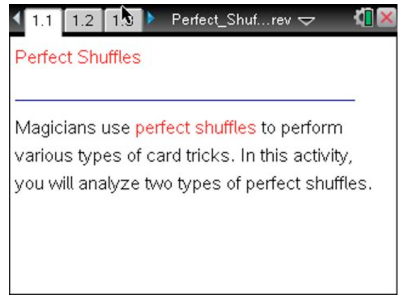
Perfect Shuffles
Student Activity

Name _______________
Class _______________

Open the TI-Nspire document *Perfect_Shuffles.tns.*

Magicians use perfect shuffles to perform various types of card tricks. In this activity, you will analyze two types of perfect shuffles.



In a **perfect shuffle** of a deck of cards with an even number of cards, the magician splits the deck into an upper half and a lower half and then interlaces the cards alternately, one at a time from each half of the deck. For example, for a deck with 4 cards:

1 2 3 4

the split gives half decks 1 2 and 3 4. A perfect shuffle in which a card is taken first from the top half and then from the bottom half yields a new deck with cards in the order:

1 3 2 4

Repeating this process, the deck is split into 1 3 and 2 4 and the resulting shuffle yields a deck with cards in the order:

1 2 3 4

Two shuffles return the cards to their original order. This sequence in which a card is taken first from the **top** half is called an **out-shuffle** since the top and bottom cards remain on the outside of the deck.

Consider a deck with eight cards:

1 2 3 4 5 6 7 8.

1. Complete the table below with the order of the eight cards from top to bottom after one and two perfect out-shuffles.

original order [top-to-bottom]	1	2	3	4	5	6	7	8
order after one shuffle	1							8
order after two shuffles	1							8

Name ______________
Class ______________

Move to page 1.2.

Press ctrl ▶ and ctrl ◀ to navigate through the lesson.

This out-shuffle can be defined by a piece-wise linear function of the form $f1(x) = \begin{cases} a*x-b & 1 \le x \le 4 \\ c*x-d & 5 \le x \le 8 \end{cases}$ where each "piece" of the function corresponds to half of the deck. If x represents the position of a card before the shuffle, then $f1(x)$ represents its position after the shuffle.

2. The values *n* = 4 and a = 2 have been entered under the definition of $f1(x)$. Find and enter the values of *b*, *c*, and *d* for this out-shuffle.

Hint: To find *a* and *b*, use the facts that 1 → 1 and 2 → 3 so that *a*·1 – *b* = 1 and *a*·2 – *b* = 3.

Move to page 1.3.

The position of each card after two shuffles is given by the composite function $f2(x) = f1(f1(x))$; if *x* represents the position of a card before the first shuffle, then $f2(x)$ represents its position after the second shuffle. The position of each card after three shuffles is found using $f3(x) = f1(f1(f1(x)))$ = $f1(f2(x))$, and so on. Twelve shuffles have been defined on this page.

3. Express $f4(x)$ as the composition of two or more functions in two different ways.

Move to page 1.4.

The first two columns of the table should be identical to the first two rows of the table you completed in Question 1.

4. a. To perform the next shuffle, click on the black arrow at the top of the second column, and choose $f2$ from the menu of functions displayed. Verify that the entries in the second column are identical to those in the third row of your table from Question 1.

b. Move to the top of the third column. Click on the black arrow at the top of this column, and choose $f3$ from the menu of functions displayed. Examine these entries to check whether the cards have returned to their original order. If not, repeat this process, and choose $f4, f5, \ldots$ etc. until the entries in the column with the values of $fk(x)$ are the same as those in the first column giving the cards in their original order. (You might need to use the right arrow at the bottom of the page to display the next column.)
How many out-shuffles were needed; i.e. what is the value of *k*?

Move to page 1.5.

5. Make a conjecture about the smallest number of out-shuffles needed to return a deck of 2*n* cards to its original order when
   a. n = 8 [16 cards]:

   b. n = 16 [32 cards]:

and type your conjecture in the box on Page 1.5.

Hint: In a deck with 2 cards, an out-shuffle does not change the order of the cards so 1 out-shuffle returns the cards to their original order. We have seen that for decks of 4 and 8 cards, 2 and 3 out-shuffles, respectively, are needed to return them to their original order.

Move back to page 1.2.

c. Suppose your conjecture for the number of out-shuffles needed to return the cards in a deck with 16 cards [n = 8] to their original positions was *k*. To check your conjecture. first, redefine $f1(x)$ for a deck with16 cards - enter *n* = 8 and any new values for *a*, *b*, *c*, and/or *d*.

Move to page 1.4.

Then return to the second column, and choose $f1$. In subsequent columns, successively choose $f2, f3, \ldots fk$ until finding the value of *k* showing that the cards have returned to their original order (for the first time). Was your conjecture correct?

Return to thinking about a four-card deck 1, 2, 3, 4. If the first card is taken from the **bottom** half, the following sequence is obtained:

1 2 3 4
3 1 4 2
4 3 2 1
2 4 1 3
1 2 3 4

Four shuffles are required to return the deck to its original order. This shuffle is an **in-shuffle**.

Again, consider a deck with eight cards:

1, 2, 3, 4, 5, 6, 7, 8

6. Follow the 'movement' of each of the two cards 3 and 4 through the deck after each in-shuffle. Stop when **both** cards have returned to their original positions. Complete the table showing the position of each card after 1, 2, 3, … shuffles and describe your reasoning.

number of shuffle]		1	2	3	4	5	6
position of 3	3						
position of 4	4						

Move back to page 1.2.

7. Let *k* be the number of in-shuffles needed to return the cards in this deck to their original positions To determine the value of *k*:
   Redefine f $f1(x)$ for an in-shuffle on a deck with 8 cards - enter *n* = 4 and then any new values for *a*, *b*, *c*, and/or *d*.

Move to page 1.4.

Then return to the second column, and choose f1. In subsequent columns, successively choose $f2, f3, ..., fk$ until you find the value of k showing that the cards have returned to their original order (for the first time). What is the value of *k*?

Move to page 1.5.

8. Make a conjecture about the smallest number of in-shuffles needed to return a deck of 2*n* cards to its original order when
   a. n = 8 [16 cards]

   b. n = 16 [32 cards].

And type your conjecture in the box on Page 1.5.

Hint: For decks of cards with 2, 4, 8, 16, .. cards, how does the number of out-shuffles compare to the number of in-shuffles?

Move back to page 1.2.

9. It can be shown that it takes 52 in-shuffles to return the cards in a standard deck of 52 cards to their original order. Determine your answer by using the same process of redefining $f1$ for an out-shuffle of a deck with 52 cards on page 1.2 and then finding the number out-shuffles needed on page 1.4. How many out-shuffles are needed?

Name ________________
Class ________________

This page intentionally left blank.