Exploring Cavalieri's Principle

About the Lesson

Students will:

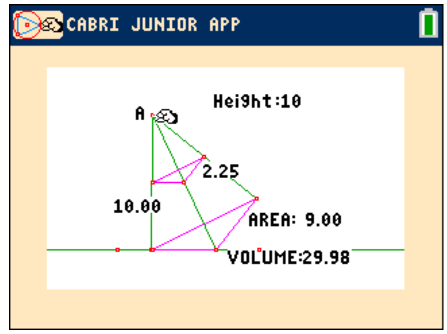
- Explore cross-sectional area and volume to discover Cavalieri's Principle.
- Propose and critique an argument using Cavalieri's principle for the formulas for the volumes of solid figures.
- Use the volume formulas for cylinders and pyramids to solve problems.

Vocabulary

- Cavalieri's Principle
- volume

Teacher Preparation and Notes

- This activity was written to be explored with the Cabri Jr. app on the TI-84 Plus family graphing calculators.
- Before beginning this activity, make sure that all students have the Cabri Jr. application, and the Cabri Jr. files Cyl1.8xv and Cav1.8xv loaded on their calculators.



Tech Tips:

- This activity includes screen captures taken from the TI-84 Plus C Silver Edition. It is also appropriate for use with the TI-84 Plus family with the latest TI-84 Plus operating system (2.55MP) featuring MathPrint™ functionality. Slight variations to these directions given within may be required if using other calculator models.
- Access free tutorials at http://education.ti.com/calculators/pd/US/Online-Learning/Tutorials
- Any required calculator files can be distributed to students via handheld-to-handheld transfer.

Compatible Devices:

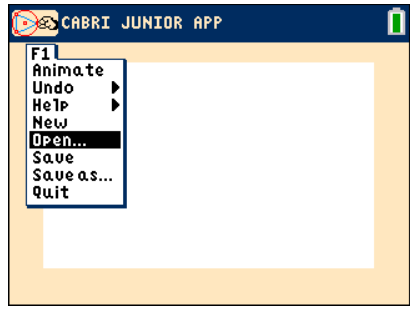
- TI-84 Plus Family
- TI-84 Plus C Silver Edition

Associated Materials:

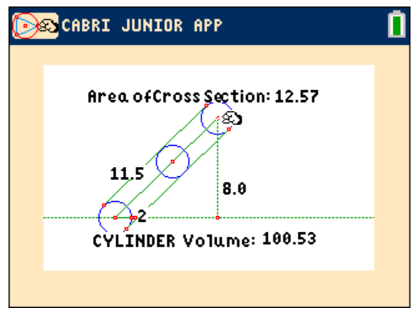
- CavalierisPrinciple_Student.pdf
- CavalierisPrinciple_Student.doc
- CYL1.8xv
- CAV1.8xv

Tech Tip: Before beginning the activity, the files CYL1.8xv and CAV1.8xv need to be transferred to the students' calculators via handheld-to-handheld transfer or transferred from the computer to the calculator via TI-Connect.

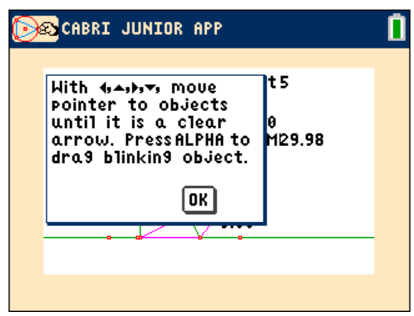
Problem 1 – Oblique Cylinder



Students will start the *Cabri Jr.* application by pressing [APPS] and selecting **CabriJr**. Have students open the file *CYL1* by pressing [Y=] for [F1], selecting **Open…**, and selecting the file.



Students will begin this activity by investigating the volume of a cylinder that is not necessarily vertical. The radius of the base and the height of the cylinder are fixed, but the angle of the cylinder changes when students move point *A*. Students will discover that the volume does not depend on the angle of the cylinder if the cross sectional area remains fixed.



In *CYL1*, students are given a cylinder with a certain radius and height. Have students move point *A* and answer the questions on their student activity sheets. To grab a point, have students press [ALPHA]. They can press it again to let go of the point, or press [CLEAR].

1. What do you notice about the dimensions of the cylinder when you move point *A*? Record the measurements of the dimensions that do not change when *A* is moved.

   Answer: The radius is 2 units. The cross-sectional area is 12.57 units squared. On the TI-84C the height is 8.0 and the volume is 100.53 cubic units. On the TI-84 the height is 4 the volume is 50.26 units cubed.

2. Using the formulas for the area of a circle and volume of a cylinder, show your work to explain how you could calculate the area and volume of the oblique cylinder.

   Answer: The area of a circle is $A = \pi r^2$, and the radius is 2 units. This means the area of the cross-section of the cylinder is $4\pi \approx 12.57$ units^2. The area of a three-dimensional object is the area of the base multiplied by the height. So, the volume of a cylinder is $V = \pi r^2 h$. If the height is 8.0, $V =$ $\pi(2)^2 8 \approx 100.53$ units^3.

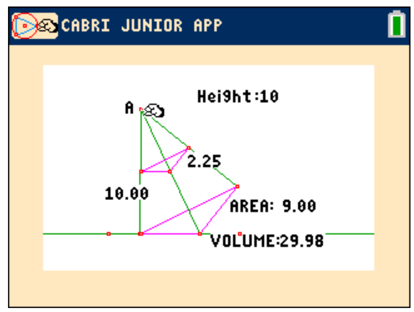
Teacher Tip: Students should be careful to use precision in their vocabulary and attend to precision with the number of digits that they keep. Have students determine whether their calculated answer agrees with the value given on their geometric figure on their device. For example, if students use the value 3.14 for pi, they will calculate a volume of 100.48 cubic units as opposed to 100.53. Students should recognize that the difference is due to how the values are rounded.

3. How does the angle of the slant of the cylinder affect the volume of the cylinder?

   Answer: No matter the angle of the cylinder, the volume remains the same.

Problem 2 – Triangular Pyramid

Students will next explore the volume of a triangular pyramid. When students move point *A*, they will discover that the area of the parallel cross-sections remain fixed. This means that the volume of the pyramid does not depend on where *A* is as long as the height of the pyramid remains fixed at 5 units.



Have students open the *Cabri Jr.* file *CAV1*. In *CAV1*, students are given a triangular pyramid and two parallel cross-sections. Have students move point *A* and answer the questions on their student activity sheets.

4. What do you notice about the cross-sectional areas of the triangles when you move point *A*?

   Answer: The cross-sectional triangle areas remain the same.

5. What do you notice about the volume when you move point *A*?

   Answer: The volume remains the same.

6. Your observations should be leading you to understand Cavalieri's Principle. This principle relates the height, cross-sectional area, and volume of two figures (such as two cylinders). Make a conjecture as to what Cavalieri's Principle is.

   Answer: Teachers will want to ensure that students discover Cavalieri's Principle. Cavalieri's Principle states that if two figures have the same height and the same cross-sectional area at every level, then they have the same volume. Teachers should show students a variety of cross sections with the same area, but all have the same height to demonstrate Cavalieri's Principle to students.

Teacher Tip: Have students critique each other's conjectures. Or, present a hypothetical conjecture to students and have them discuss and critique the conjecture. For example, ask students, "What would you say of the following conjecture: Cavalieri's Principle applies to right triangular pyramids and states that, no matter what the slant angle of pyramid is, the volume stays the same?" There are a few things incorrect about this statement. The condition that must be satisfied for Cavalieri's Principle to hold true is that the cross-sectional area remains the same. It also applies to all kinds of three-dimensional volume figures, not just right triangular pyramids. Encourage students to research or look up Cavalieri's Principle to confirm their conjecture. Have them write a paragraph comparing their conjecture with Cavalieri's.

7. A construction worker is setting up cylindrical barrels along the interstate. The diameter of each barrel is 3.0 feet and the height is 4.0 feet. What volume of sand would be needed to fill a single barrel? Show your formula and work.

Answer: $V = pr^2h = p(3.0)^2 4.0 \text{»} 113\ \text{ft}^3$

8. If the barrel from Question 7 is hit by a car and leaves the barrel slanted at a 45º angle (without deforming the barrel at all), how will the height of the sand in the barrel compare?

Answer: The height of the sand remains the same.

9. The Great Pyramid of Giza has a height of 146.5 meters. One side of the square base measures 230.4 meters. What is the volume of this pyramid?

Answer: $V = \frac{1}{3}Ah = \frac{1}{3}(230.4)^2 146.5 \approx$ 2,592,000 meters^3

10. If the vertical height of that pyramid were doubled, how would volume compare?

Answer: If the height is increased, the volume will increase proportionally. The volume would double.