



### Math Objectives

- Students will conduct simulations with multiple trials.
- Students will use relative frequencies to compute experimental probabilities.

### Vocabulary

- Law of Large Numbers
- frequency
- relative frequency
- experimental probability
- theoretical probability
- scatter plot
- connected scatter plot
- trial
- simulation

### About the Lesson

- This lesson involves investigating simulations used to observe long-run relative frequencies.
- As a result, students will:
  - Simulate drawing blue and red marbles one at a time from a bag.
  - Generate a scatter plot of the relative frequencies of blue marbles.
  - Examine variations in the scatter plots generated by the simulations.
  - Determine and discuss any consistent patterns or trends found in the results of these simulations.

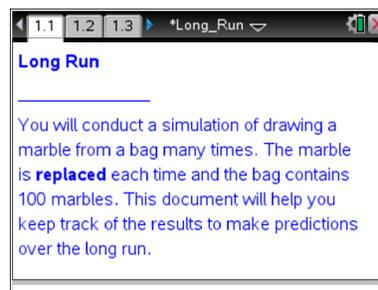


### TI-Nspire™ Navigator™ System

- Send out the *Long\_Run.tns* file.
- Monitor student progress using Class Capture.
- Use Live Presenter to spotlight student answers.

### Activity Materials

Compatible TI Technologies:  TI-Nspire™ CX Handhelds,  
 TI-Nspire™ Apps for iPad®,  TI-Nspire™ Software



### Tech Tips:

- This activity includes screen captures taken from the TI-Nspire CX handheld. It is also appropriate for use with the TI-Nspire family of products including TI-Nspire software and TI-Nspire App. Slight variations to these directions may be required if using other technologies besides the handheld.
- Watch for additional Tech Tips throughout the activity for the specific technology you are using.
- Access free tutorials at <http://education.ti.com/calculators/pd/US/Online-Learning/Tutorials>

### Lesson Files:

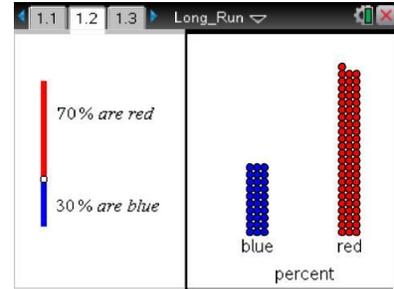
- Student Activity*
- Long\_Run\_Student.pdf
  - Long\_Run\_Student.doc
- TI-Nspire document*
- Long\_Run.tns
  - Long\_Run\_Assessment.tns.



### Discussion Points and Possible Answers

Move to page 1.2.

1. On the left-hand side of this page, grab and move the open point on the slider until the percent of blue marbles in the bag is 30%.



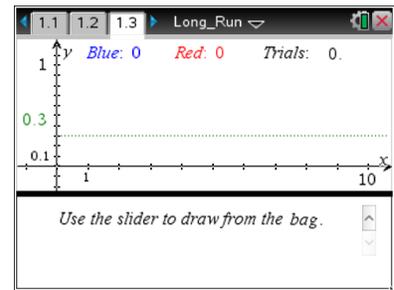
2. The theoretical probability of drawing a blue marble from the bag is the ratio of the number of blue marbles to the total number of marbles. What is the theoretical probability of drawing a blue marble? Explain your reasoning.

**Answer:** The probability is 30% or 0.3 since we assume that any one of the 100 marbles could be drawn with equal likelihood. Since there are 30 blue marbles, the theoretical probability is 30 out of 100.

Move to page 1.3.

On this page, simulate the draw of a marble with replacement.

- Each time you select an “up” arrow, one marble is removed from the bag, the result is recorded at the bottom of Page 1.3, and the marble is placed back into the bag.
- The relative frequency of blue draws is graphed at the top of Page 1.3.
- Notice the green horizontal line—it shows the constant percent of blue marbles out of the 100 marbles in the bag. In this situation, the constant percent is 30%, which is 0.3 when written as a decimal.



3. Select the up arrow five times to draw one marble out of the bag five times.
  - a. What is the sequence of your first five draws? Use **R** for red and **B** for Blue.  
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**Sample Answer:** The sequence shown on Page 1.3 below would be: B, B, R, R, R.

- b. How many blue marbles did you draw in 5 trials? (This is the **frequency** of blue marbles drawn in 5 trials.)

**Sample Answer:** The value can be any integer from 0 to 5, but the results shown in Screen 1.3 below show 2 blue marbles were drawn from the bag.

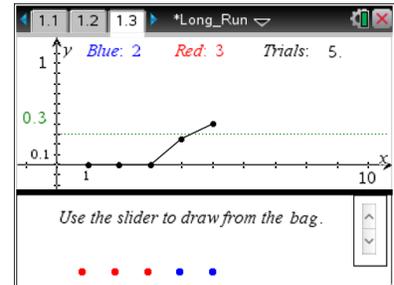


- c. What is the percent of blue marbles in your simulation after 5 trials? (This is the **relative frequency**.)

**Sample Answer:** Using the result above, the percentage would be 40%.

- d. Compare this percent with the total percent of blue marbles in the bag.

**Sample Answer:** The sequence percentage is 10% above the percentage of blue marbles in the bag.



**Teacher Tip:** Students will get different results in this simulation, so be ready to lead the discussion with your students. They need to know that their experimental results will vary from one student to another even if the theoretical probability is the same for all students.



**TI-Nspire Navigator Opportunity: Class Capture**

See Note 1 at the end of this lesson.

4. For the first five marble draws, record the relative frequency of drawing a blue marble after each draw.

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**Sample Answer:** Given the sequence B, B, R, R, R, the relative frequency sequence would be 1 out of 1, 2 out of 2, 2 out of 3, 2 out of 4, and 2 out of 5.



**Tech Tip:** The points in the scatter plot can be traced revealing the relative frequency after each trial as the  $y$ -coordinate. Select an open space in the upper work area. Then select **MENU > Trace > Graph Trace**. Use the left and right arrow to trace from point to point. Use the up and down arrows to switch between the scatter plot and the green line representing the theoretical probability.



**Tech Tip:** Select  **> Trace > Graph Trace** to trace points along the plot. Use the left and right arrows that appear in the menu bar to trace from point to point. Longer taps may be necessary.



- 5. Use the sequence of marbles you have drawn in the first five trials to explain the scatter plot.

**Sample Answer:** The connected scatter plot in the upper portion of the screen shows the relative frequency plotted for each drawing of a blue marble out of the bag. The first percentage graphed is 0% since 1 out of 1 marbles were red. The second percentage is 0% since 2 out of 2 of the first two marbles were red. The third percentage is also 0% because 3 out of 3 marbles drawn were red. The fourth percentage is 25% because 1 blue marble was drawn in the first four trials. The fifth percentage is 40% since there were 2 blue marbles drawn out of the first five.

- 6. Make a prediction of the outcome for the next 5 trials. Does the outcome depend on the results of the first 5 trials? Explain your reasoning.

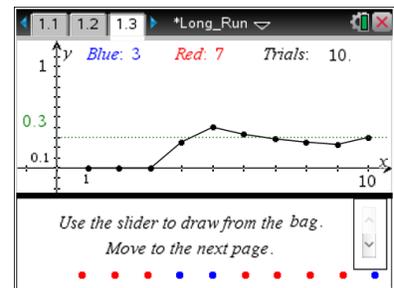
**Sample Answer:** The answers will vary for the predictions. The outcome of the second set of five trials does not depend on the first five. These outcomes are independent of each other.

**Teacher Tip:** If the outcome in the first five trials were 4 blue marbles, a student will often mistakenly believe that in the next five trials, it is more likely that few, if any, blue marbles will be drawn so that the results after 10 trials will be close to 30%. Emphasize that the next 5 draws are totally independent from the first five draws. Marbles have no memory!

- 7. Select the up arrow five more times.
  - a. What is the red-blue sequence of your second five draws?

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**Sample Answer:** The sequence shown in the screenshot to the right would be: R, R, R, R, B.



- b. How does the scatter plot reveal this sequence?

**Sample Answer:** If the plot goes down, a red marble was drawn. If the plot goes up, a blue marble was drawn. The plot moves down starting at the third draw and continues to move down until the last two draws.

- c. Compare the frequency of blue marbles in the first five trials with the frequency in the second five trials.

**Sample Answer:** Answers will vary.



- d. Compare the shape of the scatter plot for the first five trials with the shape of the plot for the second five trials. Explain any differences or similarities.

**Sample Answer:** Answers will vary. The scatter plot shown above in Question 7a shows an increasing relative frequency after the first three marbles are drawn. The second set of five draws shown in the scatter plot reveals a decreasing relative frequency for the first four marbles then an increase for the last marble.

- e. What is the frequency of blue marbles drawn in all 10 trials? Compare this frequency with the frequency of blue marbles in the first 5 trials (Question 3c).

**Sample Answer:** The screenshot in Question 7a shows that 3 blue marbles were drawn.

- f. What is the relative frequency (percent) of blue marbles in your simulation (sequence) after 10 trials? Compare this relative frequency with the relative frequency of blue marbles after the first 5 trials (Question 3b).

**Sample Answer :** 30%. Answers will vary. For the data displayed, the relative frequencies decreased from 40% to 30%.

- g. Compare this percent with the percent of blue marbles in the bag.

**Sample Answer:** Answers will vary. This experimental result is equivalent to the actual percent of blue marbles in the bag.

**Teacher Tip:** Make sure the students compare their sequences with others. Begin the discussion about the varied outcomes of several students. Make sure to discuss the connection of each simulation's scatter plot of relative frequencies. Remind students that each drawing is independent of the previous and subsequent drawing. You can do this by asking questions like: "Does one trial influence the next trial?", "Does the bag of marbles keep track of what has happened?" and "What could influence the likelihood that a blue marble will be drawn next?" It is crucial that your students realize that each draw is independent of the one before and the one that follows. Prior results don't influence the next result. Only the percent of blue marbles in the bag, which is constant at 30% when drawing with replacement, influences each result.



8. Compare your experimental results with several other students. Explain why your answers differ from others.

**Sample Answer:** The other students did a simulation and other students' results differ because each of their trials are random and independent of each other and independent of mine. Their results and my results should not be the same unless by chance.



#### TI-Nspire Navigator Opportunity: *Class Capture*

See Note 1 at the end of this lesson.

**Teacher Tip:** Make sure to discuss the differences in each student's relative frequency and the selected probability of 30%. Not all of the students will have 30%. Find out how many have a relative frequency below, at, or above 30%.



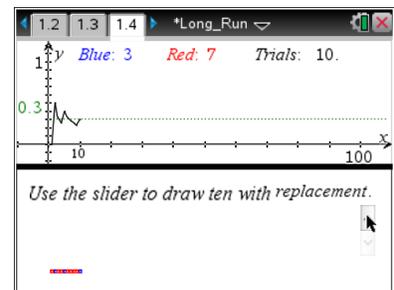
#### TI-Nspire Navigator Opportunity: *Quick Poll*

See Note 2 at the end of this lesson.

**Teacher Tip:** Discuss with the students why each of their results might vary from person to person. Some of the scatter plots will show the relative frequency changing dramatically around the dotted line while others will show little, if any, relationship to the dotted line. Remind students that each drawing is independent. Ask some of them to share their ideas about independence. Some of the scatter plots will show steady progress toward the dotted line.

### Move to page 1.4.

In this part of the activity, each time you select an "up" arrow, you complete 10 trials of removing one marble from the bag and replacing it before the next draw. The previous 10 trials you completed on Page 1.3 are already shown on this page. Notice the scatter plot is compressed on the x-axis.





9. Use the up arrow (nine) times to complete total of 100 trials.  
 a. Record the results in this chart.

Number of Trials	Frequency of Blue draws	Relative frequency
10		
20		
30		
40		
50		
60		
70		
80		
90		
100		

- b. Describe what happens to the graph as you increase the number of trials.

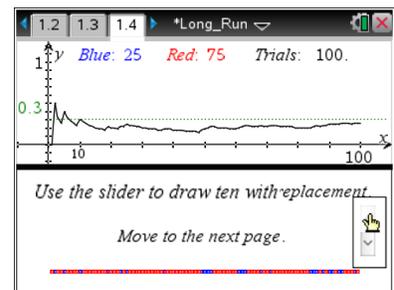
**Sample Answer:** The scatter plot of the relative frequency moves about the 30% line in a random fashion. The relative frequency must remain between 0 and 1.

- c. What is the relative frequency of blue marbles in your simulation after 100 trials?

**Sample Answer:** Answers will vary. For the case shown below, it is 37% or 0.37.

- d. Compare this percent with the percent of blue marbles in the bag.

**Sample Answer:** Answers could vary considerably. For the cases shown, it is 37%—7% above the percent of blue marbles, 30%.



**Tech Tip:** Students can use the Trace feature described earlier to reveal the relative frequency after any given number of trials. A handy trick is to use the up arrow to trace the horizontal theoretical probability line. Enter the digits for the number of trials for which you want to know the relative frequency, then down arrow to the scatter plot.



**TI-Nspire Navigator Opportunity: Class Capture**

See Note 1 at the end of this lesson.



**Tech Tip:** If students select a blank place in the bottom frame and then move their pointer to any one of the blue marbles, all of the blue marbles will become bolder.

- 10. Compare the outcome of the experiment with 10 trials (Question 7) to the outcome of the experiment with 100 trials (Question 9). What did you observe?

**Sample Answer:** The relative frequency seems to be getting closer to 30% after 100 trials than it was after 10 trials. It fluctuated more when there were fewer trials, and it varies less with more trials.

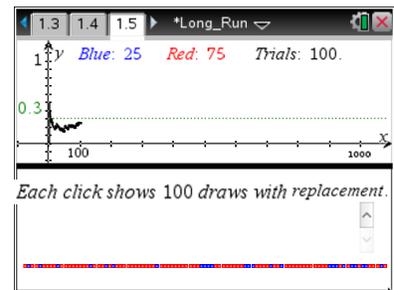
- 11. If you were to start this experiment over and draw one marble from the bag 100 times, with replacement, would you get the similar results? Explain your reasoning.

**Sample Answer:** The results should be different each time the simulation is conducted because the trials are independent. However, most of the time the scatter plot of the relative frequencies after each trial will show a similar trend of approaching the theoretical probability of 30%.

**Teacher Tip:** Many of the students' plots show a lot of similarity as many will oscillate close to the 30% line after 100 trials. The value of examining a variety of scatter plots will allow students to see different outcomes that display a steady trend to get closer to the theoretical probability, yet there will be some that do not close in as fast. Point out that 100 trials might not yield a result that is close to the theoretical probability for all of the simulations. One hundred trials really is not a very large number of trials! However, they should notice that more of the scatter plots likely show results that are closer to the theoretical probability after 100 trials than after only 10 trials.

**Move to page 1.5.**

In this part of the activity, each time you select an "up" arrow you complete 100 trials of drawing one marble from the bag and replacing it before drawing another one. The previous 100 trials you completed on Page 1.4 are already shown on this page. Notice the compression of the x-axis on the scatter plot.





12. Use the up arrow nine times to complete total of 1000 trials.  
 a. Record your results in this table.

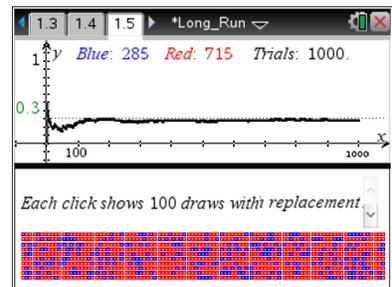
Number of Trials	Frequency of Blue draws	Relative frequency
100		
200		
300		
400		
500		
600		
700		
800		
900		
1000		

- b. Describe what happens to the graph as you increase the number of trials.

**Sample Answer:** The relative frequency changed less dramatically about the 30% line than it did before.

- c. What is relative frequency of blue marbles after 1000 trials in your simulation? Compare this percent with the percent of blue marbles in the bag.

**Sample Answer:** In the screenshot, the value is 30.9%.  
 Answers will vary yet be near 30%.



13. Compare the outcome of your experiment with 100 trials to the outcome of the experiment with 1000 trials. What did you observe?

**Sample Answer:** The graph gets much closer to the dotted line while still oscillating around it. That means the trend is that the relative frequency gets closer and closer to the theoretical probability as the number of trials increases. In this simulation, over a great many trials, the relative frequency should approach 30%.



**TI-Nspire Navigator Opportunity: Class Capture**

**See Note 3 at the end of this lesson.**



**Teacher Tip:** The relative frequency should approach the theoretical probability as more trials are conducted. It is possible, for some students, that fewer draws could provide closer values to the selected theoretical probability, but for most of the students their results will improve in the long-run. Make sure to have students discuss the difference between the probability of a drawing out a single blue marble and the cumulative effect that is shown by relative frequencies. Mention “the law of large numbers” as idea that people have noticed for centuries in a variety of probabilistic settings.

14. Summarize the results of your simulated experiment, and complete the first row in the table below. Compare theoretical probability with experimental relative frequency for each case.

**Sample Answer:** The relative frequency will get closer to the theoretical probability after several hundred trials.

Theoretical Probability	Relative frequency after 10 trials	Relative frequency after 100 trials	Relative frequency after 1000 trials
30%			

15. What is the general trend for the relative frequency of drawing a blue marble from the bag after many trials?

**Sample Answer:** The relative frequency will tend to approach the theoretical probability.

**Move back to page 1.2.**

16. Change the blue marble percentage, and repeat the experiment above with your new percentage for blue marbles. The previous results will be erased. After 10, 100, and 1000 trials, record your results in the second row of the table above.

17. Does your experiment support your earlier conclusion about the general trend for relative frequency after many trials? Explain your reasoning.

**Sample Answer:** Yes, the relative frequency gets closer to the theoretical probability when there have been several hundred trials.



18. Allow other students to examine your scatter plots for 100 and 1000 trials. Examine their scatter plots as well. Do their experiments support your earlier conclusion about the general trend for relative frequency after 100 and 1000 trials?

**Sample Answer:** Yes, their relative frequencies get closer to their newly selected theoretical probabilities.



**TI-Nspire Navigator Opportunity: *Class Capture***

**See Note 3 at the end of this lesson.**

**Teacher Tip:** This opportunity will allow some students to provide explanations for their answers in their own words supported by the results found on each of their TI-Nspire document's Pages 1.3, 1.4, and 1.5. The students might show some results of relative frequencies that come close to or exactly match the theoretical probabilities while others might show the long-run trend where the relative frequency approaches the theoretical probability.

## Wrap Up

Upon completion of the lesson, the teacher should ensure that students are able to understand:

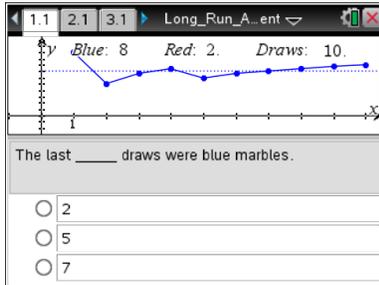
- How to conduct and explain the results of this simulation.
- How to explain the long-run trend of relative frequency when compared to the theoretical probability.
- How to explain the variation seen in other students' results for relative frequencies and for varied selections of theoretical probabilities.



### Assessment

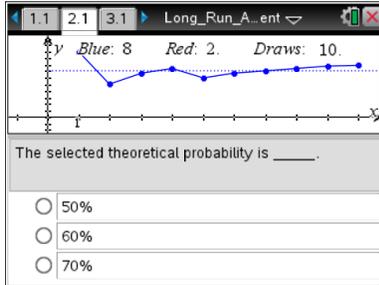
Use the provided TI-Nspire assessment document with five questions to assess students understanding of the concepts of this lesson. Explanations to the answers:

#### Question 1:



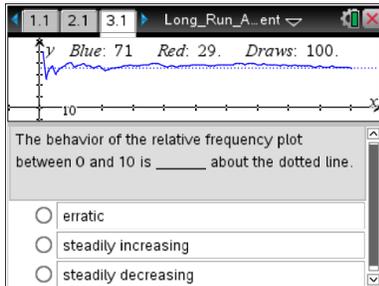
The scatter plot reveals that the relative frequency is constantly increasing after the fifth marble draw. This happens because the last five marbles drawn were blue.

#### Question 2:



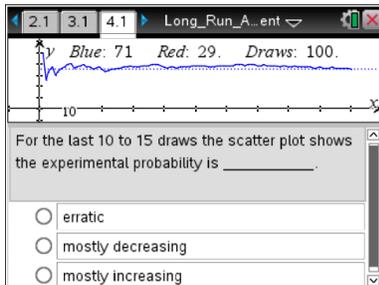
The screen shows the theoretical probability is 70%. Each tic mark on the y-axis is 10%.

#### Question 3:



The screen shows that the early behavior of relative frequency is erratic about the dotted line.

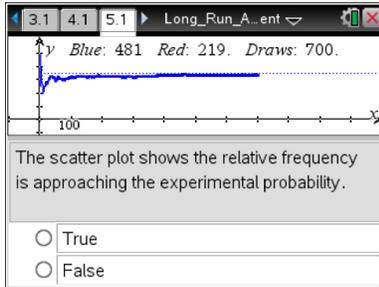
#### Question 4:



The screen shows the relative frequency is mostly decreasing in the last 10 – 15 draws.



### Question 5:



The scatter plot shows the relative frequency approaching the theoretical probability, so the answer is true.



## TI-Nspire Navigator

### Note 1

#### Name of Feature: Class Capture

Use Class Capture to display scatter plots created by the students and let them compare scatter plots with different outcomes. It is important that students recognize there are different outcomes for simulations of random events.

### Note 2

#### Name of Feature: Quick Poll

Use Quick Poll with open response questions for students to share their results after ten and 100 trials. Ask them for their percents for blue marbles drawn after 10 and 100 trials. This will allow them to see variation in results. Ask them how far their experimental results are from their selected theoretical values.

### Note 3

#### Name of Feature: Class Capture

Use Class Capture to provide students the opportunity to show and explain their results and to allow students to compare a variety of outcomes leading them to understand that over the long-run, the relative frequency approaches the selected probability. In some cases, this approach occurs faster for some simulations than for others.