



Math Objectives

- Students will be able to state and apply the properties of logarithms:
 - $\log_a(mn) = \log_a(m) + \log_a(n)$
 - $\log_a\left(\frac{m}{n}\right) = \log_a(m) - \log_a(n)$
 - $\log_a(m^n) = n \log_a(m)$
- Students will make comparisons between the properties of exponents and the properties of logarithms.
- Students will try to make a connection with how to understand these topics in IB Mathematics courses and on their final assessments.

Vocabulary

- logarithmic expressions
- exponential properties
- equivalent expressions

About the Lesson

- This lesson is aligning with the curriculum of IB Mathematics Applications and Interpretations SL/HL and IB Mathematics Approaches and Analysis SL/HL
- This falls under the IB Mathematics Content Topic 1 Algebra:
 - AI HL 1.9** Laws of Logarithms (see above)
 - AA 1.7b** Laws of Logarithms (see above)
- As a result, students will:
 - Apply this information to real world situations

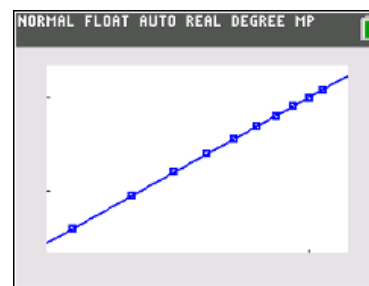
Teacher Preparation and Notes

- This activity is done with the use of the TI-84 family as an aid to the problems, specifically using Lists to input and generate graphs using stat plots and linear regression commands.
- This activity serves as a nice introduction to properties of logarithms.

Activity Materials

- Compatible TI Technologies:
 - TI-84 Plus*, TI-84 Plus Silver Edition*, TI-84 Plus C Silver Edition, TI-84 Plus CE

* with the latest operating system (2.55MP) featuring MathPrint™ functionality.



Tech Tips:

- This activity includes screen captures taken from the TI-84 Plus CE. It is also appropriate for use with the rest of the TI-84 Plus family. Slight variations to these directions may be required if using other calculator models.
- Watch for additional Tech Tips throughout the activity for the specific technology you are using.
- Access free tutorials at <http://education.ti.com/calculators/pd/US/Online-Learning/Tutorials>

Lesson Files:

Student Activity
 Properties_of_Logarithms_Student-84.pdf
 Properties_of_Logarithms_Student-84.doc


Discussion Points and Possible Answers

Teacher Tip: At some point during the investigation, you may want to discuss some of the historical implications of the properties of logarithms. The usefulness of logarithms in calculations is based on these properties. The first two properties are the basis for the slide rule, a mechanical computation device that preceded the electronic calculator.

For this activity, the expression used is $\log_2(x)$. The investigations also work for any base > 0 and base $\neq 1$.

1. Fill in the following table, and with a classmate, discuss and answer the following questions.

(m, n)	$\log_2(m \cdot n)$	$\log_2(m) \cdot \log_2(n)$	$\log_2(m + n)$	$\log_2(m) + \log_2(n)$
(4, 2)	3	2	2.5850	3
(4, 3)	3.5850	3.1699	2.8074	3.5850
(4, 4)	4	4	3	4
(4, 5)	4.3219	4.6439	3.1699	4.3219
(4, 6)	4.5850	5.1699	3.3219	4.5850

- a. Find which expressions, if any, appear to be equivalent independent of the values of m and n .

Answer: $\log_2(mn)$ and $\log_2(m) + \log_2(n)$

- b. Using $m = 8$ and $n = 4$, substitute these values into the logarithmic expressions you found to be equivalent in part 1a, and simplify these expressions to show they are indeed equivalent.

Answer: $\log_2(8 \cdot 4) = \log_2(32) = 5$ and $\log_2(8) + \log_2(4) = 3 + 2 = 5$

- c. Use the expressions you found in parts 1a and 1b to write a general logarithmic property for $\log_a mn$, where a is a real number, $a > 0$ and $a \neq 1$.

Answer: $\log_a(mn) = \log_a(m) + \log_a(n)$

- d. Explain how the operations in the logarithmic property in part 1c relate to the operations in the exponential property $a^m a^n = a^{m+n}$.



Answer: For the exponential rule, if the bases are the same and the expressions are being multiplied, the exponents are added. Because a logarithm is also an exponent, the rule applies similarly.

Teacher Tip: If desired, you may want students to prove question 1d instead. Here is a sample answer:

Using the definition of logarithms that states $\log_a x = y \Leftrightarrow a^y = x$ and the stated exponential property, you can verify the logarithm property from question 1c.

$$\log_a(m) = y_1 \Leftrightarrow a^{y_1} = m$$

$$\log_a(n) = y_2 \Leftrightarrow a^{y_2} = n$$

$$\log_a(m) + \log_a(n) = y_1 + y_2$$

$$mn = a^{y_1}a^{y_2} = a^{y_1+y_2} \Leftrightarrow \log_a(mn) = y_1 + y_2$$

$$\text{Thus, } \log_a(m) + \log_a(n) = \log_a(mn)$$

Now let's look at this same idea but graphically. Suppose you wanted to simplify the logarithm of a product, like $\log 6a$. Think about how you might go about doing this. Let's start by defining a new variable $b = 6a$.

Step a: On your handheld, press **stat**, **Edit**, and in **L₁** enter at least 10 values for a , that are in the domain of the logarithmic function.

Step b: At the top of **L₂**, enter a formula that will calculate $b = 6a$ from the values of **L₁**.

Step c: Make a scatter plot of these values. Press **2nd**, **y =**, and select **Plot1**. Adjust the settings to display the a-values in **L₁** along the x-axis and the b-values in **L₂** along the y-axis.

Step d: Press **zoom**, **9: ZoomStat** to view the plot in an appropriate window.

2. Describe the shape of the graph. Discuss with a classmate if it is what you expected. Share your results with the class.

Answer: The shape of the graph is linear.

Step e: Now we will define two new variables, x and y . Let $x = \log a$ and $y = \log b$. At the top of **L₃** enter a formula that calculates x from the values of a in **L₁**. At the top of **L₄** enter a formula that calculates y from the values of b in **L₂**.

Step f: Make a scatter plot of y vs. x . Press **2nd**, **y =**, and select **Plot1** again. Adjust the settings to display the x-values in **L₃** along the x-axis and the y-values in **L₄** along the y-axis. Press **zoom**, **9:ZoomStat**.

3. Describe the shape of the graph. Discuss with a classmate if it is what you expected. Share your



results with the class.

Answer: The shape of the graph is linear.

Step g: The data appear linear. Find the equation of a line through these points with the **LinReg(ax + b)** command. Press **stat**, **CALC**, **4: LinReg(ax + b)**. Make sure to fill in the appropriate Xlist and Ylist.

4. Write down the equation of the line through these points.

Answer: $y = x + 0.778151$

5. Find the y-intercept of the line.

Answer: y-intercept = 0.778151

You should have found that the equation of the line was $y = x + 0.778151$. Think about where this y – *intercept* comes from. (Here’s a hint: Try raising 10 to the 0.778151 power.)

6. Using logs, find what 0.778151 is.

Answer: $\log_{10} 6 \approx 0.778151$

7. Since $10^{0.778151} \approx$ _____, $\log(6) \approx$ _____.

Answer: $10^{0.778151} \approx 6$ and $\log_{10}(6) \approx 0.778151$

You have found that $y = \log 6 + x$. Think about what this means. Substitute to rewrite this as an equation in terms of a . The explanation for each step is given to the right.

$y = \log 6 + x$	Equation of the line
	$x = \log a$ and $y = \log b$
	$b = 6a$

Product Property of Logarithms

For $a > 0$ and $b > 0$, $\log ab = \log a + \log b$.

Examples

$\log xy$ is written in *expanded form* as $\log x + \log y$

$\log 7 + \log z$ is written as a single logarithm as $\log 7z$



8. Fill in the following table, and with a classmate, discuss and answer the following questions.

(m, n)	$\log_2\left(\frac{m}{n}\right)$	$\frac{\log_2(m)}{\log_2(n)}$	$\log_2(m - n)$	$\log_2(m) - \log_2(n)$
(4, 2)	1	2	1	1
(4, 3)	0.4150	1.2619	0	0.4150
(4, 4)	0	1	Undef	0
(4, 5)	-0.3219	0.8614	Undef	-0.3219
(4, 6)	-0.5850	0.7737	Undef	-0.5850

- a. Find which expressions, if any, appear to be equivalent independent of the values of m and n .

Answer: $\log_2\left(\frac{m}{n}\right)$ and $\log_2(m) - \log_2(n)$

- b. Using $m = 8$ and $n = 4$, substitute these values into the logarithmic expressions you found to be equivalent in part 8a, and simplify these expressions to show they are indeed equivalent.

Answer: $\log_2\left(\frac{8}{4}\right) = \log_2(2) = 1$ and

$$\log_2(8) - \log_2(4) = 3 - 2 = 1$$

- c. Use the expressions you found in parts 8a and 8b to write a general logarithmic property for $\log_a\left(\frac{m}{n}\right)$ where a is a real number, $a > 0$ and $a \neq 1$.

Answer: $\log_a\left(\frac{m}{n}\right) = \log_a(m) - \log_a(n)$

- d. Explain how the operations in the logarithmic property in part 8c relate to the operations in the exponential property $\frac{a^m}{a^n} = a^{m-n}$.

Answer: The log of a quotient is the difference of the logs. Since logarithms are exponents, the exponent of a quotient is the difference of the exponents.



Teacher Tip: If desired, you may want students to prove question 2d instead. Here is a sample answer:

Using the definition of logarithms that states $\log_a x = y \Leftrightarrow a^y = x$ and the stated exponential property, you can verify the logarithm property from question 2c.

$$\begin{aligned}\log_a(m) &= y_1 \Leftrightarrow a^{y_1} = m \\ \log_a(n) &= y_2 \Leftrightarrow a^{y_2} = n \\ \log_a(m) - \log_a(n) &= y_1 - y_2 \\ \frac{m}{n} &= \frac{a^{y_1}}{a^{y_2}} = a^{y_1 - y_2} \Leftrightarrow \log_a\left(\frac{m}{n}\right) = y_1 - y_2\end{aligned}$$

$$\text{Thus, } \log_a(m) - \log_a(n) = \log_a\left(\frac{m}{n}\right)$$

Again, let's look at this same idea but graphically. Suppose you wanted to simplify the logarithm of a quotient, like $\log \frac{8}{a}$. Think about how you might go about doing this. Let's start by defining a new variable $b = \frac{8}{a}$.

Step a: Going back to your lists, clear the data from **L₂**, **L₃**, and **L₄** by going to the top of the list and pressing clear, enter. You will be leaving the data in **L₁** in as is.

Step b: At the top of **L₂**, enter a formula that will calculate $b = \frac{8}{a}$ from the values of **L₁**.

Step c: Make a scatter plot of these values. Press **2nd**, **y =**, and select **Plot1**. Adjust the settings to display the a-values in **L₁** along the x-axis and the b-values in **L₂** along the y-axis.

Step d: Press **zoom**, **9: ZoomStat** to view the plot in an appropriate window.

9. Describe the shape of the graph. Discuss with a classmate if it is what you expected. Share your results with the class.

Answer: The shape of the graph is linear.

Step e: Now we will define two new variables, x and y . Let $x = \log a$ and $y = \log b$. At the top of **L₃** enter a formula that calculates x from the values of a in **L₁**. At the top of **L₄** enter a formula that calculates y from the values of b in **L₂**.

Step f: Make a scatter plot of y vs. x . Press **2nd**, **y =**, and select **Plot1** again. Adjust the settings to display the x-values in **L₃** along the x-axis and the y-values in **L₄** along the y-axis. Press **zoom**, **9:ZoomStat**.



Step g: The data appear linear. Find the equation of a line through these points with the **LinReg(ax + b)** command. Press **stat**, **CALC**, **4: LinReg(ax + b)**. Make sure to fill in the appropriate Xlist and Ylist.

10. Write down the equation of the line through these points.

Answer: $y = 0.90309 - x$

11. Find the y-intercept of the line.

Answer: y-intercept = 0.90309

You should have found that the equation of the line was $y = 0.90309 - x$. Think about where this $y - intercept$ comes from.

12. Using logs, find what 0.90309 is.

Answer: $\log_{10} 8 \approx 0.90309$

13. Since $10^{0.90309} \approx \underline{\hspace{2cm}}$, $\log(8) \approx \underline{\hspace{2cm}}$.

Answer: $10^{0.90309} \approx 8$ and $\log_{10}(8) \approx 0.90309$

You have found that $y = \log 8 - x$. Think about what this means. Substitute to rewrite this as an equation in terms of a . The explanation for each step is given to the right.

$y = \log 8 - x$	Equation of the line
	$x = \log a$ and $y = \log b$
	$b = \frac{8}{a}$

Quotient Property of Logarithms

For $a > 0$ and $b > 0$, $\log ab = \log a + \log b$.

Examples $\log \frac{x}{y}$ is written in *expanded form*

as $\log x - \log y$

$\log 7 - \log z$ is written as a single logarithm as $\log \frac{7}{z}$

14. Fill in the following table, and with a classmate, discuss and answer the following questions.



(m, n)	$\log_2(m^n)$ or $\log_2(m)^n$	$(\log_2 m)^n$	$n \cdot \log_2(m)$
(4, 2)	4	4	4
(4, 3)	6	8	6
(4, 4)	8	16	8
(4, 5)	10	32	10
(4, 6)	12	64	12

- a. Find which expressions, if any, appear to be equivalent independent of the values of m and n .

Answer: $\log_2(m^n)$ and $n \log_2(m)$

- b. Using $m = 4$ and $n = 3$, substitute these values into the logarithmic expressions you found in part 14a, and simplify these expressions to show they are equivalent.

Answer: $\log_2(4^3) = \log_2(64) = 6$ and
 $3\log_2(4) = 3 \cdot 2 = 6$

- c. Use the expressions you found in parts 14a and 14b to write a general logarithmic property for $\log_a(m)^n$ where a is a real number, $a > 0$ and $a \neq 1$

Answer: $\log_a(m)^n = n \log_a(m)$

- d. Explain how the operations in the logarithmic property in part 14c relate to the operations in the exponential property $(a^m)^n = a^{mn}$.

Answer: When an expression with an exponent is raised to an exponent, the exponents are multiplied. Roughly speaking, this is a restatement in logarithm form of the law of exponents.

Teacher Tip: If desired, you may want students to prove question 3d instead. Here is a sample answer:

$$\begin{aligned} \log_a(m^n) = y &\Leftrightarrow a^y = m^n \\ (a^y)^{1/n} &= (m^n)^{1/n} \\ (a)^{y/n} = m &\Leftrightarrow \log_a m = \frac{y}{n} \\ n \log_a m &= y \\ \text{Or } \log_a(m^n) &= \underbrace{\log_a(m \cdot m \cdot m \cdot \dots \cdot m)}_{n \text{ times}} = \underbrace{\log_a m + \log_a m + \dots + \log_a m}_{n \text{ times}} = n \log_a(m) \end{aligned}$$



- e. Use the logarithmic property you proved in part 14c to show that $\log_a a = 1$ for all values of a where $a > 0$ and $a \neq 1$.

Answer: Since $a = a^1$ then $\log_a a^1 = 1$.

- f. Use the logarithmic property you proved in part 14c to show that $\log_a 1 = 0$ for all values of a where $a > 0$ and $a \neq 1$.

Answer: Since $1 = a^0$ then $\log_a a^0 = 0$.

Teacher Tip: The last page does not allow for the n value to be negative, so this is a perfect opportunity to discuss with students how this can be

applied as well. In other words, discuss why $\log_2 \left(\frac{1}{x}\right) = -\log_2 x$. This

discussion can provide review of other laws of exponents as well.

One final time, let's look at this same idea but graphically. Suppose you wanted to simplify the logarithm of a power, like $\log a^2$. Think about how you might go about doing this. Let's start by defining a new variable $b = a^2$.

Step a: Going back to your lists, clear the data from **L₂**, **L₃**, and **L₄** by going to the top of the list and pressing clear, enter. You will be leaving the data in **L₁** in as is.

Step b: At the top of **L₂**, enter a formula that will calculate $b = a^2$ from the values of **L₁**.

Step c: Make a scatter plot of these values. Press **2nd**, **y =**, and select **Plot1**. Adjust the settings to display the a-values in **L₁** along the x-axis and the b-values in **L₂** along the y-axis.

Step d: Press **zoom**, **9: ZoomStat** to view the plot in an appropriate window.

15. Describe the shape of the graph. Discuss with a classmate if it is what you expected. Share your results with the class.

Answer: The shape of the graph is linear.

Step e: Now we will define two new variables, x and y . Let $x = \log a$ and $y = \log b$. At the top of **L₃** enter a formula that calculates x from the values of a in **L₁**. At the top of **L₄** enter a formula that calculates y from the values of b in **L₂**.



Step f: Make a scatter plot of y vs. x . Press **2nd**, **y =**, and select **Plot1** again. Adjust the settings to display the x -values in **L₃** along the x -axis and the y -values in **L₄** along the y -axis. Press **zoom**, **9:ZoomStat**.

Step g: The data appear linear. Find the equation of a line through these points with the **LinReg(ax + b)** command. Press **stat**, **CALC**, **4: LinReg(ax + b)**. Make sure to fill in the appropriate Xlist and Ylist.

16. Write down the equation of the line through these points.

Answer: $y = 2x$

17. Find the y -intercept of the line.

Answer: y -intercept = 0

You should have found that the equation of the line was $y = 2x$. Think about what this means.

You have found that $y = \log 6 + x$. Think about what this means. Substitute to rewrite this as an equation in terms of a . The explanation for each step is given to the right.

$y = 2x$	Equation of the line
$\log b = 2 \log a$	$x = \log a$ and $y = \log b$
$\log a^2 = 2 \log a$	$b = a^2$

Power Property of Logarithms

For $a > 0$, $\log a^b = b \log a$.

Examples $\log x^3$ can be written as $3 \log x$

$8 \log x$ can be written as $\log x^8$

Further IB Math Extension

Using the properties discussed in this activity, find the solution of:

$$\log_3 x - 2 \log_3 2 = 3 - \log_3 2$$

Answer: Using the power property, simplify $2 \log_3 2$

$$\log_3 x - \log_3 2^2 = 3 - \log_3 2$$

Using the quotient property, combine the logs on the left



$$\log_3 \frac{x}{4} = 3 - \log_3 2$$

Add the $\log_3 2$ to both sides.

$$\log_3 \frac{x}{4} + \log_3 2 = 3$$

Using the product property, combine the logs on the left

$$\log_3 \left(\frac{x}{4} \cdot 2 \right) = 3$$

$$\log_3 \left(\frac{x}{2} \right) = 3$$

Rewrite the log in exponential form and solve for x

$$3^3 = \frac{x}{2}$$

$$27 = \frac{x}{2}$$

$$x = 54$$

Wrap Up

Upon completion of the discussion, the teacher should ensure that students know and can apply the following logarithm properties:

- $\log_a(mn) = \log_a(m) + \log_a(n)$
- $\log_a\left(\frac{m}{n}\right) = \log_a(m) - \log_a(n)$
- $\log_a(m^n) = n \log_a(m)$

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