Monday Night Calculus, November 15, 2021

1. If $x > 1$, then find $\dfrac{d}{dx}\displaystyle\int_e^x \frac{1}{t}\,dt$ (Ahsley Labrucherie)

Solution

> **The Fundamental Theorem of Calculus, Part 1**
> If f is a continuous function on $[a, b]$, then the function g defined by
> $$g(x) = \int_a^x f(t)\,dt \qquad a \le x \le b$$
> is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

$$\frac{d}{dx}\int_e^x \frac{1}{t}\,dt = \frac{1}{x}$$

Extras

(a) $g(x) = \displaystyle\int_{\sin x}^1 \sqrt{1+t^4}\,dt$

$$\frac{d}{dx}\left[\int_{\sin x}^1 \sqrt{1+t^4}\,dt\right] = -\frac{d}{dx}\left[\int_1^{\sin x} \sqrt{1+t^4}\,dt\right] \qquad \text{Property of Integrals}$$

$$= -\frac{d}{dx}\left[\int_1^{u} \sqrt{1+t^4}\,dt\right] \qquad \text{Let } u = \sin x$$

$$= -\frac{d}{du}\left[\int_1^{u} \sqrt{1+t^4}\,dt\right]\frac{du}{dx} \qquad \text{Chain Rule}$$

$$= -\sqrt{1+u^4}\,\frac{du}{dx} \qquad \text{FTC1}$$

$$= -\sqrt{1+\sin^4 x}\cdot\cos x \qquad \text{Use } u = \sin x$$

(b) $g(x) = \int_{-x^2}^{x^2} \frac{1}{1+t^2}\,dt$

$$\begin{aligned}
\frac{d}{dx}\left[\int_{-x^2}^{x^2} \frac{1}{1+t^2}\,dt\right] &= \frac{d}{dx}\left[\int_{-x^2}^{0} \frac{1}{1+t^2}\,dt + \int_{0}^{x^2} \frac{1}{1+t^2}\,dt\right] \\
&= \frac{d}{dx}\left[-\int_{0}^{-x^2} \frac{1}{1+t^2}\,dt + \int_{0}^{x^2} \frac{1}{1+t^2}\,dt\right] \\
&= -\frac{1}{1+(-x^2)^2}\cdot(-2x) + \frac{1}{1+(x^2)^2}\cdot 2x \\
&= \frac{2x}{1+x^4} + \frac{2x}{1+x^4} \\
&= \frac{4x}{1+x^4}
\end{aligned}$$

2. Find the equation(s) of any vertical tangent lines of $4x^3 - 4y = y^4$. (Denise Wright)

Solution

$$4x^3 - 4y = y^4$$

$$12x^2 - 4 \cdot \frac{dy}{dx} = 4y^3 \cdot \frac{dy}{dx}$$

$$-4 \cdot \frac{dy}{dx} - 4y^3 \cdot \frac{dy}{dx} = -12x^2$$

$$\frac{dy}{dx}(4 + 4y^3) = 12x^2$$

$$\frac{dy}{dx} = \frac{12x^2}{4 + 4y^3} = \frac{3x^2}{1 + y^3}$$

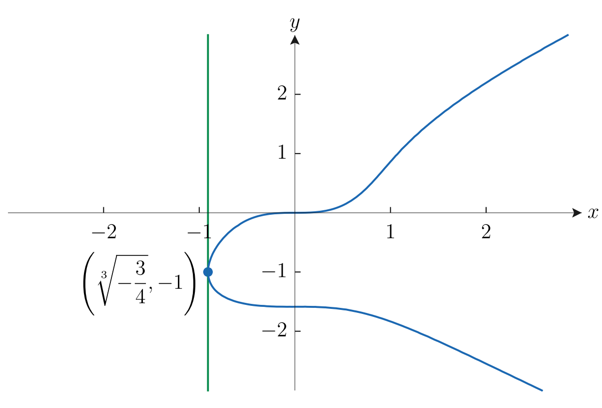
Possible vertical tangent: $1 + y^3 = 0 \quad \Rightarrow \quad y = -1$

Find the corresponding value of x (make sure that $3x^2 \neq 0$).

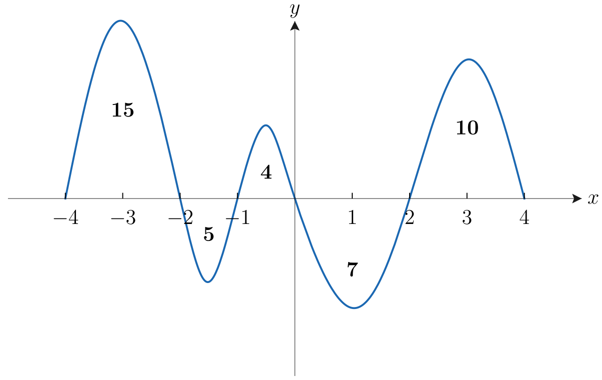
$$4x^3 - 4(-1) = (-1)^4$$

$$4x^3 = -3 \quad \Rightarrow \quad x = \sqrt[3]{-\frac{3}{4}} = -0.909$$

$$3x^2 = 3\left(\left(-\frac{3}{4}\right)^{1/3}\right)^2 \neq 0$$



3. The figure below shows the graph of f. The areas of the regions between the graph of f and the x-axis are labeled.



Determine the value of each definite integral.

(a) $\displaystyle\int_0^4 f(x)\,dx =$

(b) $\displaystyle\int_{-2}^2 f(x)\,dx =$

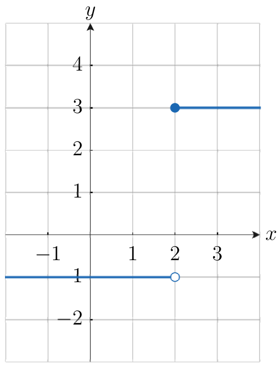
(c) $\displaystyle\int_{-4}^0 |f(x)|\,dx =$

(d) $\displaystyle\left|\int_{-4}^0 f(x)\,dx\right| =$

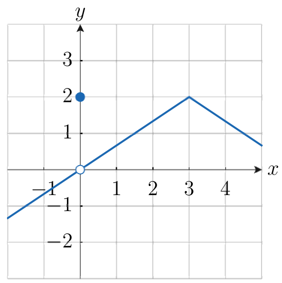
(e) $\displaystyle\int_{-4}^4 f(|x|)\,dx =$

(f) $\displaystyle\int_1^4 f(-x)\,dx =$

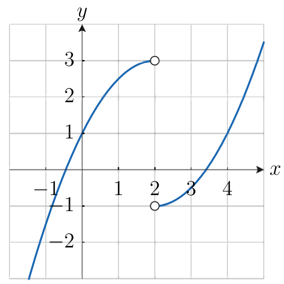
4. The graphs of four functions are shown in the figures. (Judy Mitchell Barnette)



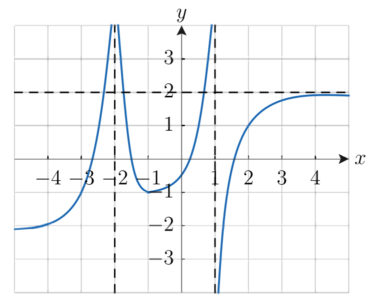
$y = f(x)$



$y = g(x)$



$y = h(x)$



$y = j(x)$

Find the following limits or state that the limit does not exist.

(a) $\lim\limits_{x\to -2} j(x) =$

(b) $\lim\limits_{x\to 1} j(x) =$

(c) $\lim\limits_{x\to -1} \dfrac{f(x)-2}{[j(x)]^2} =$

(d) $\lim\limits_{x\to \infty} h(j(x)) =$

(e) $\lim\limits_{x\to 3} f(g(x)) =$

(f) $\lim\limits_{x\to 0} f(|x|+2) =$

(g) $\lim\limits_{x\to 0}[g(x)\cdot f(x+2)] =$

(h) $\lim\limits_{x\to 2} \left[(f(x)-1)^2 - 6\right] =$

(i) $\lim\limits_{x\to 2}[h(x)+f(x)] =$

(j) $\lim\limits_{x\to -2} j(j(x)) =$

(k) $\lim\limits_{x\to 1.5} \dfrac{g(x)-1}{2x-3} =$

(l) $\lim\limits_{x\to 0} g(f(x)+1) =$

5. What values of a and b make the function differentiable at $x = 4$? (Amandeep Heer)

$$f(x) = \begin{cases} a\sqrt{x} + bx^2 - 1 & \text{if } x < 4 \\ \dfrac{16}{x} + bx & \text{if } x \geq 4 \end{cases}$$

Solution

Continuous at $x = 4$

$$\lim_{x \to 4} f(x) = f(4) = \frac{16}{4} + b \cdot 4 = 4 + 4b \quad \left(= \lim_{x \to 4^+} f(x) \right)$$

$$\lim_{x \to 4^-} f(x) = a\sqrt{4} + b \cdot 4^2 - 1 = 2a + 16b - 1 = 4 + 4b$$

$$\Rightarrow \quad 2a + 12b = 5$$

Differentiable at $x = 4$

$$x < 4: \; f'(x) = \frac{1}{2}ax^{-1/2} + 2bx = \frac{a}{2\sqrt{x}} + 2bx$$

$$x \geq 4: \; f'(x) = -16x^{-2} + b = -\frac{16}{x^2} + b$$

$$\frac{a}{2\sqrt{4}} + 2b \cdot 4 = -\frac{16}{4^2} + b \quad \Rightarrow \quad \frac{a}{4} + 7b = -1$$

$$\begin{cases} 2a + 12b = 5 \\ \dfrac{a}{4} + \quad 7b = -1 \end{cases} \quad \Rightarrow \quad \begin{cases} 2a + 12b = 5 \\ -2a - 56b = 8 \end{cases} \quad \Rightarrow \quad -44b = 13 \quad \Rightarrow \quad b = -\frac{13}{44}$$

$$2a + 12\left(-\frac{13}{44}\right) = 5 \quad \Rightarrow \quad 2a = \frac{94}{11} \quad \Rightarrow \quad a = \frac{47}{11}$$