# Infinite Series – Fractals

## Activity 1

## NCTM Standards

- Problem Solving Standard – Solve problems that arise in mathematics and other contexts.
- Connections Standard – Recognize and apply mathematics in contexts outside of mathematics.

## Materials

- TI-89

**Topics in Calculus:**

Infinite Series, Limits

**Overview:**

Fractal geometry provides a new way for mathematicians and scientists to explore nature. Euclidean geometry models many regularly shaped natural phenomena such as cubic crystals, spherical planets, and elliptic orbits. Fractal geometry models irregular objects such as coastlines, mountains, clouds, plants, and the human brain. Models in nature are only finite approximations of fractals.

A fractal is a self-similar geometric figure resulting from beginning with an initial figure and iterating a process an infinite number of times. This procedure is called recursion. A fractal has irregular (rough, crinkled) edges and fractal (fractional) dimension.

**Exercises**

The Koch Snowflake was discovered by Helge Von Koch (1870-1924). Construct the Koch Snowflake in the following way:

a) Begin with an equilateral triangle.
b) Let the length of each side be s units (For example, one unit of measure such as inches).
c) Remove the middle third of each side and replace each with two segments (outside the original triangle) forming an equilateral triangle with the missing piece as depicted in Figure 1.
d) Repeat the process an infinite number of times. Iterations of the Koch Snowflake are illustrated in Figure 1.

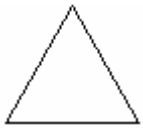
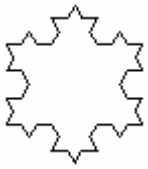
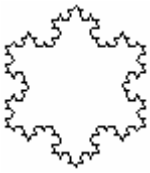
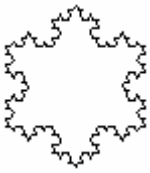
| Generating the Koch Snowflake | Perimeter | Area |
|---|---|---|
|  Level 0 | P(0) = | A(0) = |
|  Level 1 | P(1) = | A(1) = |
|  Level 2 | P(2) = | A(2) = |
|  Level n | P(n) (n is an arbitrary positive integer) = | A(n) (n is an arbitrary positive integer) = |
|  Level infinity The Koch Snowflake | $\lim_{n\to\infty} P(n)$ = | $\lim_{n\to\infty} A(n)$ = |

Figure 1

**Exercise 1**.

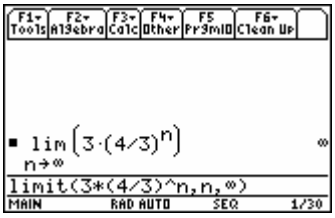
a) Use the table in Figure 1 as a guide to find the perimeter and area of the Koch Snowflake. Recall that the fractal is the object at <u>infinity</u>.

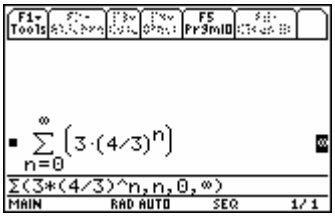
b) Explain what your expectations were before doing each calculation and describe your reaction to the result, including surprises, if any.

**Solution (Perimeter of the Koch Snowflake)**:

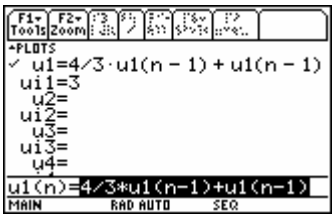
Let s = 1 unit. The $n^{th}$ term of the sequence P($n$) = 3(4/3)$^n$. P($n$) is an infinite geometric sequence with a common ratio greater than 1. The limit of P($n$) as $n$ approaches infinity, $\lim_{n\to\infty} P(n) = \infty$. The sequence of partial sums diverges. The sum of the infinite series, $\sum_{n=1}^{\infty} 3(4/3)^n = \infty$.

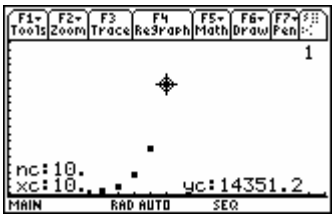
You can calculate the limit of the nth term of the sequence as n approaches infinity and the sum of the infinite series on the home screen of the TI-89 (TI92 Plus) as illustrated in the screens below.

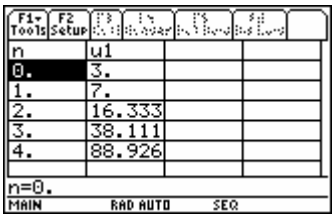


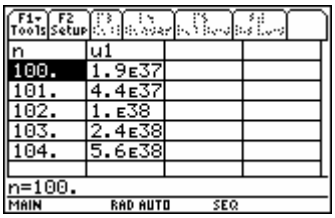




You can graph the sequence of partial sums of the series and look at the table for grahical and numerical evidence. Set the TI-89 (TI-92 Plus) in **SEQUENCE** mode and enter the recursive definition generating the series in the equation (**Y=**) editor.




**Answer**: The perimeter of the Koch snowflake is infinite.

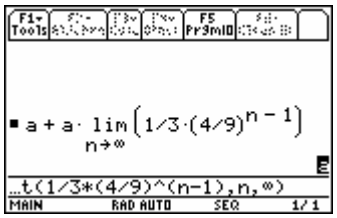
**Exercise 2**.

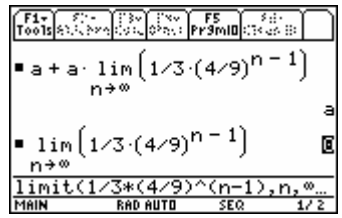
a) Use the table in Figure 1 as a guide to find the area of the Koch Snowflake. Recall that the fractal is the object at <u>infinity</u>.

b) Explain what your expectations were before doing each calculation and describe your reaction to the result, including surprises, if any.

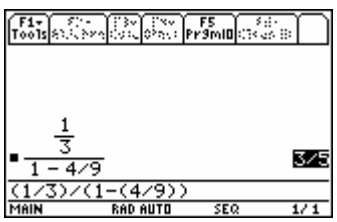
**Solution (Area of the Koch Snowflake)**:

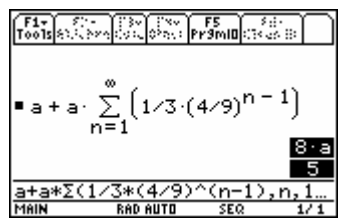
The $n^{\text{th}}$ term of the sequence A($n$) = $A_0 + A_0(\frac{1}{3})(\frac{4}{9})^{n-1}$. The limit of A($n$) as $n$ approaches infinity, $\lim_{n\to\infty} A(n)$ = $A_0$, that is, $\lim_{n\to\infty}(\frac{1}{3})(\frac{4}{9})^{n-1} = 0$.



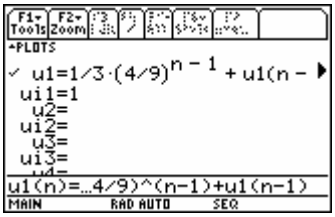

Since $(\frac{1}{3})(\frac{4}{9})^{n-1}$ is an infinite geometric series with common ratio less than one, the series converges. The sum S of the infinite series, $(\frac{1}{3})(\frac{4}{9})^{n-1} = \dfrac{\frac{1}{3}}{1-\frac{4}{9}} = \frac{3}{5}$. The sum of the infinite series, $A_0 + A_0 \sum_{n=1}^{\infty}(\frac{1}{3})(\frac{4}{9})^{n-1} = \frac{8}{5}A_0$.

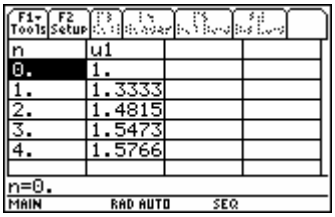


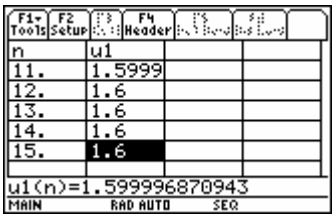



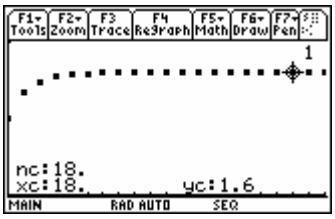

The area of the Koch Snowflake is $A_0 + \frac{3}{5}A_0 = \frac{8}{5}A_0$. In other words, the area of the Koch Snowflake is $\frac{8}{5}$ times $A_0$, the area of the original triangle.

You can graph the sequence of partial sums of the series and look at the table for grahical and numerical evidence. Set the TI-89 (TI-92 Plus) in **SEQUENCE** mode and enter the recursive definition generating the series in the equation (**Y=**) editor. Let $A_0$ = 1.






Note that the sum is approacing 1.6 = 8/5.

**Answer**: The area of an equilateral triangle with sides of length s units = $\frac{\sqrt{3}}{4}$ s square units.

Therefore, the area of the corresponding Koch snowflake equals $\frac{8\sqrt{3}}{5*4}s = \frac{2\sqrt{3}}{5}s$ square units.

**Exercise 3**.

Another version of the Koch Snowflake can be generated by replacing the middle third of each side of the original equilateral triangle with two segments inside the original triangle forming an equilateral triangle with the missing piece.

a) Draw a sketch.
b) Find the perimeter and area of the fractal.
c) Explain what your expectations were before doing each calculation and describe your reaction to the result, including surprises, if any.

**Exercise 4**.

The Sierpinsky triangle named for Waclaw Sierpinsky (1882-1969) is depicted in Figure 2. The following procedure describes one of many ways to generate it. Begin with an equilateral triangle. Remove the center triangle formed by the segments joining the midpoints of the sides of the original triangle. Repeat the process with the remaining three triangles. Repeat again forever.

Figure 2

a) Find the perimeter and area of the Sierpinsky triangle.
b) Explain what your expectations were before doing each calculation and describe your reaction to the result, including surprises, if any.

**Ideas For Further Study**:

**Exercise 5**.

a) Generate fractals in a manner similar to that of the Koch snowflake by increasing the number of sides of the regular polygon to 4, 5 and 6. For example, begin with a square. Replace the middle third of each segment with three segments to form a square with the missing piece on the middle third.
b) Use computer software or a graphing calculator to generate the fractal. (Optional)
c) Predict the perimeter and area.
d) Find the perimeter and area.
e) Report your observations and/or discoveries.

**Extension**

Investigate other fractals, and properties of fractals on the World Wide Web:
*Exploring Fractals* by Mary Ann Connors,
http://www.math.umass.edu/~mconnors/fractal/fractal.html
Write a report of your procedures and results.