Investigations with Stacked Fractions and Ans *to Generate Patterns and Foster Modeling*

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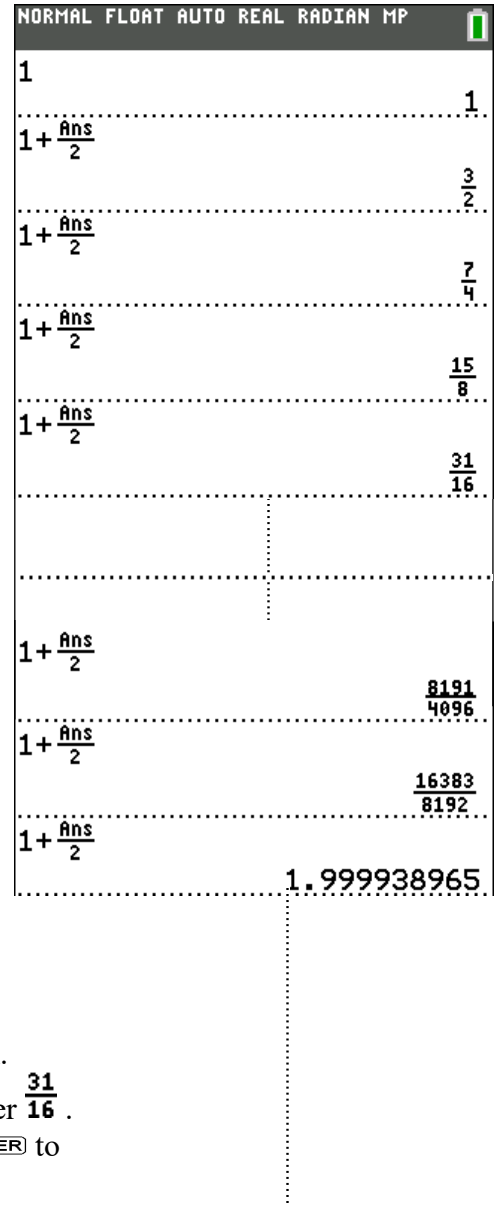
Investigation 1: A gift that keeps on giving!

Reset the defaults on your calculator. (These are calculator settings it has when it first comes out of the box.)
Press [2nd] [MEM] `7:Reset…` `2:Defaults…` `2:Reset`



1. Clear the home screen.
   Press the number 1 followed by [ENTER].
   This number will be the initial seed.

2. Next build the expression $1+\frac{\text{Ans}}{2}$
   For the shortcut FRAC menu, press [ALPHA] [F1] and use `n/d` for stacked fractions instead of the division key.

3. Once you build the expression, continue pressing [ENTER] to create the screen to the right. Describe any patterns.

4. Conjecture what three expressions will come next.

   $1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16}$, _____ , _____ , _____

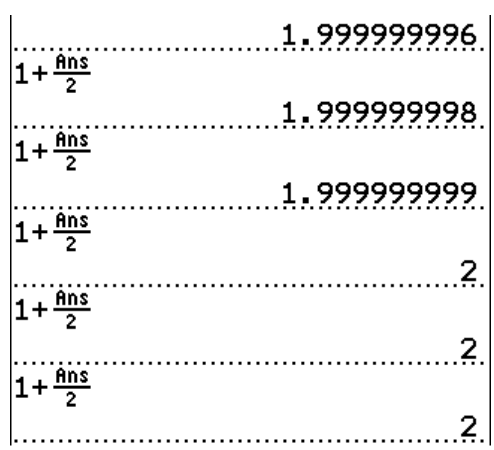
5. After pressing [ENTER] many, many times, the TI-84 Plus will eventually stop displaying a number as a stacked fraction. (It resigns from duty once the number's denominator exceeds 5 digits. Alas, using `▸Frac` on `1.999938965` will not help.)

   a. If we were write the number which comes after $\frac{16383}{8192}$ as a stacked fraction, what would be its denominator?

   b. What would be its numerator?

   c. Verify your claim by entering your fraction on the home screen.

   d. A student had pressed the [ENTER] key 5 times to reach the number $\frac{31}{16}$. What is the least number of times they would have pressed [ENTER] to reach `1.999938965`? Create a formula for the *n*th term.



6. Eventually the expression $1+\frac{\text{Ans}}{2}$ will converge to 2.
   This means when `Ans` = 2, then $1+\frac{\text{Ans}}{2} = \text{Ans}$.

   a. Solve the equation $1+\frac{x}{2}=x$ to show that $x=2$ is the one and only value to which this expression converges.
   b. Conjecture what would happen if the initial seed were 2. Then see what happens.
   c. Precalculus:
      Show your formula in 5d is equivalent to $2-\frac{1}{2^{n-1}}$.
      Then show why $2-\frac{1}{2^{n-1}} \to 2$ as $n \to \infty$.

Investigation 2: Exploring $1+\frac{\text{Ans}}{3}$, $1+\frac{\text{Ans}}{4}$, $1+\frac{\text{Ans}}{5}$, and beyond!

In the previous investigation, we have seen that the expression $1+\frac{\text{Ans}}{2}$ can build the sequence 1, $\frac{3}{2}$, $\frac{7}{4}$, $\frac{15}{8}$, $\frac{31}{16}$, … , 2, 2, ….
We now explore similar expressions.

1. Explore $1+\frac{\text{Ans}}{3}$ with 1 as an initial seed.
   a. Use your calculator to complete the next three blanks: 1, $\frac{4}{3}$, $\frac{13}{9}$, $\frac{40}{27}$, _____ , _____ , _____
   b. What is the **denominator** of the *n*th term:

   c. What is the **numerator** of the *n*th term: _____
   Hint: Look back at Question 3 on Investigation 1.

   d. Solve the equation $1+\frac{x}{3}=x$ to find what this sequence converges to.
   Validate your claim by pressing ENTER on the home screen.

   e. Show that the *n*th term of this sequence is equivalent to $\frac{3}{2}\cdot\frac{3^n-1}{3^n}$.

   Complete the box: As $n\to\infty$, $\frac{3}{2}\cdot\frac{3^n-1}{3^n}\to\boxed{}$. Explain your reasoning.

2. Explore $1+\frac{\text{Ans}}{4}$ with 1 as an initial seed.
   a. Use your calculator to complete the next three blanks: 1, $\frac{5}{4}$, $\frac{21}{16}$, $\frac{85}{64}$, _____ , _____ , _____
   b. What is the denominator of the *n*th term? The numerator of the *n*th term? What is the *n*th term?
   Look for similarities with Question 1.
   c. Solve the equation $1+\frac{x}{4}=x$ to find what this sequence converges to.
   Validate your claim by pressing ENTER continually on the home screen.
   d. Show that the *n*th term from parts b and c of this question is equivalent to $\frac{4}{3}\cdot\frac{4^n-1}{4^n}$.

   Complete the box: As $n\to\infty$, $\frac{4}{3}\cdot\frac{4^n-1}{4^n}\to\boxed{}$. Explain your reasoning.

3. Similarly explore $1+\frac{\text{Ans}}{5}$. Then replace 5 with any real number and make a conjecture.
   For example, what would you expect for $1+\frac{\text{Ans}}{99}$? Show that your conjecture holds.

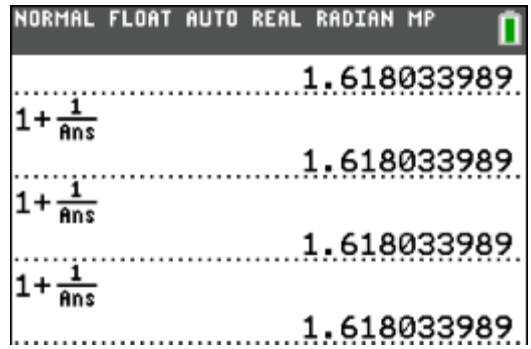
Investigation 3: Quadratic fun with $1+\frac{1}{\text{Ans}}$, $1+\frac{2}{\text{Ans}}$, $2+\frac{3}{\text{Ans}}$, $2+\frac{2}{\text{Ans}}$, and beyond!

In the previous investigation, we have seen that the expression $1+\frac{\text{Ans}}{2}$ can build the sequence 1, $\frac{3}{2}$, $\frac{7}{4}$, $\frac{15}{8}$, $\frac{31}{16}$, … , 2, 2, … We now explore similar expressions which lead to solving quadratic equations and building patterns..

1. Explore $1+\frac{1}{\text{Ans}}$ with 1 as an initial seed.
   a. Complete the next three blanks:
      1, 2, $\frac{3}{2}$, $\frac{5}{3}$, $\frac{8}{5}$, $\frac{13}{8}$, _____ , _____ , _____

   b. Describe any patterns you see in this sequence.

   c. This will converges to a famous number. (Google "phi".)
      Solve the equation $1+\frac{1}{x}=x$ to find its exact value.
      (You will need the quadratic formula.)



2. Explore $1+\frac{2}{\text{Ans}}$ with 1 as an initial seed.
   a. Use your calculator to complete the next three blanks:
      1, 3, $\frac{5}{3}$, $\frac{11}{5}$, $\frac{21}{11}$, $\frac{43}{21}$, _____ , _____ , _____

   b. Consider this conjecture to predict the next numerator.
      For the 3rd term $\frac{5}{3}$, its numerator 5 = **3** + 2×**1** where 3 and 1 are the two previous numerators.
      For the 4rd term $\frac{11}{5}$, its numerator 11 = **5** + 2×**3** where 5 and 3 are the two previous numerators.
      For the 5rd term $\frac{21}{11}$, its numerator 21 = **11** + 2×**5** where 11 and 5 are the two previous numerators.
      Does it hold for the next term?
   c. If 1 is the seed, this will converges to a positive integer. Solve the equation $1+\frac{2}{x}=x$ to find its value.
      Hint: Solve the equivalent equation $x+2=x^2$. It factors! ☺
      Validate your claim by pressing ENTER continually on the home screen.
   d. There are two solutions to $x+2=x^2$. What is the negative solution? What initial seed will have $1+\frac{2}{\text{Ans}}$ converge to this negative integer? (Hint: pick the solution as the seed.)

3. Explore $2+\frac{3}{\text{Ans}}$ with 1 as an initial seed.
   a. Use your calculator to complete the next three blanks:

      1, 5, $\frac{13}{5}$, $\frac{41}{13}$, $\frac{121}{41}$, $\frac{365}{121}$, _____ , _____ , _____

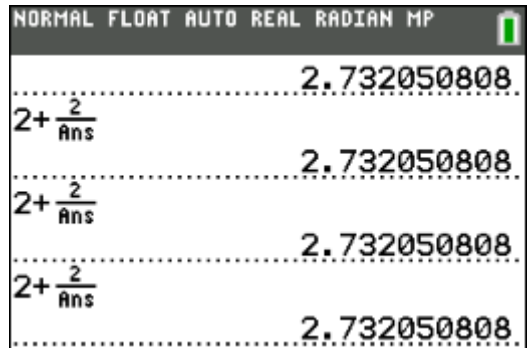
   b. Consider this conjecture to predict the next numerator.
      For the 3rd term $\frac{13}{5}$, its numerator 13 = 2×**5** + 3×**1** where 5 and 1 are the two previous numerators.
      For the 4rd term $\frac{41}{13}$, its numerator 41 = 2×**13** + 3×**5** where 13 and 5 are the two previous numerators.
      For the 5rd term $\frac{121}{41}$, its numerator 121 = 2×**41** + 3×**13** where 41 and 13 are the two previous numerators.
      Does it hold for the next term?
   c. If 1 is the seed, this will converges to a positive integer. Solve the equation $2+\frac{3}{x}=x$ to find its value.
      Hint: Solve the equivalent equation $2x+3=x^2$. It factors too! ☺ Then validate your claim by pressing ENTER continually on the home screen. What initial seed will have $2+\frac{3}{\text{Ans}}$ converge to a negative integer?

4. The expression $2+\frac{2}{\text{Ans}}$ with 1 as an initial seed will gives us the sequence of terms:

$1,\ \frac{4}{1},\ \frac{(1+4)\times 2}{4}=\frac{5}{2},\ \frac{(5+2)\times 2}{5}=\frac{14}{5},\ \frac{(14+5)\times 2}{14}=\frac{19}{7},\ \frac{(19+7)\times 2}{19}=\frac{52}{19},\ \frac{(52+19)\times 2}{52}=\frac{71\times 2}{52}=\frac{71}{26},$

a. What pattern do you see?

b. Use your pattern to conjecture the next three terms:
$1,\ \ldots,\ \frac{71}{26},$ _____ , _____ , _____
Validate with technology.



c. When we solve the equation $2+\frac{2}{x}=x$
to find the values at which it will converge,
we could solve the equivalent equation $2x+2=x^2$.
Solve this by completing the square.

d. Complete the boxes.
The solution in part c will be equivalent to $(x-1)^2=\square$
When 1 is the initial seed, $2+\frac{2}{\text{Ans}}$ will converge to the value $1+\sqrt{\square}$

e. It can be shown that the continued fraction $1+\cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{2+\ldots}}}=\sqrt{3}$.

How does this investigation validate this amazing fact?
Hint: Add 1 to both sides of this equation and compare with 4d.

5. Find the value of the continued fraction $1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ldots}}}$. Start by exploring how $2+\frac{1}{\text{Ans}}$ converges.

6. Find the value of the continued fraction $1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}$. Hint: See Question 1

7. Find the value of the continued fraction $1+\cfrac{3}{2+\cfrac{3}{2+\cfrac{3}{2+\ldots}}}$. Hint: See Question 3.

8. Make conjectures for what will happen for expressions of the form $b+\frac{c}{\text{Ans}}$ for any values of b and c (both positive and negative.) For what values of b and c will you have the sequence converge to rational solutions?

Teacher's Notes

Parts of these activity can be used in Algebra 1, Algebra 2, or Precalculus. Selected answers are below.

Investigation 1

3. The denominators are powers of 2, or double the previous fraction's denominator: $1,\ \frac{3}{\boxed{2}},\ \frac{7}{\boxed{4}},\ \frac{15}{\boxed{8}},\ \frac{31}{\boxed{16}},\ \frac{\square}{\boxed{32}}\ \ldots$

   Also each denominator is 1 more than the previous numerator: $\frac{\boxed{3}}{2}$, $\frac{7}{\boxed{4}}$ and $\frac{\boxed{7}}{4}$, $\frac{15}{\boxed{8}}$ and $\frac{\boxed{15}}{8}$, $\frac{31}{\boxed{16}}$, and $\frac{\boxed{31}}{16}$, $\frac{\square}{\boxed{32}}$

   Students may notice that if the denominator of a fraction is *d*, its numerator is twice its denominator less 1, or $2d - 1$:

   $1,\ \frac{3}{2}=\frac{2\cdot\boxed{2}-1}{\boxed{2}},\ \frac{7}{4}=\frac{2\cdot\boxed{4}-1}{\boxed{4}},\ \frac{15}{8}=\frac{2\cdot\boxed{8}-1}{\boxed{8}},\ \frac{31}{16}=\frac{2\cdot\boxed{16}-1}{\boxed{16}},\ \frac{?}{32}=\frac{2\cdot\boxed{32}-1}{\boxed{32}}=\frac{64-1}{32}=\frac{63}{32},\ \ldots$

   So students might get the next fraction by first finding its denominator, then shave off 1 to get the current numerator. Or they might see that if you added 1 to each numerator in the sequence $1,\ \frac{3}{2},\ \frac{7}{4},\ \frac{15}{8},\ \frac{31}{16}$, you would have exactly a 2:1 ratio such as $\frac{2}{1},\ \frac{4}{2},\ \frac{8}{4},\ \frac{16}{8},\ \frac{32}{16},\ \ldots$ So undo this by *removing* 1 from the numerator of these ratio of 2's.

   There is also a pattern among numerators. 7 is double the previous numerator plus 1: $1,\ \frac{3}{2},\ \frac{2\cdot\boxed{3}+1}{4}=\frac{\boxed{7}}{4},\ \ldots$

   15 is double the previous numerator plus 1: $1,\ \frac{3}{2},\ \frac{7}{4},\ \frac{2\cdot\boxed{7}+1}{8}=\frac{\boxed{15}}{8},\ \ldots$

   31 is double the previous numerator plus 1: $1,\ \frac{3}{2},\ \frac{7}{4},\ \frac{15}{8},\ \frac{2\cdot\boxed{15}+1}{8}=\frac{\boxed{31}}{8},\ \ldots$

   Yet another pattern occurs: add the numerator of one term to the denominator of the next term and you will have:

   $\frac{\boxed{3}}{2},\ \frac{7}{\boxed{4}}$ $\quad 3+4=7$

   $\frac{\boxed{7}}{4},\ \frac{15}{\boxed{8}}$ $\quad 7+8=15$

   $\frac{\boxed{15}}{8},\ \frac{31}{\boxed{16}}$ $\quad 15+16=31$

   $\frac{\boxed{31}}{16},\ \frac{\square}{\boxed{32}}$ $\quad 31+32=63$

4. $1,\ \frac{3}{2},\ \frac{7}{4},\ \frac{15}{8},\ \frac{31}{16},\ \frac{63}{32},\ \frac{127}{64},\ \frac{255}{128}$.

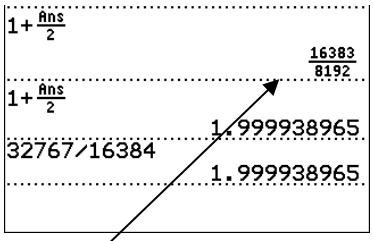
5. a. One approach is to add 1 to the numerator 16383: $\frac{\boxed{16383}}{8192},\ \frac{\square}{\boxed{16364}}$

   Another approach is to double the denominator 8192 to get 16384. $\frac{16383}{\boxed{8192}},\ \frac{\square}{\boxed{16364}}$

   b. Take the new denominator 16384, double it, and subtract 1, so we have 32768 – 1 = 32767.

   The fraction is therefore $\frac{32767}{16384}$. Or add 16334 + 16384 $\frac{\boxed{16383}}{8192},\ \frac{\square}{\boxed{16364}}$

   c. Verify your claim by entering your fraction on the home screen. See to the right.



   d. Each denominator in the sequence is 1, 2, 2^2, 2^3, 2^4, …
   So the *n*th denominator is $d = 2^{n-1}$.

   The numerator is twice the denominator less 1, so $2d-1=2\cdot(2^{n-1})-1$.

   This can be simplified using laws of exponents: 2^n-1.
   Or you could have just noticed the *n*th numerator is 1 less than a power of 2.

   The *n*th term is therefore $\frac{2^n-1}{2^{n-1}}$. If $n = 5$, then the fraction is $\frac{31}{16}$. If the fraction is $\frac{16383}{8192}$, then $n = 13$.

   The least number of times they would have pressed ENTER to reach 1.999938965 is 14 times.

6. a. Solve the equation $1+\frac{x}{2}=x$: $1+\frac{x}{2}=x$

$$1+\tfrac{x}{2}-\tfrac{x}{2}=x-\tfrac{x}{2}$$
$$1=\tfrac{x}{2}$$
$$x=2$$

b. Students might conjecture that the sequence will converge to 2 immediately, which it does.

c. $$\frac{2^n-1}{2^{n-1}} = \frac{2^n}{2^n 2^{-1}} - \frac{1}{2^{n-1}}$$
$$= 2 - \frac{1}{2^{n-1}}$$

As $n \to \infty$, the expression $\frac{1}{2^{n-1}} \to 0$ so $2 - \frac{1}{2^{n-1}} \to 2$.

Note the discussion in #3 how this sequence is *nearly* like the sequence of 2's: $\frac{2}{1}, \frac{4}{2}, \frac{8}{4}, \frac{16}{8}, \frac{32}{16}, \ldots$

So as the terms increase, the ratios get closer and closer to 2.

Investigation 2

1. a. The sequence is, $1, \frac{4}{3}, \frac{13}{9}, \frac{40}{27}, \frac{121}{81}, \frac{364}{243}, \frac{1093}{729}$

b. The denominators are powers of 3. The denominator of the first term is 3^0. The denominator of the second term is 3^1. The denominator of the third term is 3^2, and so on. So the denominator of the *n*th term is 3^{n-1}.

c. The numerator of the *n*th term is he sum of the numerator of the *n*–1st term and the denominator of the *n*th term. For example:

$\frac{\boxed{1}}{1}, \frac{4}{\boxed{3}}$ $\quad 1+3=4$ The numerator of the second term is $3^0 + 3^1$ or $\frac{3^2-1}{3-1}$

$\frac{\boxed{4}}{3}, \frac{13}{\boxed{9}}$ $\quad 4+9=13$ The numerator of the third term is $3^0 + 3^1 + 3^2$ or $\frac{3^3-1}{3-1}$

$\frac{\boxed{13}}{9}, \frac{40}{\boxed{27}}$ $\quad 13+27=40$ The numerator of the fourth term is $3^0 + 3^1 + 3^2 + 3^3$ or $\frac{3^4-1}{3-1}$

$\frac{\boxed{40}}{27}, \frac{121}{\boxed{81}}$ $\quad 40+81=121$ The numerator of the fifth term is $3^0 + 3^1 + 3^2 + 3^3 + 3^4$ or $\frac{3^5-1}{3-1}$

Therefore the numerator of the *n*th term is $\frac{3^n-1}{3-1}$ or $\frac{3^n-1}{2}$. The *n*th term is $\frac{\frac{1}{2}(3^n-1)}{3^{n-1}}$

d. This is a linear equation with one solution. The solution is x = $\frac{3}{2}$.

e. $\frac{\frac{1}{2}(3^n-1)}{3^{n-1}} = \frac{1}{2}\cdot\frac{3^n-1}{3^{-1}\cdot 3^n} = \frac{1}{2\cdot 3^{-1}}\cdot\frac{3^n-1}{3^n} = \frac{3}{2}\cdot\frac{3^n-1}{3^n}$. As $n \to \infty$, $\frac{3}{2}\cdot\frac{3^n-1}{3^n} \to \boxed{\frac{3}{2}}$ since

$$\frac{3^n-1}{3^n} \to \frac{3^n}{3^n} - \frac{1}{3^n} \to 1 - \frac{1}{3^n} \to 1$$

2. a. The sequence is $1, \frac{5}{4}, \frac{21}{16}, \frac{85}{64}, \frac{341}{256}, \frac{1093}{729}, \frac{3280}{2187}$

b. The denominator of *n*th term is 4^{n-1}. The numerator of *n*th term is $\frac{4^n-1}{4-1}$. The *n*th term is $\frac{\frac{1}{3}(4^n-1)}{4^{n-1}}$

c. This is a linear equation with one solution. The solution is x = $\frac{4}{3}$.

d. Similar to 1e but as $n \to \infty$, $\frac{4}{3}\cdot\frac{4^n-1}{4^n} \to \boxed{\frac{4}{3}}$

3. $1+\frac{Ans}{5}$ will produce a sequence which will converge to the value x, where

x is the solution to the equation as $1 + \frac{x}{5} = x$. The solution is x = $\frac{5}{4}$ and as $n \to \infty$, $\frac{5}{4} \cdot \frac{5^n - 1}{5^n} \to \boxed{\frac{5}{4}}$.

In general $1 + \frac{Ans}{c}$ will produce a sequence of n terms which will converge to the value x, where

x is the solution to the equation as $1 + \frac{x}{c} = x$ and as $n \to \infty$, $\frac{c}{c-1} \cdot \frac{c^n - 1}{c^n} \to \boxed{\frac{c}{c-1}}$.

$1+\frac{Ans}{99}$, will produce a sequence which will converge to the value x, where

x is the solution to the equation as $1 + \frac{x}{99} = x$. The solution is x = $\frac{99}{98}$ and as $n \to \infty$,

$\frac{99}{98} \cdot \frac{99^n - 1}{99^n} \to \boxed{\frac{99}{98}}$.

These investigations were adapted from the presentation *Embark on the Voyage of Discovery with the TI-84 Plus C Silver Edition and the CCSS* at the 26th Annual 2014 T^3 International Conference, Las Vegas NV posted on the Website http://users.ipfw.edu/lamaster/technology/