Exploring the Equation of a Circle

MATH NSPIRED

Math Objectives

- Students will understand the definition of a circle as a set of all points that are equidistant from a given point.
- Students will understand that the coordinates of a point on a circle must satisfy the equation of that circle.
- Students will relate the Pythagorean Theorem and Distance Formula to the equation of a circle.
- Given the equation of a circle $(x-h)^2 + (y-k)^2 = r^2$, students will identify the radius r and center (h, k).
- Students will derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation
- Students will look for and make use of structure (CCSS Mathematical Practice).

Vocabulary

- Pythagorean Theorem
- Distance Formula
- radius

About the Lesson

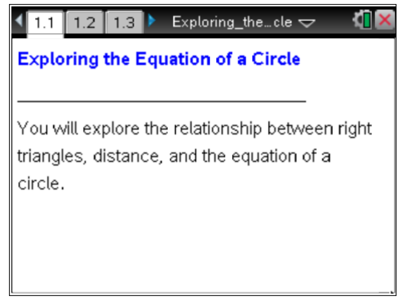
- This lesson involves plotting points that are a fixed distance from the origin, dilating a circle entered on the origin, translating a circle away from the origin, and dilating and translating a circle while tracing a point along its circumference.
- As a result students will:
  - Visualize the definition of a circle and the relationship between the radius and the hypotenuse of a right triangle
  - Observe the consequence of this manipulation on the equation of the circle
  - Infer the relationship between the equation of a circle and the Pythagorean Theorem, and infer the relationship between the equation of a circle and the Distance Formula
  - Identify the radius r and center (h, k) of the circle $(x-h)^2 + (y-k)^2 = r^2$, and deduce that the coordinates of a point on the circle must satisfy the equation of that circle

TI-Nspire™ Navigator™

- Live Presenter
- Class Capture
- Quick Poll

Activity Materials

- Compatible TI Technologies: TI-Nspire™ CX Handhelds, TI-Nspire™ Apps for iPad®, TI-Nspire™ Software



Tech Tips:

- This activity includes screen captures taken from the TI-Nspire CX handheld. It is also appropriate for use with the TI-Nspire family of products including TI-Nspire software and TI-Nspire App. Slight variations to these directions may be required if using other technologies besides the handheld.
- Watch for additional Tech Tips throughout the activity for the specific technology you are using.
- Access free tutorials at http://education.ti.com/calculators/pd/US/Online-Learning/Tutorials

Lesson Files:

Student Activity

- Exploring_the_Equation_of_a_Circle_Student.pdf
- Exploring_the_Equation_of_a_Circle_Student.doc

TI-Nspire document

- Exploring_the_Equation_of_a_Circle.tns

Discussion Points and Possible Answers

Tech Tip: If students experience difficulty dragging the point, check to make sure that they have moved the arrow until it becomes a hand ( ) getting ready to grab the point. Then press ctrl to grab the point and close the hand ( ).

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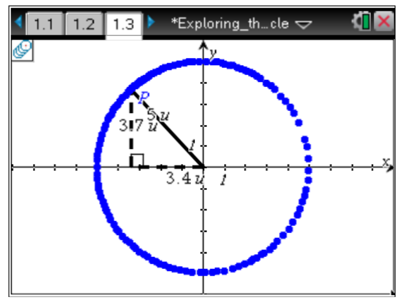

1. Select **Menu > Trace > Geometry Trace**. Select point *P*. Then grab and drag point *P* to observe the path it traces.
   a. What do these points have in common?

   Answer: All points are 5 units from the origin. All points are at the end of the hypotenuse of the right triangle. The points appear to form a circle.

Tech Tip: Select > **Trace > Geometry Trace**.

TI-Nspire Navigator Opportunity: *Live Presenter*
See Note 1 at the end of this lesson.

Teacher Tip: Make sure students drag point *P* and place points in all quadrants.

   b. As you drag point *P*, a triangle moves along with the point. What changes about the triangle? What stays the same?

   Answer: The lengths of the legs change. The length of the hypotenuse stays the same.

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2. Read the definition of a circle given on this page. What does the word locus mean?

   Answer: A locus is a set of points satisfying a given condition.

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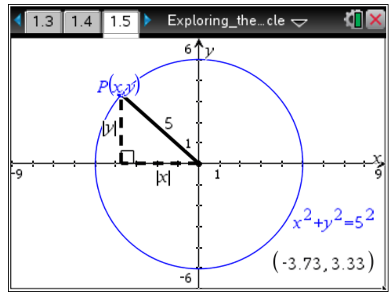


3. Drag point *P* around the circle. The equation of the circle and the coordinates of point *P* are given.
   a. What is the relationship between the hypotenuse of the right triangle and the radius of the circle?

      Answer: The hypotenuse of the right triangle is the radius of the circle. The hypotenuse and the radius have the same length.

   b. What is the relationship between the legs of the right triangle and point *P*?

      Answer: The absolute value of the *x*-coordinate is the length of the horizontal leg. The absolute value of the *y*-coordinate is the length of the vertical leg.

      Teacher Tip: The absolute value operation is needed for the measurements of the legs of the right triangle because the *x*- and *y*-coordinates are directed distances.

   c. When given any right triangle and the lengths of its legs, what formula is used to find the length of its hypotenuse? Why is that helpful in this situation?

      Answer: The Pythagorean Theorem: $a^2 + b^2 = c^2$; $c = \sqrt{a^2 + b^2}$ is used to find the length of the hypotenuse. From the earlier questions, students should see that as point *P* moves around the circle, it also moves the right triangle around. The sides of the right triangle and the hypotenuse are used in building the circle and thus its equation.

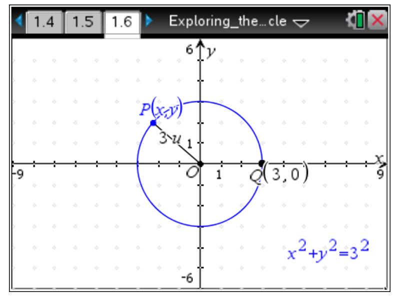
   d. Since point *P* lies on the circle, what must be true about its coordinates? Pick a point and verify.

      Answer: The coordinates of point *P* must satisfy the equation of the circle. If the *x*- and *y*-values are substituted into the equation, then $x^2 + y^2$ must equal 25. Answers will vary depending on the point students pick.

      Tech Tip: Because of rounding, if students do not pick an integer value, their point may not quite fit into the equation. This is a good opportunity to discuss rounding issues on the handheld. If desired, you could have students hover the cursor over the coordinates and press [+] to show more digits.

Tech Tip: Because of rounding, if students do not pick an integer value, their point may not quite fit into the equation. This is a good opportunity to discuss rounding issues on the iPad® . If desired, you could have students select the coordinate, select , and drag the Custom Precision slider to show more digits.

Move to page 1.6.



4. Change the radius of circle *O* by dragging point *Q* along the *x*-axis.
   a. When the radius of the circle changes, what changes in the equation? What stays the same?

      Answer: When the radius changes, the constant term on the right side of the equation changes. The x^2 and y^2 stay the same.

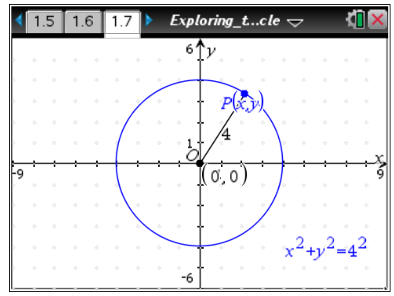
TI-Nspire Navigator Opportunity: ***Class Capture***
See Note 2 at the end of this lesson.

   b. Why does the constant in the equation change?

      Answer: The constant term changes because, in the Pythagorean Theorem, it corresponds to the length of the hypotenuse of a right triangle, which is the radius of the circle.

Teacher Tip: Encourage students to drag point *Q* to the left of the y-axis. Though point *Q*'s *x*-value will be negative, the square of this value remains positive. To reinforce what it means to satisfy the equation of the circle, consider substituting the coordinates of point *Q* into the equation. Make sure students see the relationship between *Q* and the hypotenuse of the triangle.

Move to page 1.7.



5. Move the center of the circle away from the origin by dragging point *O*.
   a. How are the coordinates of the center of the circle related to the equation?

   <u>Answer:</u> The constant terms in the parentheses are opposites of the coordinates of the center of the circle. The constant terms are additive inverses of the coordinates of the center of the circle.

   b. What formula is used to find the length of radius $\overline{OP}$?

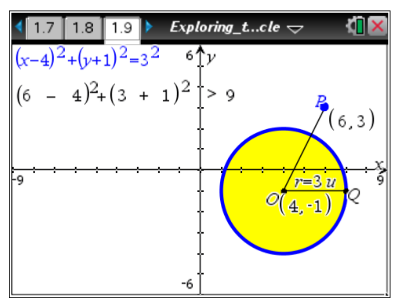
   <u>Answer:</u> The Distance Formula: $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

   c. Why is this formula similar to the equation of the circle?

   <u>Answer:</u> All of the points on a circle are the same distance away from its center. This distance is a fixed number, which is why there is a number on one side of the equation of a circle instead of a "*d*." The center of the circle is a known point, so its coordinates take the place of one set of coordinates in the Distance Formula.

   Teacher Tip: Have students move the center to the origin and point *P* to integer values. Work through the Distance Formula with them so that they see how the equation of the circle resembles the Distance Formula.

Move to page 1.9.



6. Move the center of the circle by dragging point *O.* Change the radius of the circle by dragging point *Q*. Drag point *P* outside the circle, on the circle, and in the interior of the circle.
   a. What values are being substituted into the equation?

   <u>Answer:</u> The coordinates of point *O* and point *P* are being substituted into the equation.

Teacher Tip: Students tend to think that the center of a circle is a point on the circle. You may wish to point out that, though the center is a critical point when dealing with circles, it does not lie on the circumference of the circle. The coordinates of point *P* are approximations or more precise values if on the circle. Therefore, the values substituted into the equation of the circle may result in a value that differs from an expected value by a few hundredths.

b. Describe the location of point *P* when the inequality statement shows "**>**". Describe the location of point *P* when the inequality statement shows "**<**".

Answer: When point *P* is on the exterior of the circle, the inequality is "**>**" because the values substituted into the equation result in a value that is greater than the radius squared value. The reasoning can be used for the interior of the circle and the "**<**" (less than the radius squared value).

c. Drag point *P* until the statement becomes an equality (=). Where is point *P*?

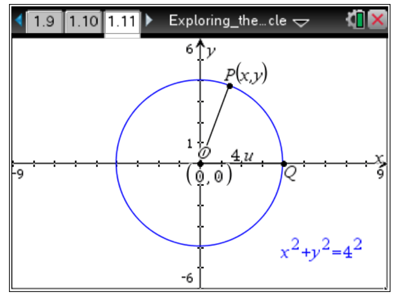
Answer: The statement becomes an equality when point *P* is anywhere on the circle.

d. Why do the constants within the parentheses in the equation and the coordinates of the center of the circle have opposite signs?

Answer: The length of the radius of a circle is the distance between the center and all points that lie on the circle. The horizontal and vertical distance between two points is the respective difference between their *x*- and *y*-values, which requires subtraction. Since subtraction is the same as the addition of a number's opposite, the constant terms and coordinates of the center of the circle must be opposites.

Teacher Tip: The idea of "opposites" is sometimes confusing, so continue to relate the idea of directed distance and the Distance Formula to the equation of a circle.

Move to page 1.11.



7. Move the center of the circle by dragging point *O*. Change the radius of the circle by dragging point *Q*. Drag point *P* around the circle. What do the *x*- and *y*-variables in the equation represent?

Answer: The *x*- and *y*-variables in the equation represent the coordinates of all points that lie on the circumference of the circle. The *x*- and *y*-variables represent the set of all points whose coordinates satisfy the equation of the circle.

8. Suppose a circle has the equation $(x - 12)^2 + (y + 4)^2 = 25$.

   a. What is the radius of the circle?

   Answer: 5

   b. What are the coordinates of the center of the circle?

   Answer: (12, –4)

   c. How can you determine whether the point (12, –9) lies on the circle?

   Answer: Substitute the *x*- and *y*-values of the coordinates into the equation of the circle. If the values satisfy the equation, then the point lies on the circle.

   $(12-12)^2+(-9+4)^2=25$

   $(0)^2+(-5)^2=25$

   $25=25$

   The point (12, –9) lies on the circle.

TI-Nspire Navigator Opportunity: *Quick Poll*
See Note 3 at the end of this lesson.

Move to page 2.1.

9. Follow the discussion led by your teacher for Problem 2 of this activity.

Teacher Tip: Problem 2 of the TI-Nspire document provides an opportunity for a formal look at the derivation of the center-radius form of the equation of a circle based upon the locus definition.

Teacher Tip: Discuss what is meant by "locus of points" in the definition of a circle. How is the distance formula going to apply to a locus of points? Make sure students understand the use of subscripts in the labeling of the coordinates of the points on this page. Discuss how the application of the Pythagorean Theorem in this situation results in the distance formula shown. You might wish to have students complete the right triangle either on the TI-Nspire page or on graph paper.

Move to page 2.2.

Teacher Tip: Have students verbalize their understanding that this form of the distance formula is the same as the previous. Only the variables have been changed.

Move to page 2.3.

Teacher Tip: Assure that students follow the algebraic manipulation that results in the center-radius form for the equation of a circle.

Move to page 2.4.

Teacher Tip: Have students connect this equation to the specific examples that they addressed in earlier parts of this activity.

Move to page 3.1. After reading, move to page 3.2.

Teacher Tip: Another part of the Common Core standard is to complete the square of a standard form equation of a circle to determine the center and radius of a circle. The next page will have the students graph the standard form of a circle, and then the teacher can algebraically show students how to complete the square to arrive at the center-radius form of a circle equation.

10. Select **Menu > Graph Entry/Edit > Equation > Circle >** $a \cdot x^2 + a \cdot y^2 + b \cdot x + c \cdot y + d = 0$ to choose the standard form of a circle.
    - Type a 1 in the first box, then tab (a 1 will automatically be put in the second box), tab, then -4, tab, 6, tab and finally -3.
    - A circle with center (2,-3) and radius 4 will be drawn.

Wrap Up

Upon completion of the discussion, the teacher should ensure that students understand:

- A circle is the set of all points equidistant from a given point.
- The coordinates of a point on a circle must satisfy the equation of that circle.
- The relationship between the Pythagorean Theorem, Distance Formula, and the equation of a circle.
- The equation of a circle not centered at the origin: $(x-h)^2+(y-k)^2=r^2$, where r is the radius and (h, k) is the center.

Assessment

- What is the length of the radius of the circle with equation $(x + 2)^2 + (y - 6)^2 = 10$?
- What is the center of the circle with equation $(x + 2)^2 + (y - 6)^2 = 10$?
- Write the equation of a circle with center (3, –5) and a radius of length 4.

TI-Nspire Navigator

Note 1

Question 1, *Live Presenter*

Use Live Presenter to show students how to use the Geometry Trace tool.

Note 2

Question 3, *Class Capture*

Use Class Capture to view each student's circle with different radii. Discuss how the circles are similar and different.

Note 3

Question 8, *Quick Poll*

Collect student responses to questions 8a and 8b in an Open Response Quick Poll.

For 8c, send a Custom Choice Quick Poll.

The point (12, –9) lies _____ the circle with equation $(x - 12)^2 + (y + 4)^2 = 25$.

A. inside

B. on

C. outside

Answer: B. The point lies **on** the circle.