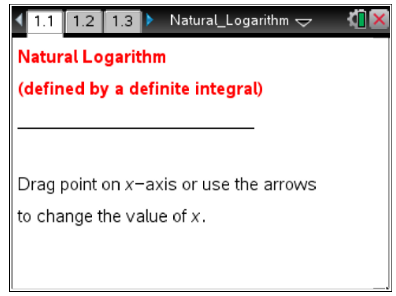


Math Objectives

- Students will understand the definition of the natural logarithm function in terms of a definite integral.
- Students will be able to use this definition to relate the value of the natural logarithm function to the area under a curve.
- Students will be able to use the Fundamental Theorem of Calculus to compute the derivative of the natural logarithm function.
- Students will be able to use the derivative $\left(\frac{1}{x}\right)$ of the natural logarithm to determine properties of its graph.
- Students will look for and make use of structure. (CCSS Mathematical Practice)
- Students will reason abstractly and quantitatively. (CCSS Mathematical Practice)

Vocabulary

- natural logarithm function
- area under a curve
- increases without bound
- derivative
- Fundamental Theorem of Calculus

About the Lesson

- This lesson involves constructing the graph of the natural logarithm function from its definition.
- As a result, students will:
- Conjecture about the value of the natural logarithm function as x increases without bound.
- Conjecture about the value of the natural logarithm function as x approaches 0 from the right.
- Use the derivative to discover properties of the graph of the natural logarithm function.

TI-Nspire™ Navigator™ System

- Send out the *Natural_Logarithm.tns* file.
- Monitor student progress using Class Capture.
- Use Live Presenter to spotlight student answers.

Activity Materials

- Compatible TI Technologies: TI-Nspire™ CX Handhelds, TI-Nspire™ Apps for iPad®, TI-Nspire™ Software

Tech Tips:

- This activity includes screen captures taken from the TI-Nspire CX handheld. It is also appropriate for use with the TI-Nspire family of products including TI-Nspire software and TI-Nspire App. Slight variations to these directions may be required if using other technologies besides the handheld.
- Watch for additional Tech Tips throughout the activity for the specific technology you are using.
- Access free tutorials at http://education.ti.com/calculators/pd/US/Online-Learning/Tutorials

Lesson Materials:

Student Activity

- Natural_Logarithm_Student.pdf
- Natural_Logarithm_Student.doc

TI-Nspire document

- Natural_Logarithm.tns

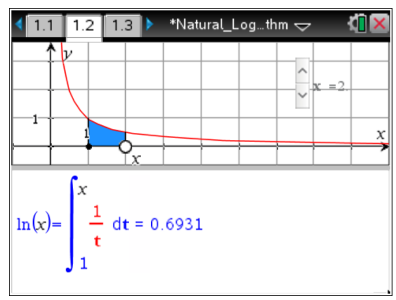
Discussion Points and Possible Answers

Tech Tip: If students experience difficulty dragging a point, make sure they have not selected more than one point. Press [esc] to release points. Check to make sure that they have moved the cursor (arrow) until it becomes a hand (👆) getting ready to grab the point. Also, be sure that the word point appears. Then select [ctrl] [grab] to grab the point and close the hand (✊). When finished moving the point, select [esc] to release the point

Tech Tip: To change the value of *x*, students can touch their finger to the point and then drag it along the *x*-axis.

TI-Nspire Navigator Opportunity: *Class Capture* and/or *Live Presenter*
See Note 1 at the end of this lesson.

Move to page 1.2.



1. As students grab and drag point *x* to the right along the horizontal axis, the computed area of the shaded region is equivalent to ln*x*, the value of the natural logarithm function. Students may also use the up/down arrows in the top-right portion of the page to change the value of *x*.

   a. Complete the following table.

   <u>Answer:</u>

x	1	1.5	2	4	6	8
ln*x*	0	0.4055	0.6931	1.3863	1.7918	2.0794

   b. Explain what happens to the value of ln*x* as *x* increases.

   <u>Answer:</u> As *x* increases, the value of ln*x* also increases (at a slower rate).

   c. Explain your answer in part b geometrically.

   <u>Answer:</u> As *x* increases, the area under the graph of $y = \frac{1}{x}$, above the *x*-axis, and between the vertical lines at 1 and *x* is increasing.

2. Drag point *x* to the left of 1 (but greater than 0), or use the up/down arrows to change the value of *x*.
   a. Complete the following table.

Answer:

x	1	0.9	0.7	0.5	0.2	0.1	0.05
$\ln x$	0	–0.10534	–0.3567	–0.6931	–1.6094	–2.3026	–2.9957

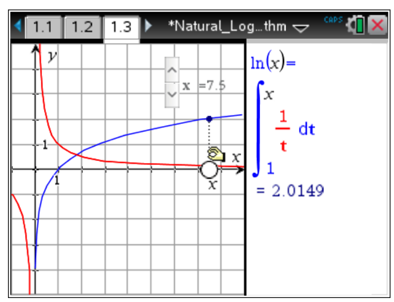
   b. Explain what happens to the value of ln*x* as *x* decreases (gets closer to 0).

Answer: As the value of *x* gets closer to 0, the value of ln*x* decreases.

   c. Explain your answer in part b geometrically.

Answer: As *x* gets closer to 0, the area under the graph of $y = \frac{1}{x}$, above the *x*-axis, and between the vertical lines at 1 and *x* is increasing. However, since the lower bound on the integral is 1, and $x < 1$, the value of the definite integral is negative. Use the following property to illustrate this: $\int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt$.

Move to page 1.3.



3. A part of the graph of $y = \ln x$ is displayed. Have students grab point *x* or use the up/down arrows to change the value and move it along the horizontal axis to the right to construct the remaining part of the graph of $y = \ln x$. The values of the natural logarithm function are displayed on the right screen.
   a. Explain what happens to the graph of $y = \ln x$ as *x* increases without bound (as $x \to \infty$).

Answer: As *x* increases without bound, the graph of $y = \ln x$ also increases, at a slower rate. Some students may say that the graph appears to be leveling off.

   b. Explain what happens to the graph of $y = \ln x$ as *x* approaches 0 from the right (as $x \to 0^+$).

Answer: As $x \to 0^+$, the graph of $y = \ln x$ decreases quickly. It appears to be approaching $-\infty$.

c. Explain why $x = 0$ is not in the domain of the function $y = \ln x$.

Answer: The natural logarithm is defined in terms of a definite integral. In order to evaluate the definite integral $\int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt$, the integrand, $\frac{1}{t}$, must be defined on the closed interval $[x, 1]$. The function $\frac{1}{t}$ is not defined for $t = 0$. Therefore, we cannot evaluate the definite integral, and the natural logarithm function is not defined for $x = 0$.

d. The function $\mathbf{f}(x) = \frac{1}{x}$ is defined for $x < 0$. For example, $\mathbf{f}(-2) = -\frac{1}{2}$. Explain why the definition of the natural logarithm function cannot be extended to include negative numbers.

Answer: If the function **f** is continuous, then the definite integral $\int_a^b f(x)dx$ exists. If the function f has only a finite number of jump discontinuities, that is, f is piecewise continuous, then the definite integral also exists. However, consider the integral $\int_{-2}^1 \frac{1}{t}\,dt$. The integrand, $\frac{1}{t}$, has an infinite discontinuity at 0. As $t \to 0^+$, $\frac{1}{t} \to +\infty$. And as $t \to 0^-$, $\frac{1}{t} \to -\infty$. For any value of $x < 0$, the interval $[x, 1]$ will include 0. Therefore, this definition of the natural logarithm function cannot be extended to negative numbers.

e. Use the Fundamental Theorem of Calculus to find the derivative of $\mathbf{f}(x) = \ln x$. Determine the intervals on which the graph of $y = \mathbf{f}(x)$ is increasing and the intervals on which it is decreasing. Find the absolute extreme values for **f.** Determine the intervals on which the graph of $y = \ln x$ is concave up and the intervals on which it is concave down.

Answer: Using the Fundamental Theorem of Calculus, $\mathbf{f}'(x) = \frac{d}{dx}\left[\int_1^x \frac{1}{t}\,dt\right] = \frac{1}{x}$. Since

$\mathbf{f}'(x) = \frac{1}{t} > 0$ for $x > 0$, the graph of the natural logarithm function is always increasing.

Since the graph of $y = \mathbf{f}(x)$ is always increasing, there is no absolute maximum value.
Since the domain is $x > 0$, there is no absolute minimum value.

$$\mathbf{f}''(x) = \frac{d}{dx}\left[\frac{1}{x}\right] = -\frac{1}{x^2}$$

Since $\mathbf{f}''(x) < 0$ for $x > 0$, the graph of the natural logarithm function is always concave down.

Wrap Up

Upon completion of the discussion, the teacher should ensure that students understand:

- The relationship between the area under the curve and the definition of the natural logarithm function.
- The properties of the graph of the natural logarithm function.

At the end of this activity, students discover that the graph of the natural logarithm function is always increasing and that there is no absolute maximum or minimum value. However, we do not know whether there is an upper bound or lower bound for the function. Some related topics for discussion include the area of an unbounded region and comparing the area under the graph of $y = \frac{1}{x}$, above the *x*-axis, greater than $x = 1$, with the harmonic series.

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Note 1

Question 1, *Class Capture* and/or *Live Presenter*

As students begin this activity, use Class Capture to ensure that each student is able to grab and drag the appropriate point.