randInt* in the math PROB menu generates a sequence of random integers between two specified limits. To generate birth days for a class of 25,

$$\text{randInt}(1, 31, 25) \rightarrow \text{L}_1,$$

where $\rightarrow$ denotes the sto key.

To generate birth months: $\text{randInt}(1, 12, 25) \rightarrow \text{L}_2$.

Note that you may end up with some impossible pairs here, such as 31/2, but it doesn't matter in the testing phase.

If you want to find an equation of the line of best fit, press stat CALC 3 (Med-Med) vars Y-VARS 1 1 (Y_1) enter. This calculates the median-median line of best fit (using default lists L_1 and L_2 for the data). The coefficients of the fit are shown,[20] and if you press graph, the line will be plotted over the data.

The equation of a line between the first birthday of the year, 1 January $(1, 1)$, and the last, 31 December $(31, 12)$, is $y-1=11(x-1)/30$ or $y\approx 0.367x+0.633$.

[20]If not, press catalog (2nd 0), press the D key and scroll down until you reach *Diagnostic On*. Press enter to select it and enter again to execute it. Retrieve your Med-Med command by pressing entry (2nd enter) as many times as is necessary and press enter to execute it again.

15 Temperature Conversions

Year 9, Levels 1 & 2; Strand: Algebra; Sub-Strand: Coordinate Geometry.

From 45 Single Lessons, University of Melbourne, 1996. Downloaded from the Casio Australia website. Modified by Peter McIntyre.

An application of linear functions to conversions between degrees Celsius and degrees Fahrenheit.

Celsius and Fahrenheit

John and David have recently been on a trip to the United States, where temperature is measured using the Fahrenheit scale rather than the Celsius or Centigrade scale used in Australia.

They were told, when converting from Celsius to Fahrenheit, to apply the rough rule: *double and add thirty.*

1. On the day they left Canberra, the maximum temperature was 20°C. To prepare for the US, John and David estimated this temperature in degrees Fahrenheit using the rough rule. What was this temperature?

2. They flew in to Los Angeles, where the predicted maximum temperature was 81°F. Using the rough rule, what is this temperature in degrees Celsius?

3. The actual conversion rule is $F = \frac{9}{5}C + 32$, where F is degrees Fahrenheit and C is degrees Celsius. What is the difference between the rough rule and the actual rule of conversion for a temperature of 20°C?

- On your graphics calculator, press [y=], put the rough rule in Y1 and the actual rule in Y2. What does X stand for? What about Y?

- Generate a table of values for Y1 and Y2.

- Next graph degrees Fahrenheit versus degrees Celsius for both rules. Think about a sensible [window]. Use your graph or table to help you answer the remaining questions.

4. On a 'hot' day, the temperature is 30°C or more.

(a) What is a 'hot' day on the Fahrenheit scale?

(b) How does the difference between the rough rule and the actual rule change as the days get hotter?

5. For what temperature in degrees Celsius will the rough rule give the same converted value as the actual rule? Describe how you find this out. Can you think of more than one way to do this?

6. The rough rule is considered to be good enough if it is in error by no more than 5°F. For what range of temperatures (degrees Celsius) would the rough rule be considered good enough? Describe how you find this out. Do this in at least two ways.

7. There is a temperature at which the numerical value is the same in degrees Celsius and in degrees Fahrenheit. What is this temperature? How far out, in degrees Fahrenheit, is the rough rule in this case?

Notes for Teachers

1. On the day they left Canberra, the maximum temperature was 20°C. To prepare for the US, John and David estimated this temperature in degrees Fahrenheit using the rough rule. What was this temperature?

Temperature $= 2 \times 20 + 30 = 70$°F.

2. They flew in to Los Angeles, where the predicted maximum temperature was 81°F. Using the rough rule, what is this temperature in degrees Celsius?

Here students should ask the question *What temperature when doubled and added to 30 gives 81?*, and answer it in one of several ways:

- by trial and error, giving the temperature as 25.5°C;
- by using some simple algebra: if x is the unknown temperature, then $2x + 30 = 81$, which, when solved for x, gives the temperature as 25.5°C;
- by thinking 'inversely': if I double and add 30 to go from Celsius to Fahrenheit, then to go from Fahrenheit to Celsius I must subtract 30 and halve the result.

3. The actual conversion rule is $F = \frac{9}{5}C + 32$, where F is degrees Fahrenheit and C is degrees Celsius. What is the difference between the rough rule and the actual rule of conversion for a temperature of 20°C?

From Question 1, the rough rule gives 70°F.

The actual rule gives $9 \times 20/5 + 32 = 68$°F, so the difference is 2°F.

- On your graphics calculator, press [y=], put the rough rule in Y_1 and the actual rule in Y_2. What does X stand for? What about Y?

  $Y_1 = 2X+30$; $Y_2 = 9X/5+32$

  X stands for degrees Celsius; Y for degrees Fahrenheit.

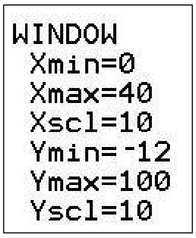
- Generate a table of values for Y_1 and Y_2.

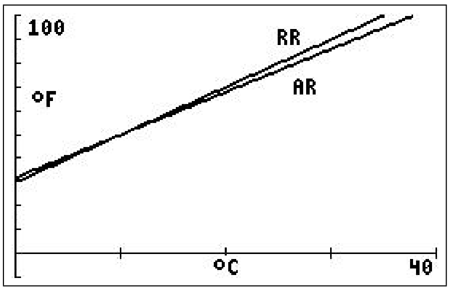
```
Y1=2X+30
Y2=9X/5+32
```

X	Y_1	Y_2
0	30	32
1	32	33.8
2	34	35.6
3	36	37.4
4	38	39.2
5	40	41
6	42	42.8
7	44	44.6
8	46	46.4
9	48	48.2
10	50	50

X=0

- Next graph degrees Fahrenheit versus degrees Celsius for both rules. Think about a sensible window.





Use your graph or table to help you answer the remaining questions.

4. On a 'hot' day, the temperature is 30°C or more.

(a) What is a 'hot' day on the Fahrenheit scale?

From the graph, table or simple calculation, a hot day is when the temperature is 86°F or more.

(b) How does the difference between the rough rule and the actual rule change as the days get hotter?

The difference becomes larger as the days get hotter, with the rough rule giving the higher temperature.

5. For what temperature in degrees Celsius will the rough rule give the same converted value as the actual rule? Describe how you find this out. Can you think of more than one way to do this?

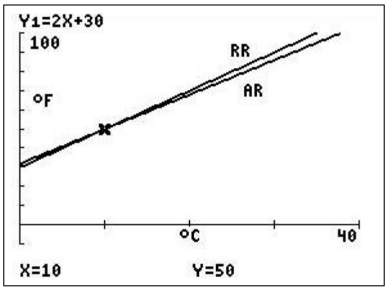
There are at least three ways to do this.

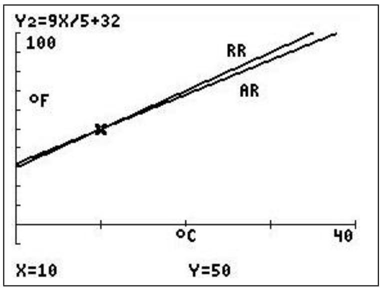
(a) The easiest way is probably from the table. Scroll up or down the table until the value for Y_1 (rough rule) is the same as that for Y_2 (actual rule). This happens when X (degrees Celsius) is 10, so the two rules give the same answer at 10°C.

X	Y_1	Y_2
2	34	35.6
3	36	37.4
4	38	39.2
5	40	41
6	42	42.8
7	44	44.6
8	46	46.4
9	48	48.2
10	50	50
11	52	51.8
12	54	53.6

X=10

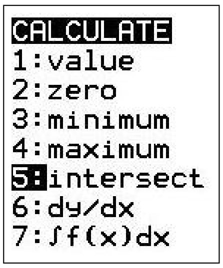
(b) The same result comes from the graph — it is the X value of the intersection point of the two lines. Find this using [trace], typing in different values and seeing if the Y value for each line is the same.

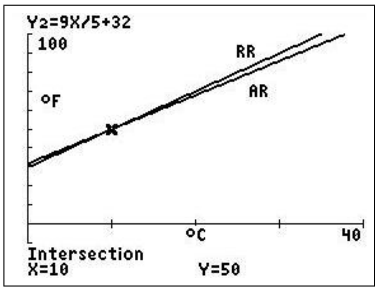




Using *intersect* in the [calc] menu is a faster way, but students may get more out of [trace].

To use *intersect*, make sure the intersection point appears on the screen, select *intersect* from the [calc] menu, press [enter] to each of the three prompts and read off the coordinates of the point of intersection at the bottom of the screen.





(c) Algebraically, set the two formulas equal and solve for C:

$$2C + 30 = \frac{9}{5}C + 32 \Rightarrow \frac{1}{5}C = 2 \Rightarrow C = 10.$$

PTO

6. The rough rule is considered to be good enough if it is in error by no more than 5°F. For what range of temperatures (degrees Celsius) would the rough rule be considered good enough? Describe how you find this out. Do this in at least two ways.

The answer is $-15 \leqslant C \leqslant 35$.

(a) The simplest way to find these values is again to use the table. By scrolling and checking the difference between Y_1 and Y_2, you can very soon find when the difference is 5 or -5 (left-hand figure below). The value -5 may need discussion. It may also not be apparent that there is a lower value for X, -15.

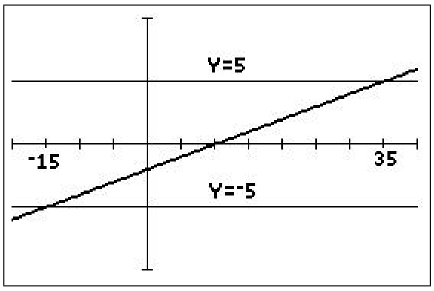
X	Y1	Y2
-17	-4	1.4
-16	-2	3.2
-15	0	5
-14	2	6.8
-13	4	8.6
-12	6	10.4
-11	8	12.2
-10	10	14
-9	12	15.8
-8	14	17.6
-7	16	19.4

X= -15

X	Y1	Y2
26	82	78.8
27	84	80.6
28	86	82.4
29	88	84.2
30	90	86
31	92	87.8
32	94	89.6
33	96	91.4
34	98	93.2
35	100	95
36	102	96.8

X=35

(b) You can do the same thing on the graph using trace, as before. This takes a little longer than the table. You could suggest graphing $Y_3 = Y_1 - Y_2$, and ask what to look for. The window will need changing to reveal the value at $X = -15$.



(c) Algebraically, you have to solve $\left|(2x+30) - (9x/5+32)\right| \leq 5$ for x. Therefore,

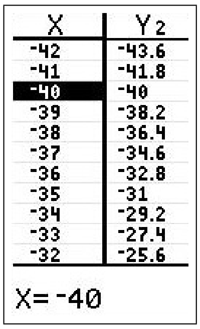
$$
\begin{aligned}
\left|\frac{x}{5} - 2\right| &\leq 5. \\
\therefore\ \frac{x}{5} - 2 &\leq 5 \quad \textbf{or} \\
-\left(\frac{x}{5} - 2\right) &\leq 5. \\
\therefore\ \frac{x}{5} &\leq 3 \quad \textbf{or} \\
-7 &\leq \frac{x}{5}. \\
\therefore\ x &\leq 15 \quad \textbf{or} \\
-35 &\leq x.
\end{aligned}
$$

The answer is $-35 \leqslant x \leqslant 15$.

7. There is a temperature at which the numeric value is the same in degrees Celsius and in degrees Fahrenheit. What is this temperature? How far out, in degrees Fahrenheit, is the rough rule in this case?

The answer is $C = F = -40$.

(a) Here the tables wins hands down! You want to find where X (degrees Celsius) is the same as Y2, the actual rule for degrees Fahrenheit. Scrolling will find this eventually.

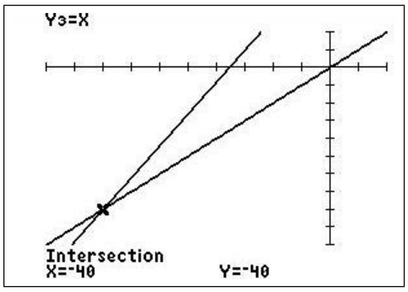


X	Y2
⁻42	⁻43.6
⁻41	⁻41.8
⁻40	⁻40
⁻39	⁻38.2
⁻38	⁻36.4
⁻37	⁻34.6
⁻36	⁻32.8
⁻35	⁻31
⁻34	⁻29.2
⁻33	⁻27.4
⁻32	⁻25.6

X= ⁻40

(b) Graphically will require some explanation and sophistication on the part of your students. You have to find where the graph of the actual rule (Y2) intersects with the graph of $y = x$.

Set Y3 = X, change the [window] and use [trace] or *intersect*.



(c) Algebraically, we start with the actual rule $F = \frac{9}{5}C + 32$, and set $F = C$, giving

$$C = \frac{9}{5}C + 32.$$

Solving for C gives $C = -40$.

For a temperature of −40°C, the rough rule gives a value of −50°F, 10°F too low.

16 What's My Line?

Years 9, 10, Levels 1, 2; Strand: Algebra; Sub-strand: Coordinate Geometry – Straight Lines.

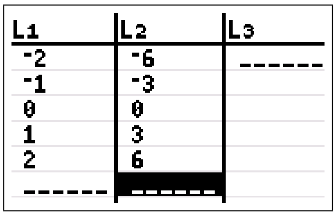
Author: Margie Smith.

These worksheets investigate the connection between a table of values, the line on a number plane and the equation of the line.

What's My Line? Worksheet 1

1. Press stat and 1 (Edit...). Enter the following data into lists L1 (x values) and L2 (y values).

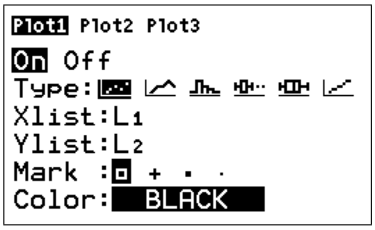
X	−2	−1	0	1	2
Y	−6	−3	0	3	6



Question: What is the rule for this relationship?

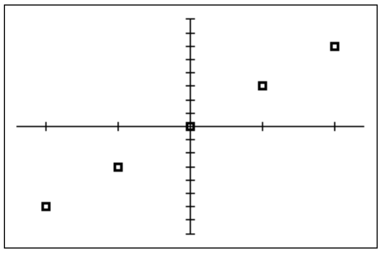
Y = ___X + ___

2. Press stat plot (2nd y=) and select 1 (Plot1). Set up your screen as shown in the figure. (*Color* is only an option on a TI-84CE.)

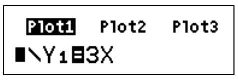


Press zoom 9 (ZoomStat) to give a graph like that shown.

Use window to adjust the viewing window if necessary.

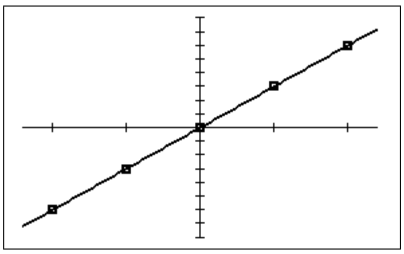


3. Press y= and and enter the relationship that you found in (1) into Y1. Remember that X is the X,T,θ n key.



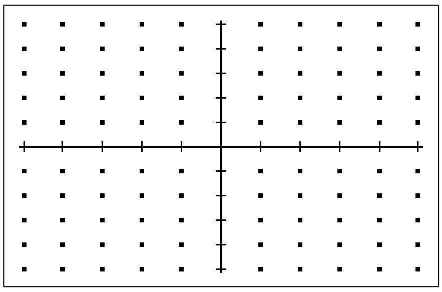
Press graph and check that the line runs through your points.

If it doesnt, your rule is incorrect!!!!
Try again.



What's My Line? Worksheet 2

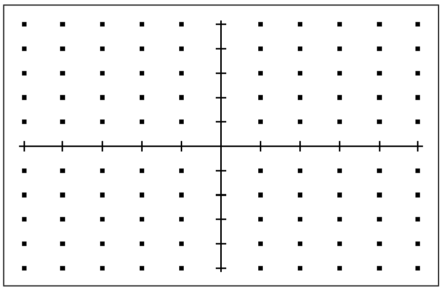
Using your graphics calculator, enter the following tables and determine the equation (rule) of the line that connects these points. Adjust the window where necessary. Sketch and label each pair of lines.



(a)

X	−2	−1	0	1	2
Y	−5	−2	1	4	7

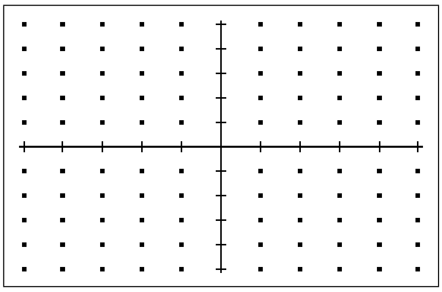
Y = __________



(b)

X	−2	−1	0	1	2
Y	−8	−5	−2	1	4

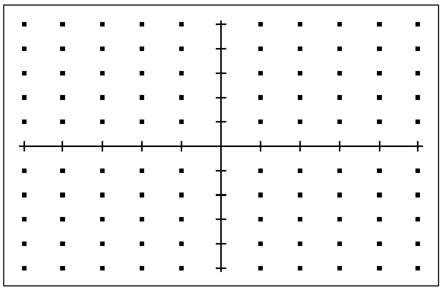
Y = __________



(c)

X	−2	−1	0	1	2
Y	−5.5	−2.5	0.5	3.5	6.5

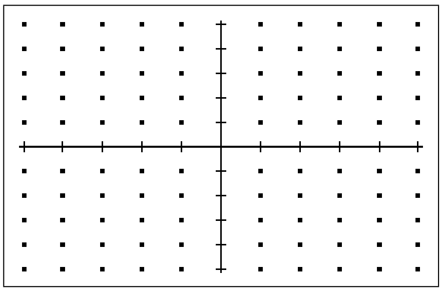
Y = __________



(d)

X	−2	−1	0	1	2
Y	−9	−6	−3	0	3

Y = __________

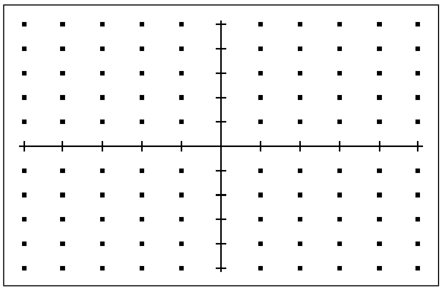


(e)

X	−2	−1	0	1	2
Y	6	3	0	−3	−6

Y = __________

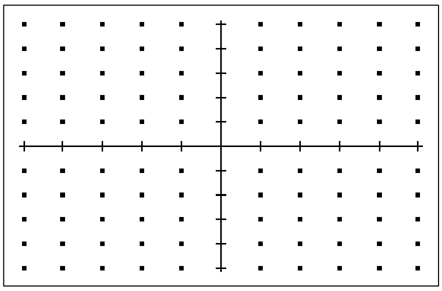
PTO



(f)

X	-2	-1	0	1	2
Y	7	4	1	-2	-5

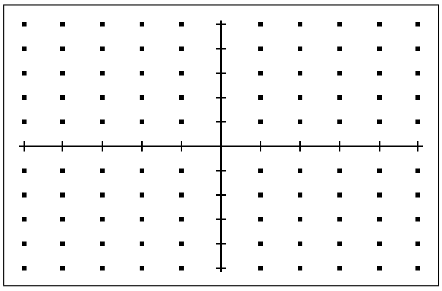
Y = ____________



(g)

X	-2	-1	0	1	2
Y	4	1	-2	-5	-8

Y = ____________



(h)

X	-2	-1	0	1	2
Y	6.5	3.5	0.5	-2.5	-5.5

Y = _________

17 Which Fuel?

Years 9 & 10, Level 1, Strand: Algebra; Sub-strand: Coordinate Geometry.

From 45 Single Lessons, University of Melbourne, 1996. Downloaded from the Casio Australia website. Modified by Peter McIntyre.

An application of linear functions to choosing whether to use petrol or LPG in your car.

A client asks her financial advisor whether it is better to install a liquid-petroleum-gas (LPG) tank in her recently acquired car or to stick with petrol. The advisor researches the cost of each option for her car and notes the following facts:

	Petrol	**LPG**
Installation cost ($)	0	2500
Average distance travelled (km/yr)	20,000	20,000
Fuel economy (litres/100 km)	10	12.5
Fuel cost ($/litre)	1.20	0.55

Construct a function $C(t)$ giving the cumulative cost of each option over t years.

Note that we are ignoring other costs such as depreciation and maintenance but we can assume these costs are the same for both options.

Make sure the units in your functions are correct.

Enter both functions into your graphics calculator.
Graph with $0 < \mathrm{X} < 5$; $0 < \mathrm{Y} < 10,000$.

1. After how many years and months does the cumulative cost of the LPG option become cheaper? How much has it cost to this time?

2. If our motorist only travels 10, 000 km a year, does this mean the LPG option takes twice as long to become the cheaper option? Explain what you did.

3. If the fuel cost for LPG doubled, does this mean the LPG option takes twice as long to become the cheaper option? Explain what you did.

4. If the installation cost for LPG doubled, does this mean the LPG option takes twice as long to become the cheaper option? Explain what you did.

5. Can we analyse this problem in a general way?

Let $\$I$ be the installation cost of the LPG option, D km be the total distance travelled, e_p and e_l be the fuel economies in litres/100 km for petrol and LPG respectively, f_p and f_l be the fuel costs in \$/litre for petrol and LPG respectively.

(a) Using your previous calculations of $C(t)$ as a guide, write down $C(t)$ for each option for this general case.

(b) By equating C for the two options, find the time at which the cumulative cost of both options is the same. Verify your answer in Question 1 by substituting the appropriate values into the expression you found here.

(c) Show that your results in Q2 and Q4 are true for any values of D and I.

(d) With the price of petrol at \$1.20, what is the minimum cost of LPG for which LPG never becomes the cheaper option? Assume e_p and e_l take the values given in the table above.

Hint: Think in terms of the graphs when this happens.

Notes for Teachers

A client asks her financial advisor whether it is better to install a liquid-petroleum-gas (LPG) tank in her recently acquired car or to stick with petrol. The advisor researches the cost of each option for her car and notes the following facts:

	Petrol	**LPG**
Installation cost (\$)	0	2500
Average distance travelled (km/yr)	20,000	20,000
Fuel economy (litres/100 km)	10	12.5
Fuel cost (\$/litre)	1.20	0.55

Construct a function $C(t)$ giving the cumulative cost of each option over t years.

Note that we are ignoring other costs such as depreciation and maintenance but we can assume these costs are the same for both options.

Make sure the units in your functions are correct.

The only tricky bit here is making sure the fuel costs are converted to litres/km. The units for each term guide us to the correct formulas.

Petrol

$$C_p(t)\,(\$) = 20,000\left(\frac{\text{km}}{\text{yr}}\right) \times \frac{10}{100}\left(\frac{\ell}{\text{km}}\right) \times 1.20\left(\frac{\$}{\ell}\right) \times t\ (\text{yr})$$

$$= 20,000 \times 0.1 \times 1.20\,t\ (\$). \qquad \text{Note units `cancel' on RHS to give \$.}$$

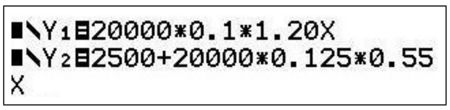
LPG

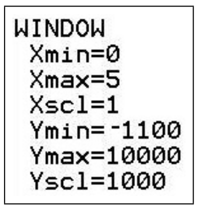
$$C_l(t)\ (\$) = 2500\,(\$) + 20,000\left(\frac{\text{km}}{\text{yr}}\right) \times \frac{12.5}{100}\left(\frac{\ell}{\text{km}}\right) \times 0.55\left(\frac{\$}{\ell}\right) \times t\,(\text{yr})$$

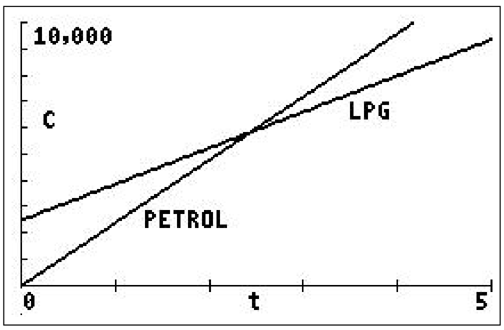
$$= 2500 + 20,000 \times 0.125 \times 0.55\,t\ (\$).$$

Enter both functions into your graphics calculator; graph with $0 < \text{X} < 5$; $0 < \text{Y} < 10,000$. The independent variable t becomes X in the calculator version of the equations.

If you enter the functions in the form shown, it makes it easier to change numbers later.







1. After how many years and months does the cumulative cost of the LPG option become cheaper? How much has it cost to this time?

Clearly the petrol option is cheaper to start with because there is no conversion cost, that is $C_p(0) < C_l(0)$. With the saving in fuel cost, eventually the LPG option becomes cheaper. We need to find when the two costs are the same.

In mathematical terms, we need to solve $C_p(t) = C_l(t)$ for t. We can do this in three ways: numerically using a table, graphically or algebraically.

(a) Generate a table of Y1 and Y2 with TblStart = 0 and ΔTbl = 1 (left-hand figure below). Scroll down to find when Y1 = Y2 : they are equal some time between $t = 2$ (Y1 < Y2) and $t = 3$ (Y1 > Y2), that is some time in the third year.

X	Y1	Y2
0	0	2500
1	2400	3875
2	4800	5250
3	7200	6625
4	9600	8000
5	12000	9375
6	14400	10750
7	16800	12125
8	19200	13500
9	21600	14875
10	24000	16250

X=2

ΔTbl = 1

X	Y1	Y2
2	4800	5250
2.1	5040	5387.5
2.2	5280	5525
2.3	5520	5662.5
2.4	5760	5800
2.5	6000	5937.5
2.6	6240	6075
2.7	6480	6212.5
2.8	6720	6350
2.9	6960	6487.5
3	7200	6625

X=2.4

ΔTbl = 0.1

X	Y1	Y2
2	4800	5250
2.0833	5000	5364.6
2.1667	5200	5479.2
2.25	5400	5593.8
2.3333	5600	5708.3
2.4167	5800	5822.9
2.5	6000	5937.5
2.5833	6200	6052.1
2.6667	6400	6166.7
2.75	6600	6281.3
2.8333	6800	6395.8

X=2.41666666666

ΔTbl = 1/12

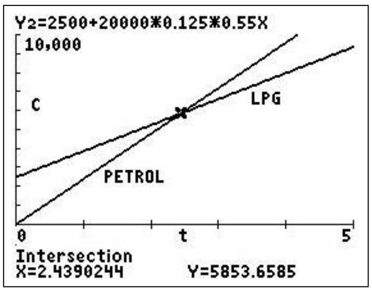
To find the answer more accurately, we need to reduce ΔTbl, with TblStart = 2. The natural thing to do is reduce ΔTbl to 0.1: we find that Y1 = Y2 between $t = 2.4$ and $t = 2.5$ (middle figure above). We then need to convert decimal years to months, giving the bounds as somewhere between 2 years 4.8 months and 2 years 6 months, still not accurate enough.

A better approach is to set TblStart = 2 and ΔTbl = 1/12 (type it in just like that): each division in the table corresponds to a month (you need to count down to find out which month — right-hand figure above). We then find that equality occurs between the end of month 5 (Y1 < Y2) and the end of month 6 (Y1 > Y2) of the second year, so the answer is after 2 years 6 months, realising that equality is sometime in the sixth month.

(b) On the graph, use trace to find the approximate intersection point. This takes a little longer than the table. Faster is *intersect* in the calc menu.

To use *intersect*, make sure the intersection point appears on the screen, select *intersect* from the calc menu, press enter to each of the three prompts and read off the coordinates of the point of intersection at the bottom of the screen.

CALCULATE
1:value
2:zero
3:minimum
4:maximum
5:intersect
6:dy/dx
7:∫f(x)dx



The intersection point is (to 4 significant digits) (2.439, 5853), and converting the time (X) in decimal years to years and months, we find the two options cost the same after 2 years 5.3 months, or in the sixth month of the second year, as we found above.

(c) Algebraically we set $C_p(t) = C_l(t)$, giving

$$\begin{aligned} 20,000 \times 0.1 \times 1.20\,t &= 2500 + 20\,000 \times 0.125 \times 0.55\,t. \\ \therefore\ t &= \frac{2500}{20,000\big(0.1 \times 1.20 - 0.125 \times 0.55\big)} \\ &= 2.439, \end{aligned}$$

the same answer we found in (b).

The cost to this time is somewhere between \$5,823 and \$5,938 from the table (use smaller ΔTbl for a more accurate value), \$5,854 from the graph and \$5,854 from the algebraic result, found by evaluating Y1 or Y2 at X = 2.439.

2. If our motorist only travels 10,000 km a year, does this mean the LPG option takes twice as long to become the cheaper option? Explain what you did.

Change the 20,000 to 10,000 in Y1 and Y2, and recalculate the time at which both options cost the same using one of the three methods above. We find $t = 4.878$, twice the previous time.

Halving the distance travelled doubles the pay-back time for the LPG option.

3. If the fuel cost for LPG doubled, does this mean the LPG option takes twice as long to become the cheaper option? Explain what you did.

Change the cost of LPG to \$1.10 in Y2 and replot the graphs or look at the table. The slope of the LPG graph is now greater than that of the petrol graph, so the two graphs never intersect when $t \geqslant 0$. The LPG option is never cheaper.

4. If the installation cost for LPG doubled, does this mean the LPG option takes twice as long to become the cheaper option? Explain what you did.

Change the 2,500 to 5,000 in Y2 and recalculate the time at which both options cost the same using one of the three methods above. We find again $t = 4.88$, twice the previous time.

Doubling the LPG installation cost doubles the pay-back time for the LPG option.

5. Can we analyse this problem in a general way?

Let $\$I$ be the installation cost of the LPG option, D km be the total distance travelled, e_p and e_l be the fuel economies in litres/100 km for petrol and LPG respectively, f_p and f_l be the fuel costs in \$/litre for petrol and LPG respectively.

(a) Using your previous calculations of $C(t)$ as a guide, write down $C(t)$ for each option for this general case.

Petrol: $C_p(t) = De_pf_pt/100$.

LPG: $C_l(t) = I + De_lf_lt/100$.

(b) By equating C for the two options, find the time at which the cumulative cost of both options is the same. Verify your answer in Question 1 by substituting the appropriate values into the expression you found here.

Setting $C_P(t){=}C_L(t)$, we have

$$\begin{aligned} De_pf_pt/100 &= I + De_lf_lt/100. \\ \therefore\ t &= \frac{100I}{D\big(e_pf_p - e_lf_l\big)}. \end{aligned}$$

Using $I = 2500,\ D = 20\,000,\ e_p = 10,\ f_p = 1.20,\ e_l = 12.5$ and $f_l = 0.55$, we find $t{=}2.439$, the value we found in Q1.

(c) Show that your results in Q2 and Q4 are true for any values of D and I.

The fact that t is proportional to I and inversely proportional to D shows that the results in Q2 and Q4 are true for any values of D and I.

(d) With the price of petrol at \$1.20, what is the minimum cost of LPG for which LPG never becomes the cheaper option? Assume e_p and e_l take the values given in the table above. *Hint*: Think in terms of the graphs when this happens.

LPG eventually becomes cheaper if the the graphs of $C_P(t)$ and $C_L(t)$ intersect for some positive t. The condition that they don't intersect for some positive t is that the two graphs are parallel, that is they have the same slope, or that the LPG graph has a larger slope.

The slope of $C_P(t)$ is $De_pf_p/100$, whereas the slope of $C_L(t)$ is $De_lf_l/100$. These slopes are the same when

$$\begin{aligned} \frac{De_pf_p}{100} &= \frac{De_lf_l}{100}, \quad \text{or} \\ e_pf_p &= e_lf_l, \quad \text{giving} \\ f_l &= \frac{e_pf_p}{e_l} \\ &= \frac{10 \times 1.20}{12.5} \\ &= 0.96, \end{aligned}$$

using the given values for e_p and e_l. When the cost per litre of LPG is 96c or more, the LPG option is never cheaper.

The same result can be seen from the expression for t in (b): when the lines are parallel, t is infinite, corresponding to the denominator being 0.

More generally,

$$f_l \;=\; \frac{e_p}{e_l}\,f_p \;=\; \frac{10}{12.5}\,f_p \;=\; 0.8f_p,$$

so that the LPG option is never cheaper if the price of LPG is greater than or equal to 80% of the price of petrol.