Slope Fields—Introduction

Math Objectives

- Students will describe the idea behind slope fields in terms of visualization of the family of solutions to a differential equation.
- Students will describe the slope of a tangent line at a point on the graph of a solution to a differential equation.
- Students will describe the general nature of a solution to a differential equation as suggested by a slope field.
- Students will use the TI-Nspire built-in function **deSolve** to find the general family of solutions to a differential equation.
- Students will match a differential equation with its corresponding slope field.

Vocabulary

- slope field
- first order differential equation
- family of solutions
- particular solution

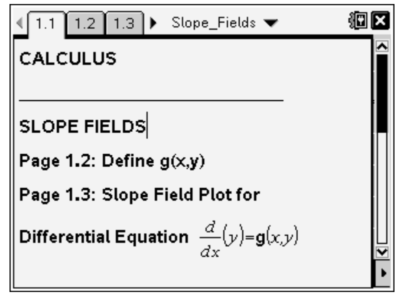
About the Lesson

- This lesson involves the concept of a slope field, a graphical representation of the family of solutions to the first order differential equation, $y' = g(x, y)$.
- As a result, students will:
  - Understand that a slope field is a visualization of the family of solutions to a differential equation.
  - Use characteristics in a slope field to describe the general nature of a solution to a differential equation.
  - Be able to match a differential equation with the appropriate slope field and find the family of solutions using the TI-Nspire.

TI-Nspire™ Navigator™ System

- Use Screen Capture to compare specific student solutions.

NOTE: This .tns file opens slowly on the handheld (about 60 sec).



TI-Nspire™ Technology Skills:

- Download a TI-Nspire document
- Open a document
- Move between pages
- Add the graph of a function to a slope field
- Use the function **deSolve**

Tech Tips:

- Make sure the font size on your TI-Nspire handheld is set to Medium.
- In Graphs, you can hide the function entry line by pressing ctrl G.

Lesson Materials:

Student Activity

Slope_Fields_Student.pdf

Slope_Fields_Student.doc

TI-Nspire document

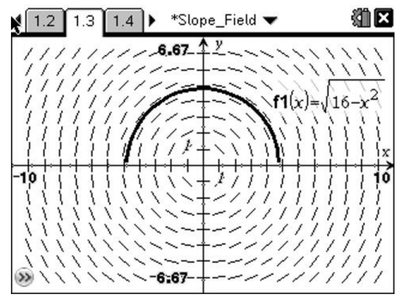
Slope_Fields.tns

Visit www.mathnspired.com for lesson updates and tech tip videos.

Discussion Points and Possible Answers

Tech Tip: The built-in TI-Nspire command **deSolve** takes three arguments: the differential equation in the form $y' = \mathbf{g}(x, y)$ (you can find the prime (′) symbol using the punctuation key ?!▸), the independent variable, and the dependent variable. The command returns, when possible, the general family of solutions to the differential equation using constants of integration of the form c_n, where n is an integer.

Move to page 1.3.



1. The slope field on this page is a visualization of the family of solutions to the differential equation $y' = -\frac{x}{y}$.

   a. Describe the slope of a tangent line to the graph of a solution at a point (0, *b*), *b* ≠ 0, on the *y*-axis. Use the differential equation to justify your answer.

   <u>Answer:</u> Each line segment in the slope field at (0, *b*) appears to be a horizontal line. Therefore, the slope of a tangent line to the graph of a solution at a point (0, *b*) appears to be 0. The differential equation supports this since at the point (0, *b*), $y' = -\frac{0}{b} = 0$.

   b. Describe the slope of a tangent line to the graph of a solution at a point (*a*, 0), *a* ≠ 0, on the *x*-axis. Use the differential equation to justify your answer.

   <u>Answer:</u> Each line segment in the slope field at (*a*, 0) appears to be a vertical line. Therefore, the slope of a tangent line to the graph of a solution at a point (*a*, 0) does not exist. The differential equation supports this since at the point (*a*, 0), $y' = \frac{a}{0}$, which is undefined.

   c. Describe a solution to the differential equation as suggested by the slope field.

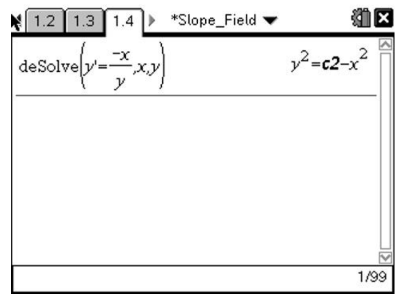
   <u>Answer:</u> The slope field suggests a solution to the differential equation is a circle centered at the origin. However, it is assumed the solution to the differential equation, *y*, is a function. Therefore, the slope field suggests a solution to the differential equation is the top half or bottom half of a circle centered at the origin.

d. Use your answers to parts 1a, b, and c to write a possible specific solution to the differential equation. Enter this function for $\mathbf{f1}(x)$. Is it consistent with the slope field? If not, try to find and graph a function that corresponds to the slope field.

Answer: A possible specific solution is the function $y = \sqrt{16 - x^2}$. The graph of this function is the top half of a circle centered at the origin. The graph of this function is consistent with the slope field.

TI-Nspire Navigator Opportunity: ***Screen Capture***
See Note 1 at the end of this lesson.

e. Add a calculator page. Use the function **deSolve** to find the general family of solutions to this differential equation. Find the specific solution to this differential equation that passes through the point (0, 5). Verify analytically that this is a solution to the differential equation.

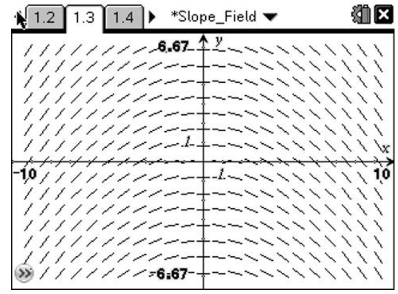


Answer: The TI-Nspire indicates that the general family of solutions to this differential equation has the form $y^2 = c_2 - x^2$. The graph of this equation is a circle centered at the origin with radius $r = \sqrt{c_2}$. A specific solution must be a function. The function whose graph passes through the point (0, 5) is $y = \sqrt{25 - x^2}$.

Verification: $y' = \frac{1}{2}(25 - x^2)^{-\frac{1}{2}}(-2x)$

$$= \frac{-x}{\sqrt{25 - x^2}}$$

$$= -\frac{x}{y}$$



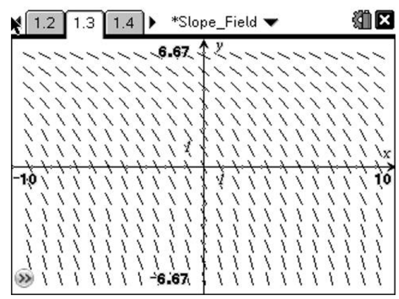
2. Consider the differential equation $y' = -\frac{x}{6}$, and on page 1.2 define $\mathbf{g}(x,y) = -\frac{x}{6}$. Move to page 1.3 and consider the corresponding slope field.

   a. Where are the slopes the same?

   <u>Answer:</u> The slopes of the line segments appear to be the same along vertical lines, or along lines that are perpendicular to the *x*-axis.

   b. Use your answer in part 2a to generalize. If **g**(*x, y*) involves only the variable *x*, then where will the slopes be the same? Justify your answer.

   <u>Answer:</u> The answer in part 2a suggests that if **g**(*x, y*) involves only the variable *x*, then the slopes will be the same along vertical lines, or along lines perpendicular to the *x*-axis. If **g**(*x, y*) involves only the variable *x*, then *y*′, the slope of the tangent line, changes only when *x* changes. Therefore, for a fixed value *a*, the slope is the same for all values (*a, y*).



3. Consider the differential equation $y' = \frac{y}{4} - 2$, and on page 1.2 define $\mathbf{g}(x,y) = \frac{y}{4} - 2$. Move to page 1.3 and consider the corresponding slope field.

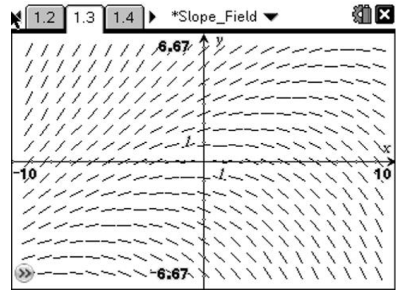
   a. Where are the slopes the same?

   <u>Answer:</u> The slopes of the line segments appear to be the same along horizontal lines, or along lines that are perpendicular to the *y*-axis.

   b. Use your answer in part 3a to generalize. If **g**(*x, y*) involves only the variable *y*, then where will the slopes be the same? Justify your answer.

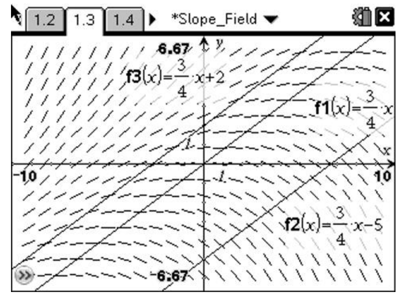
   <u>Answer</u>: The answer in part 3a suggests that if **g**(*x, y*) involves only the variable *y*, then the slopes will be the same along horizontal lines, or along lines perpendicular to the *y*-axis. If **g**(*x, y*) involves only the variable *y*, then *y*′, the slope of the tangent line, changes only when *y* changes. Therefore, for a fixed value *b*, the slope is the same for all values (*x, b*).

4. Consider the differential equation $y' = \frac{y}{6} - \frac{x}{8}$, and on page 1.2 define $\mathbf{g}(x, y) = \frac{y}{6} - \frac{x}{8}$. Move to page 1.3 and consider the corresponding slope field.



a. Where are the slopes the same?

Answer: The slopes of the line segments appear to be the same along certain straight lines with positive slope. In fact, the slopes of the line segments are the same along lines parallel to the graph of $\frac{y}{6} - \frac{x}{8} = 0$ or $y = \frac{3}{4}x$.



Teacher Tip: Add the graph of $y = \frac{3}{4}x$ and other parallel lines to the graph of the slope field. This will help to justify the answer to this question.

b. Use your answer in part 4a to generalize. If the differential equation is of the form $y' = ax + by$, where *a* and *b* are constants, then where are the slopes the same? Justify your answer.

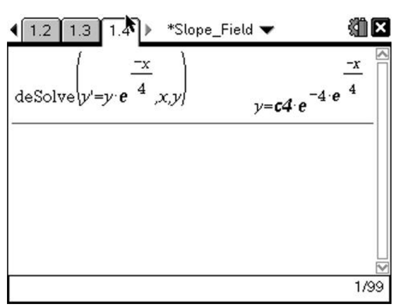
Answer: The answer in part 4a suggests that if $\mathbf{g}(x, y) = ax + by$, then the slopes are the same along lines parallel to the graph of the line $ax + by = 0$ or along lines of the form $ax + by = c$. For any point on the line $ax + by = c$, the value for y' is the same, namely $y' = ax + by = c$.

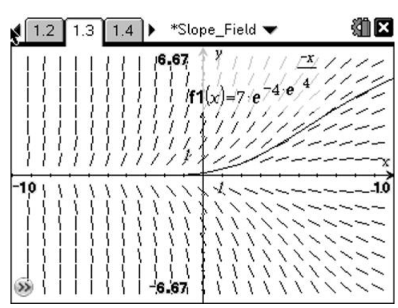
5. Match each differential equation with its corresponding slope field. Use the TI-Nspire to solve each differential equation and graph a particular solution on the corresponding slope field.

a. $y' = ye^{-\frac{x}{4}}$

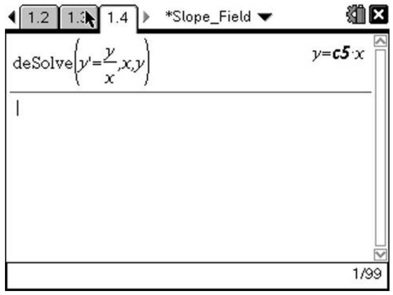
Answer: Slope field (v)

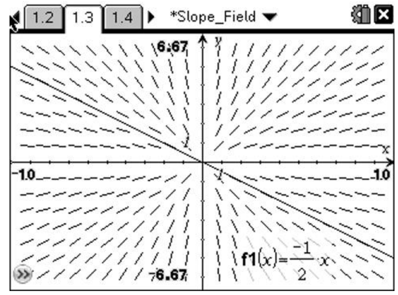
Solution: $y = ce^{-4e^{\frac{-x}{4}}}$





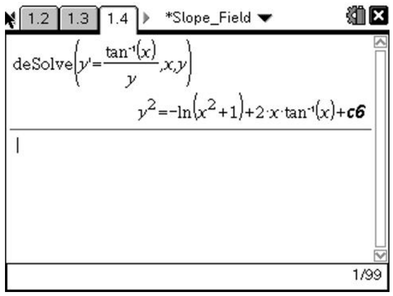
b. $y' = \dfrac{y}{x}$

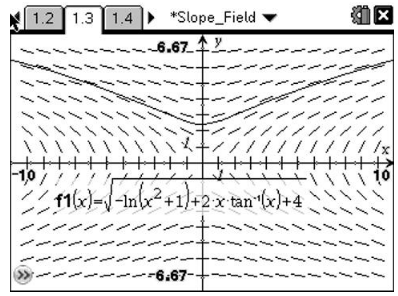




Answer: Slope field (iii)
Solution: $y = cx$

c. $y' = \dfrac{\tan^{-1} x}{y}$





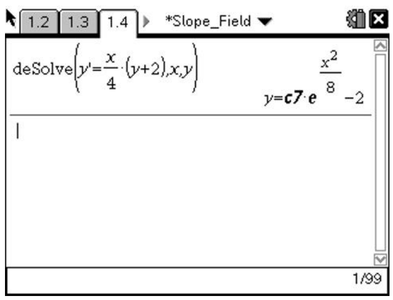
Answer: Slope field (vii)

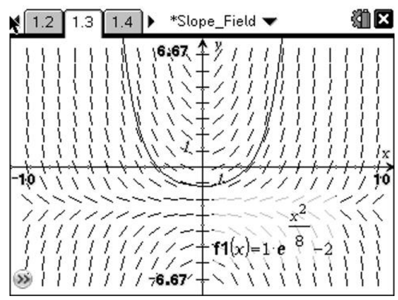
Solution:

$y = \sqrt{-\ln(x^2+1) + 2x\tan^{-1} x + c}$

$y = -\sqrt{-\ln(x^2+1) + 2x\tan^{-1} x + c}$

d. $y' = \dfrac{x}{4}(y+2)$

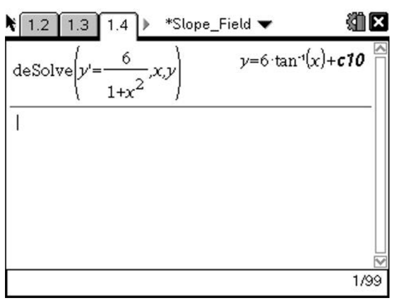


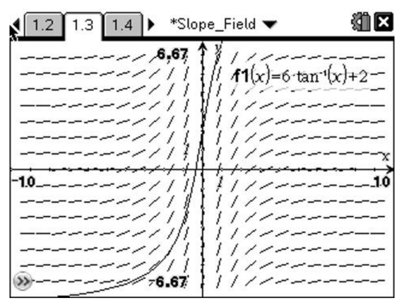


Answer: Slope field (ix)

Solution: $y = ce^{\frac{x^2}{8}} - 2$

e. $y' = \dfrac{6}{1+x^2}$



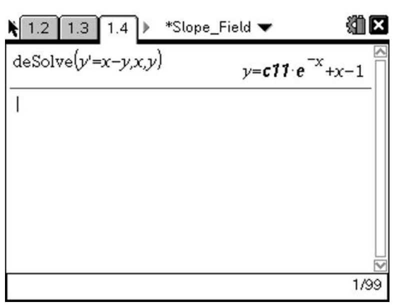


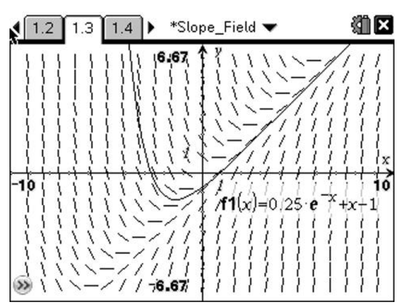
Answer: Slope field (viii)

Solution:

$y = 6\tan^{-1} x + c$

f. $y' = x - y$





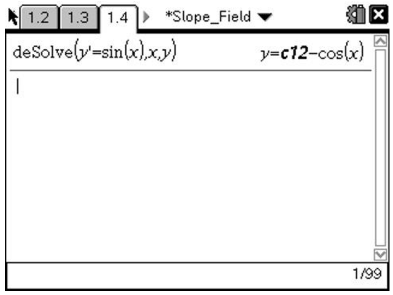
Answer: Slope field (ii)

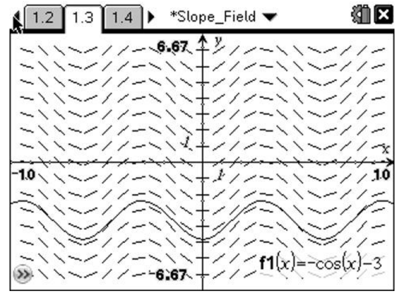
Solution:

$y = ce^{-x} + x - 1$

g. $y' = \sin x$

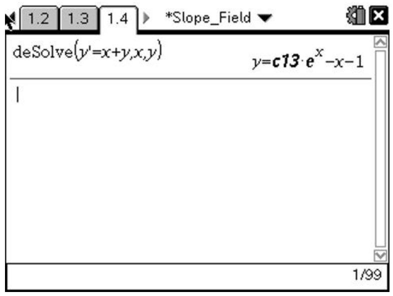
Answer: Slope field (iv)
Solution: $y = -\cos x + c$

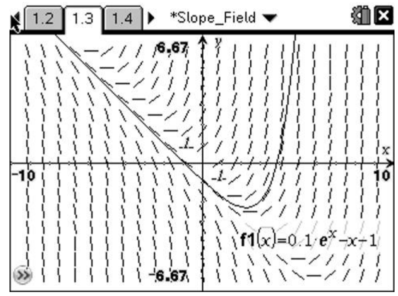




h. $y' = x + y$

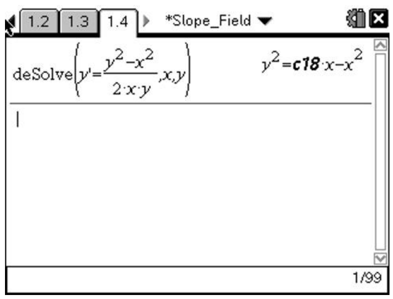
Answer: Slope field (i)
Solution: $y = ce^{x} - x - 1$

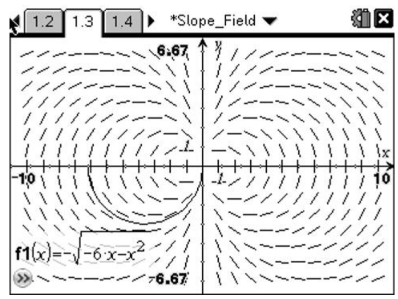




i. $y' = \dfrac{y^2 - x^2}{2xy}$

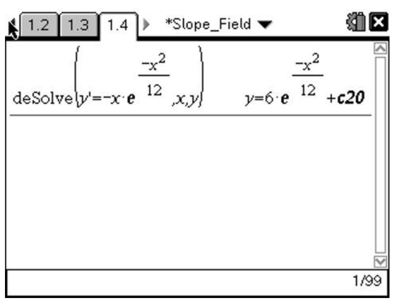
Answer: Slope field (vi)
Solution: $y = \sqrt{cx - x^2}$
$y = -\sqrt{cx - x^2}$

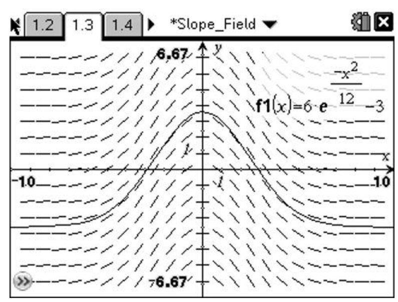




j. $y' = -xe^{-\frac{x^2}{12}}$

Answer: Slope field (x)
Solution: $y = 6e^{\frac{-x^2}{12}} + c$





TI-Nspire Navigator Opportunity: ***Screen Capture***
See Note 2 at the end of this lesson.

Wrap Up

Upon completion of this discussion, the teacher should ensure that students understand:

- How a slope field is a visualization of the family of solutions to a differential equation.
- How to use characteristics in a slope field to describe the general nature of a solution to a differential equation.
- How to match a differential equation with the appropriate slope field and find the family of solutions using the TI-Nspire.

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Note 1

Question 1d, *Screen Capture:* Compare different students' solution curves to the given slope field. For curves that do not fit the slope field, check these analytically against the differential equation (by substituting the student's function for y and its derivative for dy/dx).

Note 2

End of Lesson, *Screen Capture:* At the end of this activity, consider asking students to generate other creative slope fields. This may present a good opportunity to use *Screen Capture* and to test the limits of the TI-Nspire function **deSolve.**