Old MacDonald's Pigpen

About the Lesson

In this activity, students explore – numerically, graphically, algebraically, and verbally – the mathematics involved in maximizing the area of a rectangle with a fixed perimeter. As a result, students will:

- Graph scatter plots, analyze quadratic functions, and determine the maximum value of a parabola.
- Solve design problems that satisfy physical constraints by using geometric and algebraic methods.
- Observe numeric patterns, reason abstractly, and make connections between geometric figures and algebraic expressions.

Vocabulary

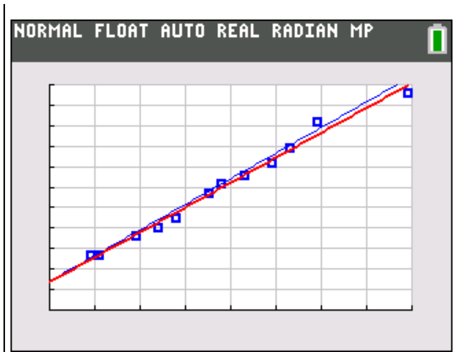
- scatter plot
- domain
- quadratic functions
- maximum
- parabola

Teacher Preparation and Notes

This activity is useful and approachable for most mathematics classes in middle school and high school. This classic maximum value problem provides students with the opportunity to use extra features on the calculator to ensure accuracy and save time.

Younger students can recognize that the area varies as the width varies. They can grasp that it is likely that there is one rectangular pigpen with the greatest area.

Students who have studied quadratic equations can be encouraged to match the mathematical theory to the results found by the calculator.



Tech Tips:

- This activity includes screen captures taken from the TI-84 Plus C Silver Edition. It is also appropriate for use with the TI-84 Plus family with the latest TI-84 Plus operating system (2.55MP) featuring MathPrint™ functionality. Slight variations to these directions given within may be required if using other calculator models.
- Access free tutorials at http://education.ti.com/calculators/pd/US/Online-Learning/Tutorials
- Any required calculator files can be distributed to students via handheld-to-handheld transfer.

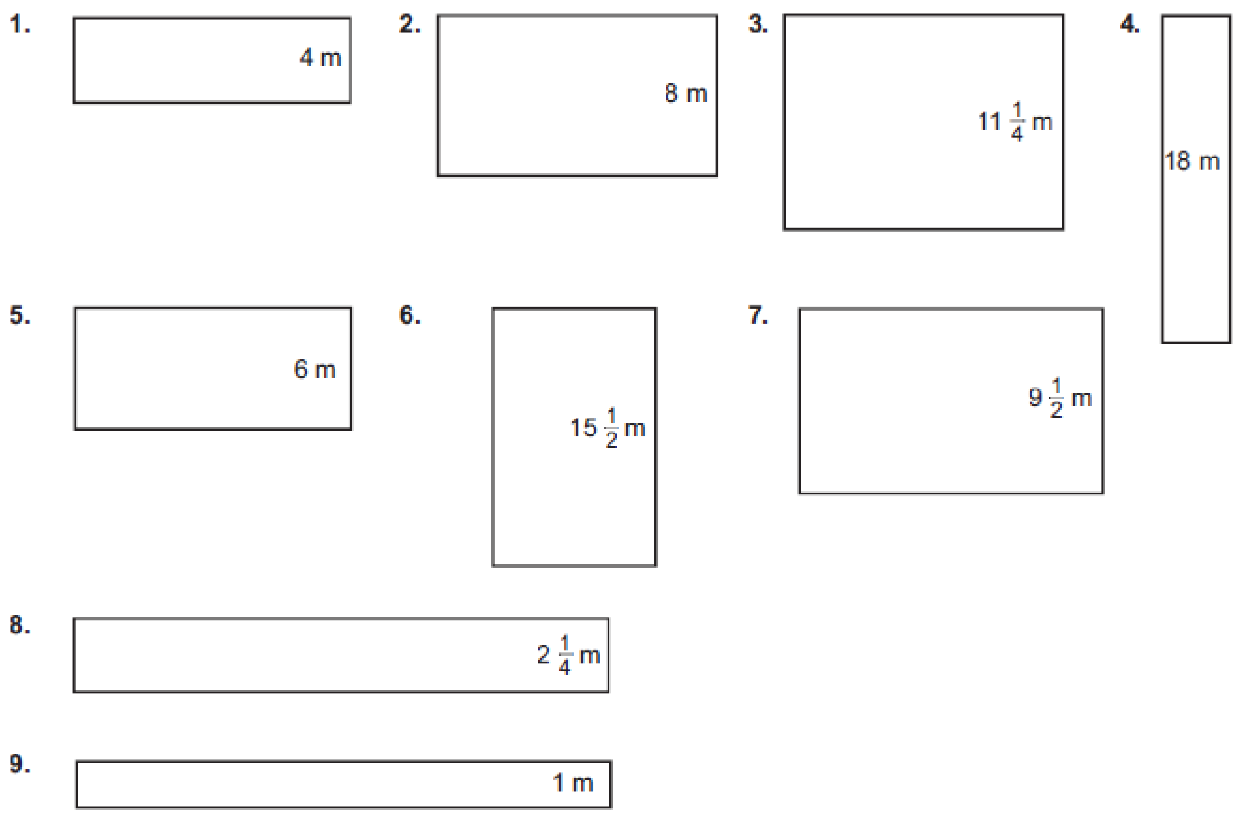
Compatible Devices:

- TI-84 Plus Family
- TI-84 Plus C Silver Edition

Associated Materials:

- OldMacDonald _Student.pdf
- OldMacDonald _Student.doc

Students will consider the following scenario: Old MacDonald has 40 meters of fencing to make a rectangular pen for his pigs. If he wants to give the little pigs as much room as possible, what dimensions should he make the pen? To figure this out, he draw a few sample rectangles with different dimensions. Have students mark the sides of the following rectangles with the lengths that will make the perimeter of each rectangle 40 meters. Students should work with a partner or small group and fill in the table on their student activity sheets. They should make sure the perimeter is always 40 meters. Have students show the explicit calculations used to determine the perimeter and area in their tables. They should show their process.



To help students find a general formala for the length, ask the student, "If 2 of the lengths plus 2 of the widths is forty, what would be the sum of one of the lengths plus one of the widths?" If students can determine that L + W = 20, they should be able to see that L = 20 – W.

Teacher Tip: Have students discuss reasons why a farmer wouldn't always want to maximize the area of a pigpen or a fenced-in zone for an animal to graze. Some reasons could include factors about the terrain of the land, such as a stream or a steeply sloped hill, or other structures on the property that would limit the dimensions of the fence.

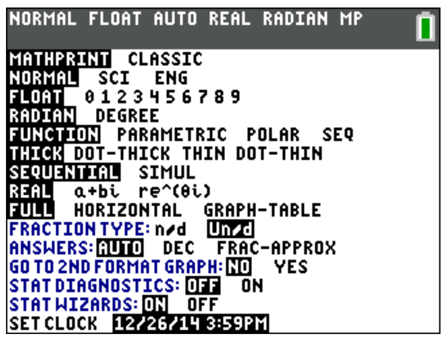
pigpen	width (m)	length (m)	perimeter (m)	area (m^2)
1.	4	16	4+4+16+16 = 40	$4 \cdot 16 = 64$
2.	8	12	8 + 8 + 12 + 12 = 40	$8 \cdot 12 = 96$
3.	$11\frac{1}{4}$	8.75	2(11.25) + 2(8.75) = 40	$11.25 \cdot 8.75 = 98.4375 =$ 98+7/16
4.	18	2	2(18) + 2(2) = 40	$18 \cdot 2 = 36$
5.	6	14	2(6) + 2(14) = 40	$6 \cdot 14 = 84$
6.	$15\frac{1}{2}$	4.5	2(15.5) + 2(4.5) = 40	$15.5 \cdot 4.5 = 69.75$
7.	$9\frac{1}{2}$	10.5	2(9.5) + 2(10.5) = 40	$9.5 \cdot 10.5 = 99.75$
8.	$2\frac{1}{4}$	17.75	2(2.75) + 2(17.75) = 40	$2.25 \cdot 17.75 = 39.9375 =$ 39+15/16
9.	1	19	2(1) + 2(19) = 40	$1 \cdot 19 = 19$
10.	x	20 – x	$2x + 2(20 - x) = 40$	$x \cdot (20 - x)$

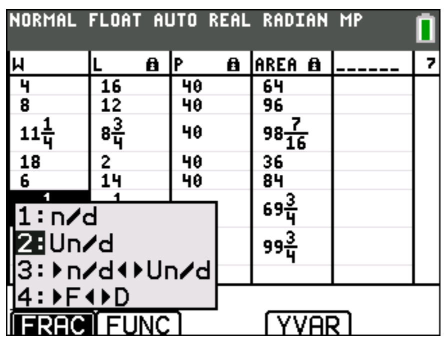
11. The table shows that the area of the rectangular pigpen is a function of the length of the four sides. Use x to denote the width and write an equation that relates the area to x. Recall the formula for area.

<u>**Answer:**</u> $A(x) = x \cdot (20 - x)$

12. What kind of equation is this? Describe the shape of the graph. This will be easier to do if you plot the data.

<u>**Answer:**</u> Quadratic equation. The shape is a parabola.

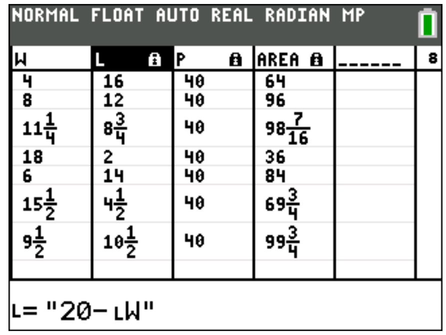




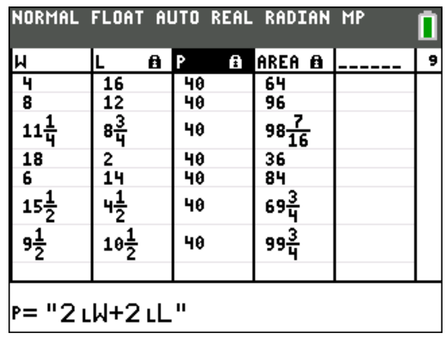
Students are instructed to use lists on their TI-84 Plus family graphing calculator to confirm their arithmetic. Have students plot the lists and explore regressions to see what equations might fit the data. Students should discover how their algebra confirms the results given from the technology.

Tech Tip: Students can change the [MODE] settings so that mixed fractions will show in the list. Have students use [ALPHA] [F1] to access the mixed fraction template.

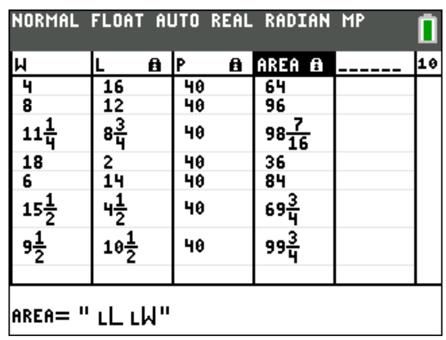
Students can use [2nd] [DEL] to insert a list with their own name. (To stay organized, have students label the width and length lists W and L, respectively.) Students could also go up to the top row and move the cursor to the right until a list appears without a name.



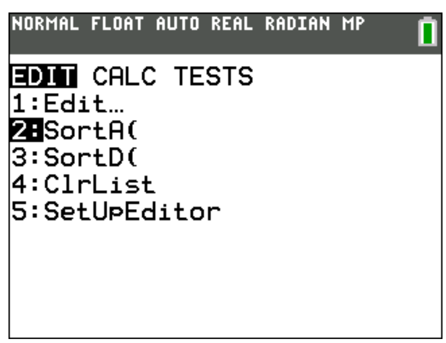
To confirm the calculations in their tables, have students move to the top row and enter the formula 20 – W in quotation marks in the column for length. Have students press [ALPHA] [+] to access the quotation mark. They should press [2nd] [STAT] for [LIST] and recall the W for width from there. Have students repeat this for the perimeter and area equations.



The perimeter P list can now be removed since the goal is to consider how the area is changing with the width. To remove a column, students can press [DEL] when in the top row.

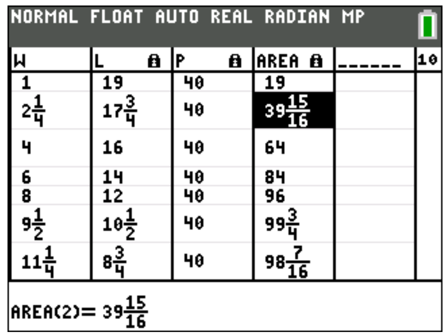


Once the students can see all of the data, it is helpful to sort the data in ascending order. To do this, have students press [STAT] and select 2:SortA(from the menu. Again, they should press [2nd] [STAT] for [LIST] and access the W for width.



Tech Tip: If students did not use the quotation marks when entering the equations into the top rows of their lists, the widths W would no longer correspond with the associated lengths L or AREA once the widths are sorted in ascending order. If lists were built without quotations marks (for example, if lists were built in L1, L2, and L4), the students would need to enter SortA(L1, L2, L4) to sort the rows such that the values in each row remain sorted together.

To see the lists again students should press STAT, 1:Edit… Notice in the screen shot to the right that the data have been sorted in ascending order. Note also that once the data are graphed in a StatPlot, the values will be displayed in order when students press TRACE.

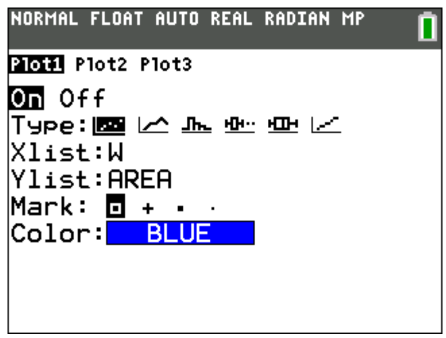


13. When plotting an equation it is important to decide what variable goes on which axes.

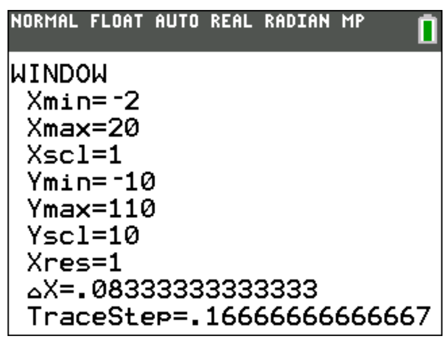
   a. What variable will be plotted on the horizontal axes? What is the domain for this variable? Explain how the domain relates to the rectangular pigpen.

   b. What variable is to be plotted on the vertical axes, the *y*-axis?

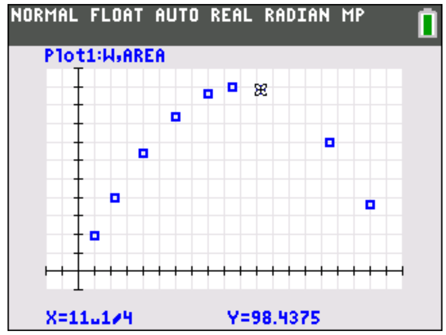
   Answer: The *x*-axis is *x*, or the width of the pigpen. The domain of the width is from 0 to 20 meters, although these are the asymptotic limits of the width (since the width can never be exactly equal to 0 or 20). The vertical axes or Ylist should be the AREA.



Students should use domain and range to set the window. This is more mathematically valuable than using ZoomStat. When students press WINDOW they should be encouraged to set the Xmin and Ymin to include some negative values so the axes can be viewed more easily. The Xscl and Yscl will leave a tick mark in the increments set.



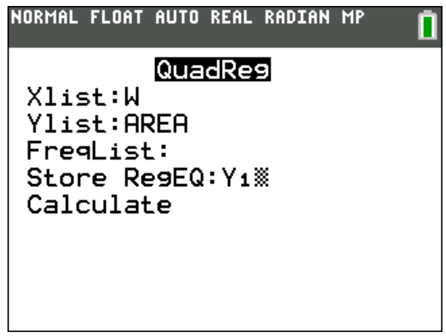
Student can press TRACE to explore the data graphically.

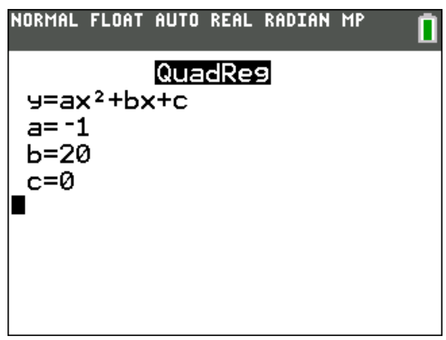


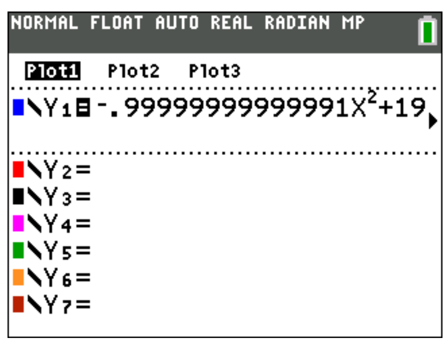
Now you can encourage students to consider again their answer for question 10. Some younger students may not be familiar with quadratic functions or describing the shape as parabolic, but they should at least recognize that the graph is non-linear.

Tech Tip: The grid lines are a unique feature on the TI-84 Plus C, as shown in the screen shot to above. Have students turn on GridLine by pressing 2nd ZOOM to change the [FORMAT] settings.

To explore the regressions, students will press STAT, CALC. Students should quickly discover that a quadratic regression is appropriate so they will choose QuadReg. Have students select ALPHA [F4] and press Y_1 to store the regression equation in Y_1 as shown to the right. Then, they will select Calculate.



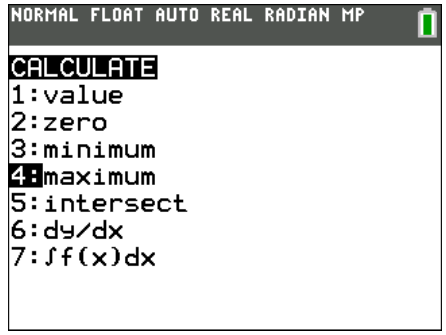


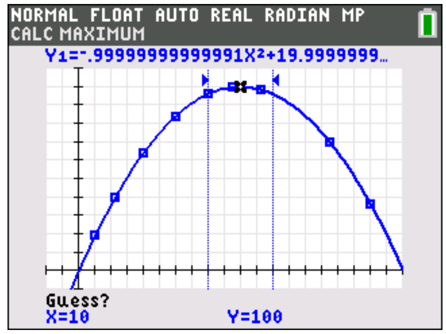


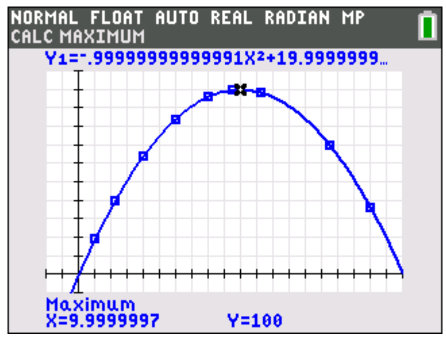
14. Using the graph, explain how you can determine the maximum area of the pigpen. What is the maximum area?

<u>Answer:</u> Graphically, the vertex will indicate the maximum value. The maximum area is 100 m^2 and occurs when both the length and width are 10 m.

Students can use the maximum tool in CALCULATE by pressing [2nd] [CALC]. Students will be prompted to give the left and right bounds and then guess the location of the maximum point on the parabola. Since the regression displays a decimal approximation, this can serve as an opportunity to discuss the precision of measured and calculated values. Have students come to the conclusion that the maximum area of 100 m^2 happens when the width is 10 m even if the value is sometimes given as 9.9999997, as shown to the right.







Younger students may need to rely on the scaffolding provided by the technology, but more mathematically advanced students should be encouraged to apply their algebra skills to confirm the vertex of the parabola. Since the area is given by the equation: $x \cdot (20 - x)$, or $A(x) = -x^2 + 20x$, the coefficients of the quadratic in general form would be $a = -1$ and $b = 20$. The x-value of the vertex is found at $-b/(2a) = -20/(2 * -1) = 10$ meters.

Substituting this into area function gives:

$A(10) = 10(20-10) = 100\ m^2$.

Teacher Tip: Be sure that students recognize what is special about this rectangle. (It is a square.) Ask students to conjecture if this is always true about the dimensions of a rectangle when area is maximized. That is, if the perimeter of a rectangle is fixed, will the maximum area always be a square? With these constraints, yes, this will always be true. Discuss other constraints where this would not be true. One example would be if Old MacDonald used the same length of 40 meters of fencing, but one of the sides of the pigpen was provided by another structure, such as a side of the barn. In this case, the maximum area of the pigpen would not be a square.