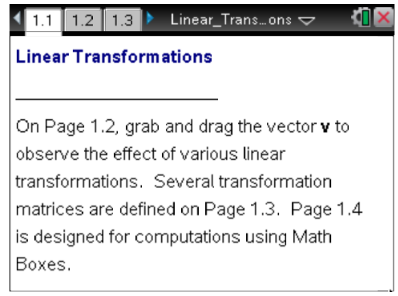

Math Objectives

- Students will use matrices to help visualize and describe linear transformations.
- Students will manipulate the input vector and observe the results of specific linear transformations geometrically.
- Students will generate output vectors.
- Students will characterize linear transformations in terms of vector magnitude and angles.
- Students will look for and express regularity in repeated reasoning (CCSS Mathematical Practice).
- Students will look for and make use of structure (CCSS Mathematical Practice).

Vocabulary

- linear transformation
- transformation matrix
- reflection
- rotation
- scaling
- projection

About the Lesson

- This lesson involves linear transformations from $\mathbf{R}^2$ to $\mathbf{R}^2$ represented by matrices. Note: $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ represents the set of all pairs of real numbers.
- As a result, students will:
  - Grab and drag the input vector and observe the effect of each linear transformation.
  - Describe each linear transformation in words.
  - Compute output vectors in order to confirm conjectures.
  - Determine general properties of certain linear transformations.

TI-Nspire™ Navigator™ System

- Transfer a File.
- Use Screen Capture to examine patterns that emerge.
- Use Live Presenter to demonstrate.
- Use Quick Poll to assess students' understanding.

TI-Nspire™ Technology Skills:

- Download a TI-Nspire document
- Open a document
- Move between pages
- Grab and drag a point

Tech Tips:

- Make sure the font size on your TI-Nspire handhelds is set to Medium.
- You can hide the function entry line by pressing ctrl G.

Lesson Files:
Student Activity
Linear_Transformations_Student.pdf
Linear_Transformations_Student.doc
TI-Nspire document
Linear_Transformations_Student.tns

Visit www.mathnspired.com for lesson updates and tech tip videos.

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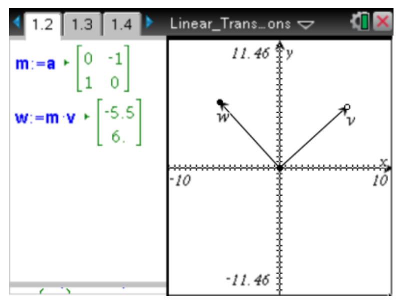
Discussion Points and Possible Answers

> **Tech Tip:** If students experience difficulty dragging a point, check to make sure that they have moved the cursor until it becomes a hand (👆) getting ready to grab the point. Also, be sure that the word *point* appears, not the word *text*. Then press ctrl 👆 to grab the point and close the hand (✊). Note: The point representing the vector $\mathbf{v}$ is an open circle.

A linear transformation from $\mathbf{R}^2$ to $\mathbf{R}^2$ can be represented by a matrix.
If T is a linear transformation that maps $\mathbf{R}^2$ to $\mathbf{R}^2$ and **v** is a 2×1 column vector, then the linear transformation can be written as $T(\mathbf{v}) = \mathbf{m} \cdot \mathbf{v}$ for some 2×2 matrix $\mathbf{m}$.

The matrix $\mathbf{m}$ is called the transformation matrix.

Move to page 1.2.

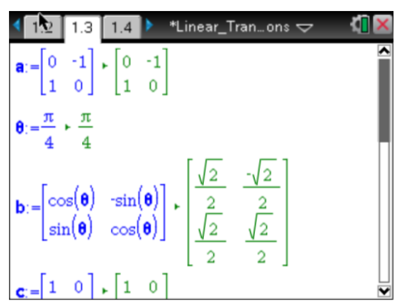


The left work area is a Notes page with two interactive Math Boxes.

- In the first Math Box, define the matrix $\mathbf{m}$ to be a transformation matrix. Note: to define $\mathbf{m},$ edit the Math Box following the assignment characters, := .
- When you open the .tns file, $\mathbf{m} = \mathbf{a}$ initially.

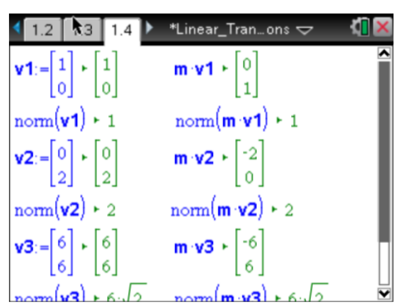
In the right work area, grab and drag the vector $\mathbf{v}$ (at the tip of the arrow).

- The product, $\mathbf{w} = \mathbf{m} \cdot \mathbf{v},$ in the left work area, and the vector $\mathbf{w},$ in the right work area, are automatically updated.



On Page 1.3, there are several defined transformation matrices and constants.

There are also several Math Boxes on Page 1.4 to compute $\mathbf{m} \cdot \mathbf{v}$ for various input vectors $\mathbf{v}$.



Note: The calculator function **norm** of a vector returns the length of the vector. Consider how each of the following transformations affects the magnitude and direction of the input vector.

1. Let $\mathbf{m} = \mathbf{a} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

   a. Describe this transformation in words.

   Sample Answers: This linear transformation rotates the input vector $\frac{\pi}{2}$, or 90°, counterclockwise. The magnitude of the output vector is the same as the input vector.

   b. Complete the following table.

   Answer:

$\mathbf{v}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 2 \end{bmatrix}$	$\begin{bmatrix} -4 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ -5 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 3 \\ -3 \end{bmatrix}$
$\mathbf{m} \cdot \mathbf{v}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -2 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ -4 \end{bmatrix}$	$\begin{bmatrix} 5 \\ 0 \end{bmatrix}$	$\begin{bmatrix} -2 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 3 \end{bmatrix}$

   c. Do the calculations in the table above support your description from part a? Why or why not?

   Sample Answers: The calculations in part (b) support the description in part (a). Each input vector $\mathbf{v}$ is rotated counterclockwise $\frac{\pi}{2}$ radians, or 90°.

2. For $\theta = \frac{\pi}{4}$, let $\mathbf{m} = \mathbf{b} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Note: To change the transformation matrix, click the Math Box in which $\mathbf{m}$ is defined (on Page 1.2). Delete the current transformation matrix (for example, $\mathbf{a}$), and type the variable representing any one of the transformation matrices defined on Page 1.3 just after the assignment characters := (for example, $\mathbf{b}$).

   a. Describe this transformation in words.

   Sample Answers: This linear transformation rotates the input vector $\frac{\pi}{4}$, or 45°, counterclockwise. The magnitude of the output vector is the same as the input vector.

b. Complete the following table.

Answer:

$\mathbf{v}$	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}2\\2\end{bmatrix}$	$\begin{bmatrix}-4\\-4\end{bmatrix}$	$\begin{bmatrix}0\\-1\end{bmatrix}$	$\begin{bmatrix}5\\5\end{bmatrix}$	$\begin{bmatrix}\sqrt{2}\\-\sqrt{2}\end{bmatrix}$
$\mathbf{m}\cdot\mathbf{v}$	$\begin{bmatrix}\frac{\sqrt{2}}{2}\\\frac{\sqrt{2}}{2}\end{bmatrix}$	$\begin{bmatrix}0\\2\sqrt{2}\end{bmatrix}$	$\begin{bmatrix}0\\-4\sqrt{2}\end{bmatrix}$	$\begin{bmatrix}\frac{\sqrt{2}}{2}\\-\frac{\sqrt{2}}{2}\end{bmatrix}$	$\begin{bmatrix}0\\5\sqrt{2}\end{bmatrix}$	$\begin{bmatrix}2\\0\end{bmatrix}$

c. Do the calculations in the table above support your description from part a? Why or why not?

Sample Answers: The calculations in part (b) support the description in part (a). Each input vector $\mathbf{v}$ is rotated counterclockwise $\frac{\pi}{4}$ units, or 45°.

d. Describe this transformation for any value of θ.

Sample Answers: For any value of θ, this linear transformation rotates the input vector θ radians counterclockwise. The magnitude of the output vector is the same as the input vector. Note: The transformation matrix $\mathbf{b}$ represents a general rotation while the transformation matrix $\mathbf{a}$ represents a specific rotation. Ask students to relate these two matrices by finding a value θ such that $\mathbf{b} = \mathbf{a}$.

3. Let $\mathbf{m} = \mathbf{c} = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$.

a. Describe this transformation in words.

Sample Answers: This linear transformation reflects the input vector across the x-axis. The magnitude of the output vector is the same as the input vector.

b. Complete the following table.

Answer:

$\mathbf{v}$	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}3\\3\end{bmatrix}$	$\begin{bmatrix}-4\\-6\end{bmatrix}$	$\begin{bmatrix}0\\-5\end{bmatrix}$	$\begin{bmatrix}3\\7\end{bmatrix}$	$\begin{bmatrix}-4\\6\end{bmatrix}$
$\mathbf{m}\cdot\mathbf{v}$	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}3\\-3\end{bmatrix}$	$\begin{bmatrix}-4\\6\end{bmatrix}$	$\begin{bmatrix}0\\5\end{bmatrix}$	$\begin{bmatrix}3\\-7\end{bmatrix}$	$\begin{bmatrix}-4\\-6\end{bmatrix}$

c. Do the calculations in the table above support your description from part a? Why or why not?

Sample Answers: The calculations in part (b) support the description in part (a). Each input vector $\mathbf{v}$ is reflected across the x-axis.

4. Let $\mathbf{m} = \mathbf{d} = \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$.

a. Describe this transformation in words.

Sample Answers: This linear transformation reflects the input vector across the y-axis. The magnitude of the output vector is the same as the input vector.

b. Complete the following table.

Answer:

$\mathbf{v}$	$\begin{bmatrix}0\\1\end{bmatrix}$	$\begin{bmatrix}3\\3\end{bmatrix}$	$\begin{bmatrix}-6\\-4\end{bmatrix}$	$\begin{bmatrix}0\\-5\end{bmatrix}$	$\begin{bmatrix}2\\5\end{bmatrix}$	$\begin{bmatrix}-3\\-5\end{bmatrix}$
$\mathbf{m}\cdot\mathbf{v}$	$\begin{bmatrix}0\\1\end{bmatrix}$	$\begin{bmatrix}-3\\3\end{bmatrix}$	$\begin{bmatrix}6\\-4\end{bmatrix}$	$\begin{bmatrix}0\\-5\end{bmatrix}$	$\begin{bmatrix}-2\\5\end{bmatrix}$	$\begin{bmatrix}3\\-5\end{bmatrix}$

c. Do the calculations in the table above support your description from part a? Why or why not?

Sample Answers: The calculations in part (b) support the description in part (a). Each input vector $\mathbf{v}$ is reflected across the y-axis.

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5. For $k = 2,$ let $\mathbf{m} = \mathbf{e} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}.$

   a. Describe this transformation in words.

   Sample Answers: This linear transformation scales the input vector by a factor of 2. The direction of the output vector is the same and the magnitude is double the magnitude of the input vector.

   b. Complete the following table.

Note: $|\mathbf{v}|$ is the magnitude, or length, of the vector $\mathbf{v}.$ The magnitude of the vector $\mathbf{v}$ can be found on Page 1.4: $\text{norm}(\mathbf{v}) = |\mathbf{v}|$

Answer:

$\mathbf{v}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \end{bmatrix}$	$\begin{bmatrix} -3 \\ 4 \end{bmatrix}$	$\begin{bmatrix} -5 \\ -12 \end{bmatrix}$	$\begin{bmatrix} 0 \\ -4 \end{bmatrix}$	$\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$
$\|\mathbf{v}\|$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \end{bmatrix}$	$\begin{bmatrix} -3 \\ 4 \end{bmatrix}$	$\begin{bmatrix} -5 \\ -12 \end{bmatrix}$	$\begin{bmatrix} 0 \\ -4 \end{bmatrix}$	$\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$
$\mathbf{m} \cdot \mathbf{v}$	$\sqrt{2}$	3	5	13	4	2
$\|\mathbf{m} \cdot \mathbf{v}\|$	$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 6 \end{bmatrix}$	$\begin{bmatrix} -6 \\ 8 \end{bmatrix}$	$\begin{bmatrix} -10 \\ -24 \end{bmatrix}$	$\begin{bmatrix} 0 \\ -8 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 2\sqrt{3} \end{bmatrix}$

   c. Do the calculations in the table above support your description from part a? Why or why not?

   Sample Answers: The calculations in part (b) support the description in part (a). Each input vector $\mathbf{v}$ is scaled by a factor of 2 in the same direction.

   d. Describe this transformation for any value of $k > 0.$

   Sample Answers: For any value of $k > 0,$ the linear transformation scales the input vector by k units in the same direction. Note: Ask students to describe this transformation for any value of $k < 0.$

6. Let $\mathbf{m} = \mathbf{h} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

   a. Describe this transformation in words.

   Sample Answers: This linear transformation projects the input vector onto the y-axis. The magnitude of the output vector is the y-coordinate of the input vector.

   b. Complete the following table.

   Answer:

$\mathbf{v}$	$\begin{bmatrix} 3 \\ 4 \end{bmatrix}$	$\begin{bmatrix} -2 \\ 7 \end{bmatrix}$	$\begin{bmatrix} -3 \\ -4 \end{bmatrix}$	$\begin{bmatrix} 0 \\ -5 \end{bmatrix}$	$\begin{bmatrix} 2 \\ -5 \end{bmatrix}$	$\begin{bmatrix} \sqrt{11} \\ 12 \end{bmatrix}$
$\mathbf{m} \cdot \mathbf{v}$	$\begin{bmatrix} 0 \\ 4 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 7 \end{bmatrix}$	$\begin{bmatrix} 0 \\ -4 \end{bmatrix}$	$\begin{bmatrix} 0 \\ -5 \end{bmatrix}$	$\begin{bmatrix} 0 \\ -5 \end{bmatrix}$	$\begin{bmatrix} 0 \\ -12 \end{bmatrix}$

   c. Do the calculations in the table above support your description from part a? Why or why not?

   Sample Answers: The calculations in part (b) support the description in part (a). Each input vector $\mathbf{v}$ is projected onto the y-axis.

Extensions

Here are some possible extensions to this activity:

1. Ask students to find a transformation matrix that projects an input vector onto the x-axis.
2. Consider the transformation matrices $\mathbf{f} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ and $\mathbf{g} = \begin{bmatrix} k & 0 \\ 0 & \frac{1}{k} \end{bmatrix}$ for various values of k. Ask students to describe each transformation in words and to check their conjectures with specific calculations.
3. Ask students to validate the following formulas by using specific values for the constants, vectors, and matrices:
   (a) $\mathbf{m} \cdot (k\mathbf{v}) = k(\mathbf{m} \cdot \mathbf{v})$
   (b) $\mathbf{m}(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{m} \cdot \mathbf{v}_1 + \mathbf{m} \cdot \mathbf{v}_2$
   (c) $\mathbf{m} \cdot \mathbf{0} = \mathbf{0}$

   Ask students to show that these properties are true in general for any linear transformation.

4. Ask students to generalize the kind of matrix that results in a rotation, a transformed vector in the same direction, or a transformed vector that is parallel to the x-axis.
5. Ask students how to use matrices to represent the composition of linear transformations. Some students might be able to construct a new calculator page that represents composition.

Wrap Up

Upon completion of the lesson, the teacher should ensure that students are able to understand:

- How to visualize linear transformations from $\mathbf{R}^2$ to $\mathbf{R}^2$ represented by matrices.
- How to explain the properties of certain linear transformations.

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