Exploration

Teacher Notes

Exploration: The Determinant of a Matrix

Learning outcomes addressed

3.7 Calculate the determinant of a matrix.
3.8 Evaluate a determinant to calculate the area of a triangle or quadrilateral in the plane, given the coordinates of its vertices.

Lesson Context

The Case of the Missing Square presented here, is one of the classic paradoxes in recreational mathematics. The recent Sherlock Holmes movie provides a "modern" context for solving this paradox as a criminal case investigation.

The assumption that the line joining the lower left corner of the rectangle to the upper right corner is a straight line results in a paradox. The sum of the areas of the four regions into which the rectangle is divided appears to be 64 square units, while the area of the original rectangle is 65 square units. The paradox is resolved when it is discovered that this "diagonal" line is not a straight line, but three line segments. Therefore the regions Gamma and Delta are not triangles, and all four regions are quadrilaterals. This sets the stage for the introduction of determinants to calculate the areas of the quadrilaterals given the coordinates of their vertices.

Lesson Launch

Have students read the introduction to *The Case of the Missing Square*.
Ask initiating questions such as:

- How did Inspector Holmes know that some of the gold block was missing?
- Did the executor divide the shares in the correct ratio?
- Did the executor distribute all of the gold to the four children?

Lesson Closure

Ensure that students understand the resolution of the paradox above. Ask students to calculate the cross-sectional areas the shares should have to total 65 square inches and to be in the ratio: 5:5:3:3. (Ans. $20^{5}/_{16}$, $20^{5}/_{16}$, $12^{3}/_{16}$, and $12^{3}/_{16}$.)

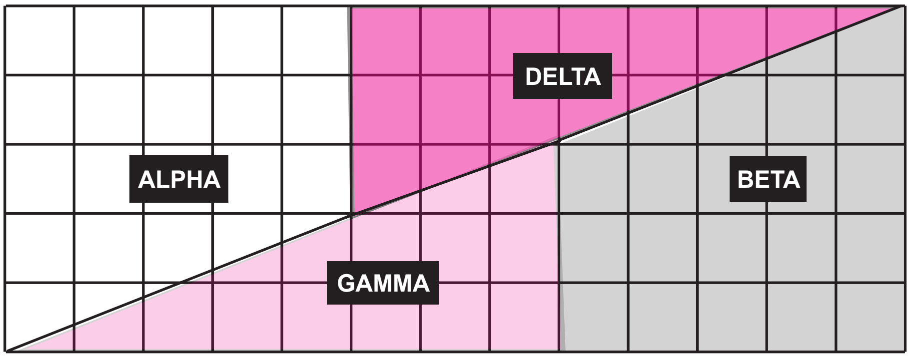
Have students evaluate several 2×2 and 3×3 determinants without technology. When they have mastered this computation, have them evaluate determinants with technology to verify their answers. Then encourage them to use technology to evaluate determinants of higher order.

Have the students use determinants to calculate the areas of the four regions. (See the TI-nspire Investigation). Check that the students understand how to derive the formula for the area of a triangle given the coordinates of its vertices (as in *Example 1* and *Example 2*.

Student Work Sheet

Exploration: The Determinant of a Matrix

❶. From squared paper:
- cut out two right triangles of base 5 units and height 3 units. Label these triangles Gamma and Delta.
- cut out two trapezoids with parallel sides 5 units apart and having lengths of 5 units and 3 units. Label these triangles Alpha and Beta.



❷. Write the formula for the area of a triangle of base *b* and height *h*.

❸. Write the areas of Gamma and Delta on your cut-outs.

❹. Write the formula for the area of a trapezoid with parallel sides of length *a* and *b* if the distance between the parallel sides is *h*.

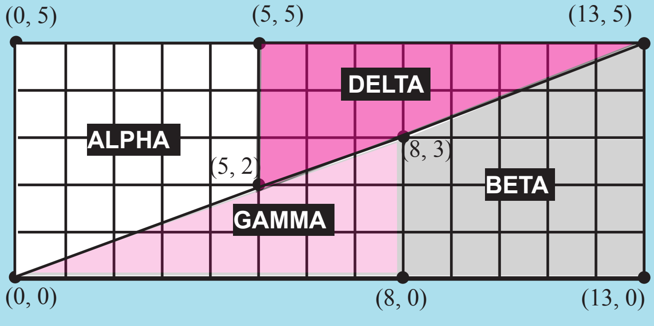
❺. Write the areas of Alpha and Beta on your cut-outs.

❻. Record the total area of the four regions and compare this with the original area of the 13 cm x 5 cm rectangle.

❼. Fit your cut-outs together to form a 13 × 5 rectangle. Explain what you discover.

TI-nspire Investigation

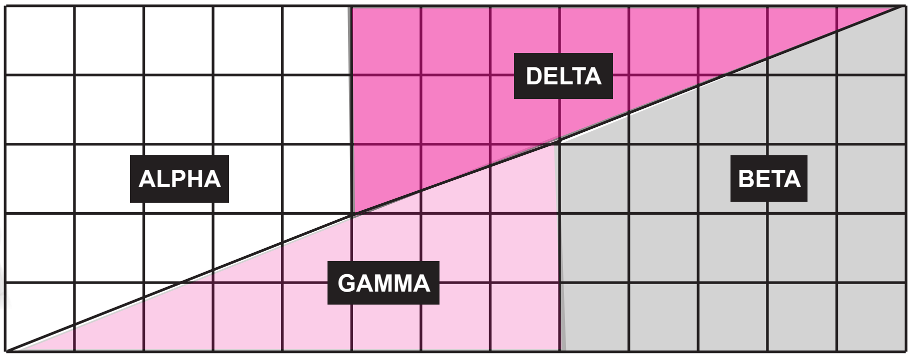
Calculate the slopes of the three line segments along the diagonal of the rectangle.



Describe what you discover and explain why there appears to be a paradox.

Exploration: The Determinant of a Matrix

The Case of the Missing Square



Mrs. Huxley's will decreed that a rectangular block of gold measuring 13" × 5" be divided among her four children Alpha, Beta, Gamma and Delta in the ratio 5:5:3:3 respectively. The executor of the will announced that he divided the block into four shares: two with triangular cross-section measuring 8" × 3" and two with trapezoidal cross-section having parallel sides of mean length 4" and width 5". Then he fit them together, as shown above, to show that all the gold was accounted for.

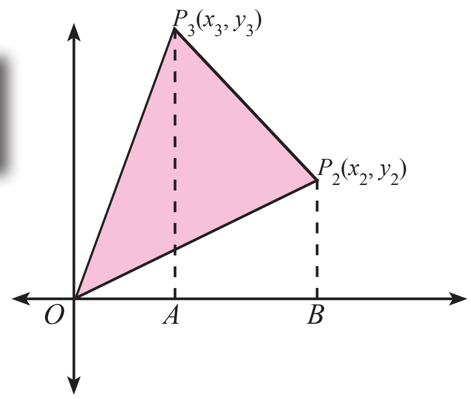
When Inspector Holmes was called in to verify that the will was executed properly, he calculated the cross sectional area of each of the four blocks of gold. He remembered that the area of a triangle is half the product of its base and height. He recalled also that the area of a trapezoid is mean length of its two parallel sides times the distance between those sides. Holmes wrote:

Area of Alpha = Area of Beta = ½(3 + 5) × 5 = 20 square inches.
Area of Gamma = Area of Delta = ½(8) × 3 = 12 square inches.
Total area of Alpha + Beta + Gamma + Delta = 64 square inches.
There's some gold missing!

How did Inspector Holmes know that some of the original gold block was missing? Where did it go?

Example 1

Find a formula for the area of ΔOP_2P_3 in terms of the coordinates of its vertices $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$.



Solution

$$\begin{aligned}\text{Area of } \Delta OP_2P_3 &= \text{Area of } \Delta OAP_3 + \text{Area of } ABP_2P_3 - \text{Area of } \Delta OBP_2\\ &= \tfrac{1}{2}x_3y_3 + \tfrac{1}{2}(y_2 + y_3)(x_2 - x_3) - \tfrac{1}{2}x_2y_2\\ &= \tfrac{1}{2}(x_2y_3 - x_3y_2)\end{aligned}$$

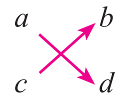
From *Example 1* we deduce the following:

Theorem: The area of ΔOP_2P_3 with vertices O(0, 0), $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$ is the absolute value of half the cross product of the coordinates of P_2 and P_3, i.e., area $\Delta OP_2P_3 = \tfrac{1}{2}|x_2y_3 - x_3y_2|$

cross product of
(x_2, y_2) and (x_3, y_3)
$$\begin{matrix} x_2 & y_2 \\ x_3 & y_3 \end{matrix}$$
means $x_2y_3 - x_3y_2$

Worked Examples

Definition: For the 2×2 matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we define the *determinant of M* by det $M = ad - bc$ and we denote the determinant M by $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.



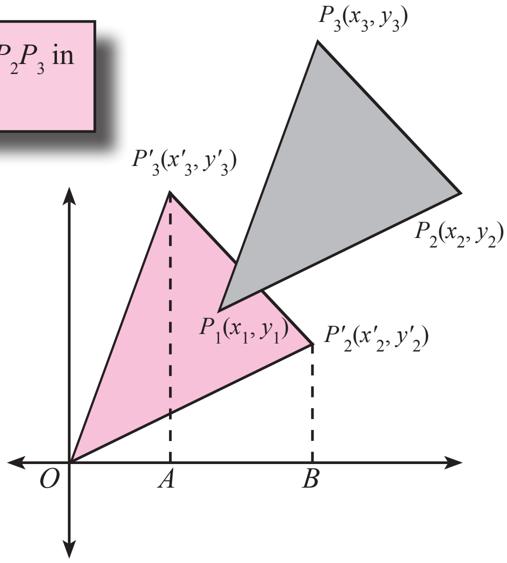
We can use the determinant to express the theorem on the previous page as follows:

Theorem: The area of ΔOP_2P_3 with vertices $O(0, 0)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$ is the absolute value of $\frac{1}{2}\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}$

Example 2

Use the theorem on the previous page to find a formula for the area of $\Delta P_1P_2P_3$ in terms of the coordinates of its vertices $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$.

Solution



To find the area of $\Delta P_1P_2P_3$ we apply the translation T that maps each point $P(x, y)$ onto the point $P'(x - x_1, y - y_1)$
This maps the vertices of $\Delta P_1P_2P_3$ onto $\Delta P'_1P'_2P'_3$, where:

P'_1 has coordinates $(0, 0)$,
P'_2 has coordinates $(x_2 - x_1, y_2 - y_1)$
P'_3 has coordinates $(x_3 - x_1, y_3 - y_1)$

Since $\Delta P'_1P'_2P'_3$ has one vertex at the origin, we can apply the theorem above to obtain:

$$\text{Area } \Delta P'_1P'_2P'_3 = \frac{1}{2}\begin{vmatrix} (x_2 - x_1) & (y_2 - y_1) \\ (x_3 - x_1) & (y_3 - y_1) \end{vmatrix}$$

$$= (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)$$

$$= x_1(y_2 - y_3) - x_2(y_1 - y_3) + x_3(y_1 - y_2)$$

To express this result in a more compact form, we introduce the following definition

Definition: For the 3×3 matrix $M = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ we define the *determinant of M*

$$|M| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & \boxed{a_{22}} & \boxed{a_{23}} \\ a_{31} & \boxed{a_{32}} & \boxed{a_{33}} \end{bmatrix} \quad \begin{bmatrix} a_{11} & \boxed{a_{12}} & \boxed{a_{13}} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & \boxed{a_{32}} & \boxed{a_{33}} \end{bmatrix} \quad \begin{bmatrix} a_{11} & \boxed{a_{12}} & \boxed{a_{13}} \\ a_{21} & \boxed{a_{22}} & \boxed{a_{23}} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

To calculate |M|, multiply a_{11} by the determinant of the 2 × 2 matrix remaining when all the elements in the same row or column as a_{11} are crossed out. Repeat this with a_{21} and then with a_{31}. Then add or subtract these products as shown.

Using a 3 × 3 determinant we can express the area of $\Delta P_1P_2P_3$ in terms of the coordinates of its vertices $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$.

Theorem: The area of $\Delta P_1P_2P_3$ with vertices $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$ is the absolute value of $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

The following example shows how we can apply this theorem.

Example 3

a) Calculate the area of $\Delta P_1P_2P_3$ where $P_1(-8, -3)$, $P_2(4, -1)$ and $P_3(2, 6)$.

b) Use the TI-*n*spire *det* command to verify your computations.

c) Graph $\Delta P_1P_2P_3$ in TI-*n*spire and measure its area to verify your answer in *part a*.

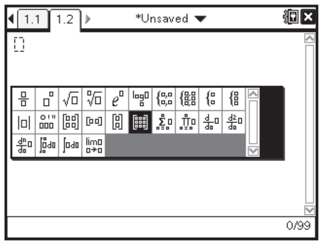
Solution

a) Applying the foregoing theorem, we have:

$$\text{Area of } \Delta P_1P_2P_3 = \tfrac{1}{2}\left|\begin{vmatrix} -8 & -3 & 1 \\ 4 & -1 & 1 \\ 2 & 6 & 1 \end{vmatrix}\right| = \tfrac{1}{2}\left| -8\underbrace{(-1-6)}_{\begin{vmatrix}-1 & 1\\ 6 & 1\end{vmatrix}} - 4\underbrace{(-3-6)}_{\begin{vmatrix}-3 & 1\\ 6 & 1\end{vmatrix}} + 2\underbrace{(-3+1)}_{\begin{vmatrix}-3 & 1\\ -1 & 1\end{vmatrix}}\right|$$

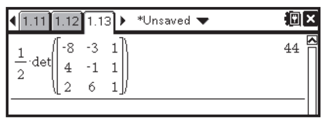
$$= \tfrac{1}{2}(88) \text{ or } 44$$

That is, the area is 44 square units.

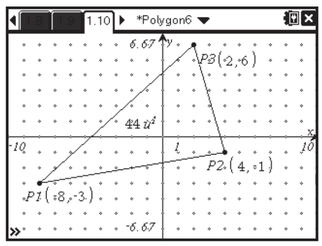


b) To verify our computation of the determinant in the *Calculator* application, we press: 1 ÷ 2 D E T (.

We select the 3 × 3 matrix template shown in the display and press enter. We press enter to accept the 3 rows and 3 columns option in the dialog box.



We enter the elements of the matrix into the template using tab.
On pressing enter, we observe that the determinant is 44, verifying *part a*.



c) We access a grid in *Graphs* by pressing: on ▶ enter menu 2 5.
To construct $\Delta P_1P_2P_3$, we press: menu 9 2 and define the vertices by clicking on grid points $P_1(-8, -3)$, $P_2(4, -1)$ and $P_3(2, 6)$.

To measure the area of $\Delta P_1P_2P_3$, we press menu 8 2 and click on $\Delta P_1P_2P_3$. The display shows the area is 44 square units, verifying the answer in *part a*.

Exercises and Investigations

❶. For what matrices is a determinant defined?

❷. Calculate the determinant of each matrix.

a) $\begin{bmatrix} 3 & -2 \\ 1 & 6 \end{bmatrix}$ b) $\begin{bmatrix} 8 & 4 \\ -3 & 0 \end{bmatrix}$ c) $\begin{bmatrix} -3 & -9 \\ 2 & 6 \end{bmatrix}$ d) $\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$

❸. a) Write matrices corresponding to the transformations R_x, R_y, R_{90}, and R_{-90}, where R_{90} and R_{-90} denote rotations of 90° and –90° about the origin and R_x and R_y denote reflections in the x-axis and y-axis respectively.

b) Calculate the determinants of R_x, R_y, R_{90}, and R_{-90}. Explain what you discover.

❹. Write each composite transformation as a 2 × 2 matrix where R_x, R_y, R_{90}, and R_{-90}, are defined as in *Exercise* ❸ and evaluate its determinant.

a) $R_x \circ R_x$ b) $R_y \circ R_y$ c) $R_x \circ R_y$

d) $R_x \circ R_{90}$ e) $R_{90} \circ R_x$ f) $R_{90} \circ R_{-90}$

Explain any pattern you discover about the determinants of rotations and reflections in the plane and their products.

❺. Define $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$.

a) Prove that $|AB| = |A| \bullet |B|$.

b) Use the *det(* command in TI-*n*spire to expand both sides of the equation in *part a* to verify the result.

❻. If P and Q are matrices such that $PQ = I$ where I is the identity matrix, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then Q is called the *inverse* of P.

Write an equation relating $|P|$ and $|Q|$.

❼. Calculate the determinant of each matrix.

a) $\begin{vmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \\ 2 & 0 & -4 \end{vmatrix}$ b) $\begin{vmatrix} 5 & -2 & -3 \\ -1 & 0 & 1 \\ 4 & 2 & -3 \end{vmatrix}$ c) $\begin{vmatrix} 2 & -3 & -1 \\ -7 & 4 & 0 \\ 5 & 0 & -2 \end{vmatrix}$

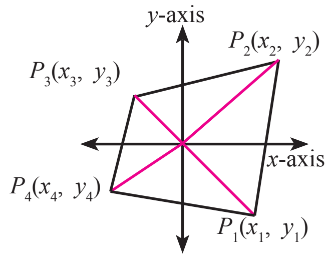
❽. Use the *det(* command in TI-*n*spire to verify your answers in *Exercise* ❼.

❾. Calculate the area of the triangles with these vertices.

a) (0, 0), (–5, 3), (6, – 4)
b) (3, –3), (6, –4), (5, 7)
c) (–2, –3), (–3, 3), (4, 4)

❿. Graph the triangle with vertices (–2, –3), (–3, 3), (4, 4) in TI-*n*spire. Follow the procedure in *Example 3 part c* to measure its area. Compare your answer with your answer in ❾ *c*.

⓫. a) Using the theorem following *Example 1*, show that the area of quadrilateral $P_1P_2P_3P_4$ with vertices $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ and $P_4(x_4, y_4)$ is given by:



$$\tfrac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) + (x_4y_1 - x_1y_4)]$$

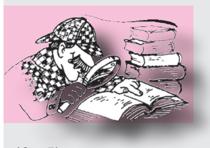
b) Using *part a*, prove that the area of quadrilateral $P_1P_2P_3P_4$ with vertices $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ and $P_4(x_4, y_4)$ is given by:

$$\tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 & 0 \\ x_2 & y_2 & 0 & 1 \\ x_3 & y_3 & 1 & 0 \\ x_4 & y_4 & 0 & 1 \end{vmatrix}$$

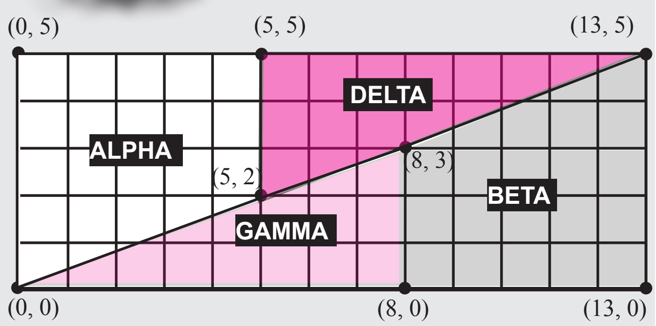
Hint:
Extend the definition of the determinant of a 3 × 3 matrix to a 4 × 4 matrix and compare the expansion with your result in *part a*.

c) Use the result in *part b* to find the area of the quadrilateral with vertices at $P_1(2,5)$, $P_2(-7,9)$, $P_3(-10,3)$, $P_4(6,-9)$.

TI-nspire Investigation



Sherlock Cracks the Case!



Sherlock Holmes labeled the vertices of the grid as shown. Then he calculated the areas of Delta and Gamma's shares as if they were quadrilaterals with these vertices:

Delta: (5, 2), (5, 5), (13, 5), (8, 3)
Gamma: (0, 0), (5, 2), (8, 3), (8, 0)

Use the formula in *Exercise* ⓫*b* to calculate the areas of Delta and Gamma's shares.

What is the total area of the four shares?

Why did the total areas of the four shares appear to be 64 square units?

Answers to the Exercises & Hints for the Investigations

Exploration

❶. The determinant is defined for square matrices only.

❷. Calculating the determinant of a 2 × 2 matrix as the difference of the cross products, we obtain these determinants:

a) 20 b) 12 c) 0 d) 1

❸. a) The matrices corresponding to R_x, R_y, R_{-90} and R_{90} are:

a) $R_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ b) $R_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ c) $R_{-90} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ d) $R_{90} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

b) The determinants of these matrices are:
$|R_x| = -1$, $|R_y| = -1$, $|R_{-90}| = 1$, and $|R_{90}| = 1$.
The determinants of all the rotations and reflections have absolute value 1. The determinants of the reflections are –1 and of the rotations are 1.

❹. The display below left shows the matrices for $R_x \circ R_x$, $R_y \circ R_y$ and $R_x \circ R_y$.

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The display above right shows the matrices for $R_x \circ R_{90}$, $R_{90} \circ R_x$ and $R_{90} \circ R_{-90}$.

a) $|R_x \circ R_x| = 1$ b) $|R_y \circ R_y| = 1$ c) $|R_x \circ R_y| = 1$
d) $|R_x \circ R_{90}| = -1$ e) $|R_{90} \circ R_x| = -1$ f) $|R_{90} \circ R_{-90}| = 1$

In *Exercise* ❸, we discovered that determinants of all the rotations and reflections have absolute value of 1. The determinants of the reflections R_x and R_y are –1 and the determinants of the rotations R_{90} and R_{-90} are 1.

The answers above indicate that the determinant of the composite transformation is the product of the determinants of its factors.

❺. a) $|A| = a_{11}a_{22} - a_{12}a_{21}$ and $|B| = b_{11}b_{22} - b_{12}b_{21}$.

$$|AB| = \begin{Vmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{Vmatrix}$$
$$= \left(a_{11}b_{11} + a_{12}b_{21}\right)\left(a_{21}b_{12} + a_{22}b_{22}\right) - \left(a_{21}b_{11} + a_{22}b_{21}\right)\left(a_{11}b_{12} + a_{12}b_{22}\right)$$
$$= \left(a_{11}b_{11}a_{21}b_{12} + a_{11}b_{11}a_{22}b_{22} + a_{12}b_{21}a_{21}b_{12} + a_{12}b_{21}a_{22}b_{22}\right) -$$
$$\left(a_{21}b_{11}a_{11}b_{12} + a_{21}b_{11}a_{12}b_{22} + a_{22}b_{21}a_{11}b_{12} + a_{22}b_{21}a_{12}b_{22}\right)$$
$$= \left(a_{11}b_{11}a_{22}b_{22} - a_{22}b_{21}a_{22}b_{12}\right) + \left(a_{12}b_{21}a_{22}b_{22} - a_{21}b_{11}a_{12}b_{12}\right)$$
$$= a_{11}a_{22}\left(b_{11}b_{22} - b_{21}b_{12}\right) - a_{12}a_{21}\left(b_{11}b_{22} - b_{21}b_{12}\right)$$
$$= \left(a_{11}a_{22} - a_{12}a_{21}\right)\left(b_{11}b_{22} - b_{21}b_{12}\right) = |A||B|$$

b) The display verifies the calculation in *part a*.

$$\det\begin{pmatrix} a11 & a12 \\ a21 & a22 \end{pmatrix} \quad a11 \cdot a22 - a12 \cdot a21$$
$$\det\begin{pmatrix} b11 & b12 \\ b21 & b22 \end{pmatrix} \quad b11 \cdot b22 - b12 \cdot b21$$
$$\det\left(\begin{bmatrix} a11 & a12 \\ a21 & a22 \end{bmatrix} \cdot \begin{bmatrix} b11 & b12 \\ b21 & b22 \end{bmatrix}\right) \quad (a11 \cdot a22 - a12 \cdot a21) \cdot (b11 \cdot b22 - b12 \cdot b21)$$

Domain of the result might be larger than the... 3/3

❻. In *Exercise* ❺, we proved that $|P||Q| = |PQ|$ for any two square 2 × 2 matrices P and Q. Therefore, if P and Q are inverse matrices,

$$|P||Q| = |PQ| = |I| = 1$$

That is, $|P| = 1/|Q|$.

Exploration *cont'd*

❼. a) 1[2(–4)] – 0[3(–4) – 2(1)] –1[3(0) – 2(2)] = –4
b) 5[0(–3) – 2(1)] + 2[(–1)(–3) – (4)(1)] –3[2(–1) – 4(0)] = –6
c) 2[4(–2) – 0(0)] + 3[(–7)(–2) – (5)(0)] –1[0(–7) – 4(5)] = 46

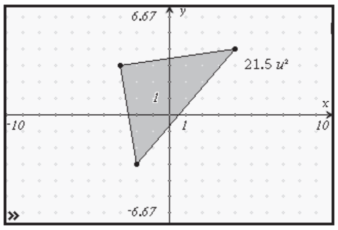
❽. The displays below verify the computations in *Exercise* ❼.

$$\det\begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \\ 2 & 0 & -4 \end{bmatrix} \quad -4$$
$$\det\begin{bmatrix} 5 & -2 & -3 \\ -1 & 0 & 1 \\ 4 & 2 & -3 \end{bmatrix} \quad -6$$
$$\det\begin{bmatrix} 2 & -3 & -1 \\ -7 & 4 & 0 \\ 5 & 0 & -2 \end{bmatrix} \quad 46$$

❾. The areas of the triangles are:
a) 1 b) 16 c) –43/2

$$\frac{\det\begin{bmatrix} 0 & 0 & 1 \\ -5 & 3 & 1 \\ 6 & -4 & 1 \end{bmatrix}}{2} \quad 1$$
$$\frac{\det\begin{bmatrix} 3 & -3 & 1 \\ 6 & -4 & 1 \\ 5 & 7 & 1 \end{bmatrix}}{2} \quad 16$$
$$\frac{\det\begin{bmatrix} -2 & -3 & 1 \\ -3 & 3 & 1 \\ 4 & 4 & 1 \end{bmatrix}}{2} \quad \frac{-43}{2}$$

❿. The display shows that the area of the triangle in *Exercise* ❾c) is 21.5 or 43/2 square units. This verifies the result in *Exercise* ❾c).



⓫. a) The area of the quadrilateral $P_1P_2P_3P_4$ is the area of:
□OP_1P_2 + □OP_2P_3 + □OP_3P_4 + □OP_4P_1
$= ½[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) + (x_4y_1 - x_1y_4)]$

b) Expanding the given determinant, we see it is equal to double the expression given in *part a*. That is, the area of quadrilateral $P_1P_2P_3P_4$ is half the determinant given in the display.

$$a \quad \begin{bmatrix} x1 & y1 & 1 & 0 \\ x2 & y2 & 0 & 1 \\ x3 & y3 & 1 & 0 \\ x4 & y4 & 0 & 1 \end{bmatrix}$$
$$\det(a)$$
$$x1 \cdot (y2 - y4) - x2 \cdot (y1 - y3) - x3 \cdot (y2 - y4) + x4 \cdot (y1 - y3)$$

c) The display shows that the area of the quadrilateral $P_1P_2P_3P_4$ is 121 square units, where P_1, P_2, P_3 and P_4 are respectively (2, 5), (–7, 9), (–10, 3), and (6, –9).

$$\frac{1}{2} \cdot \det\begin{bmatrix} 2 & 5 & 1 & 0 \\ -7 & 9 & 0 & 1 \\ -10 & 3 & 1 & 0 \\ 6 & -9 & 0 & 1 \end{bmatrix} \quad 121$$

Hint for the TI-nspire Investigation

The area of Delta is given by the determinant shown in the display. Does this mean that Delta is not a triangle of area 12?

$$\begin{bmatrix} 5 & 2 & 1 & 0 \\ 5 & 5 & 0 & 1 \\ 13 & 5 & 1 & 0 \\ 8 & 3 & 0 & 1 \end{bmatrix} \to d \quad \begin{bmatrix} 5 & 2 & 1 & 0 \\ 5 & 5 & 0 & 1 \\ 13 & 5 & 1 & 0 \\ 8 & 3 & 0 & 1 \end{bmatrix}$$
$$\frac{1}{2} \cdot \det(d) \quad \frac{-25}{2}$$